Basics of black hole physics

3. The Kerr black hole

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Centre for Strings, Gravitation and Cosmology
Chennai, India
17-22 January 2022

1. What is a black hole? *(yesterday)*
2. Schwarzschild black hole *(today)*
3. Kerr black hole *(today)*
4. Black hole dynamics *(on Wednesday)*

Home page for the lectures

https://luth.obspm.fr/~luthier/gourgoulhon/bh16/chennai/
(slides, lecture notes, SageMath notebooks)

Lecture 3: The Kerr black hole

Outline

The Kerr solution (1963)

Spacetime manifold $\mathscr{M}$

$$\mathscr{M} := \mathbb{R}^2 \times \mathbb{S}^2 \setminus \mathscr{R}$$

with $\mathscr{R} := \left\{ p \in \mathbb{R}^2 \times \mathbb{S}^2, \quad r(p) = 0 \text{ and } \theta(p) = \frac{\pi}{2} \right\}$,

(t, r) spanning $\mathbb{R}^2$ and (θ, φ) spanning $\mathbb{S}^2$

Boyer-Lindquist (BL) coordinates (t, r, θ, φ) (1967)

(t, r, θ, φ) with $t \in \mathbb{R}$, $\boxed{r \in \mathbb{R}}$, $\theta \in (0, \pi)$ and $\varphi \in (0, 2\pi)$

The Kerr solution (1963)

Spacetime metric $\boldsymbol{g}$

2 parameters (m, a) such that $0 < a \leq m$

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2,$$

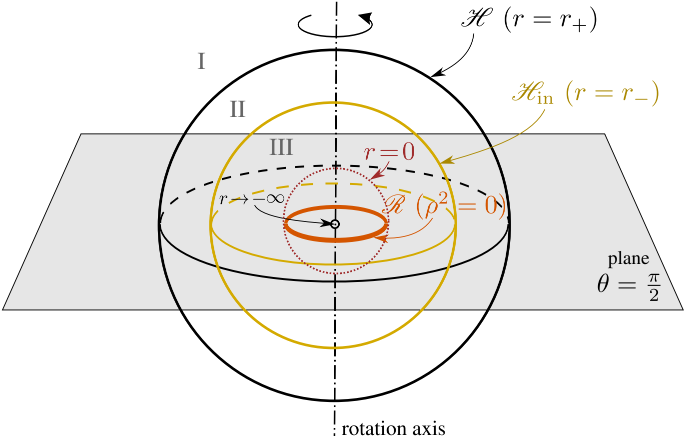
where $\rho^2 := r^2 + a^2\cos^2\theta$ and $\Delta := r^2 - 2mr + a^2$

Some metric components diverge when

- $\rho = 0 \iff r = 0$ and $\theta = \pi/2$ (set $\mathscr{R}$, excluded from $\mathscr{M}$)
- $\Delta = 0 \iff r = r_+ := m + \sqrt{m^2 - a^2}$ or $r = r_- := m - \sqrt{m^2 - a^2}$

Define $\mathscr{H}$: hypersurface $r = r_+$, $\mathscr{H}_{\mathrm{in}}$: hypersurface $r = r_-$

Section of constant Boyer-Lindquist time coordinate



View of a section $t = \mathrm{const}$ in O'Neill coord. (R, θ, φ) with $R := e^r$

NB: $r = 0$ is a sphere, not a point

Define three regions, bounded by $\mathscr{H}$ or $\mathscr{H}_{\rm in}$:

$\mathscr{M}_{\rm I}$: $r > r_+$, $\mathscr{M}_{\rm II}$: $r_- < r < r_+$, $\mathscr{M}_{\rm III}$: $r < r_-$

Basic properties of Kerr metric (1/3)

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2,$$

- $\boldsymbol{g}$ is a solution of the vacuum Einstein equation: $\mathrm{Ric}(\boldsymbol{g}) = 0$
  See this SageMath notebook for an explicit check:
  https://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_solution.ipynb

Basic properties of Kerr metric (2/3)

$$\mathrm{d}s^2 = -\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}t^2 - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\varphi + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\varphi^2,$$

- asymptotic behavior:

$$r \to \pm\infty \Longrightarrow \quad \rho^2 \sim r^2,\ \rho^2/\Delta \sim (1 - 2m/r)^{-1},$$
$$4amr/\rho^2\,\mathrm{d}t\,\mathrm{d}\varphi \sim 4am/r^2\,\mathrm{d}t\,r\mathrm{d}\varphi$$

$$\Longrightarrow \mathrm{d}s^2 \sim \quad -(1 - 2m/r)\,\mathrm{d}t^2 + (1 - 2m/r)^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) + O\left(r^{-2}\right)$$

$\Longrightarrow$ Schwarzschild metric of mass m for $r > 0$

Schwarzschild metric of mass $m' = -m$ (negative!) for $r < 0$

Basic properties of Kerr metric (3/3)

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- $\partial g_{\alpha\beta}/\partial t = 0 \Longrightarrow \boxed{\boldsymbol{\xi} := \boldsymbol{\partial}_t}$ is a Killing vector; since $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) < 0$ for r large enough, which means that $\boldsymbol{\xi}$ is timelike, $(\mathscr{M}, \boldsymbol{g})$ is pseudostationary

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- if $a \neq 0$, $g_{t\phi} \neq 0 \Longrightarrow \boldsymbol{\xi}$ is not orthogonal to the hypersurface $t = \mathrm{const} \Longrightarrow (\mathscr{M},\boldsymbol{g})$ is *not* static

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- when $a = 0$, $\boldsymbol{g}$ reduces to Schwarzschild metric (then the region $r \leq 0$ is excluded from the spacetime manifold)

The ring singularity

$$\begin{aligned} \mathrm{d}s^2 = \; & -\left(1 - \frac{2mr}{\rho^2}\right) \mathrm{d}t^2 - \frac{4amr\sin^2\theta}{\rho^2}\, \mathrm{d}t\, \mathrm{d}\varphi + \frac{\rho^2}{\Delta}\, \mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 \\ & + \left(r^2 + a^2 + \frac{2a^2 mr\sin^2\theta}{\rho^2}\right) \sin^2\theta\, \mathrm{d}\varphi^2, \end{aligned}$$

- The singularity of the metric components at $\Delta = 0$ is a mere coordinate singularity as we shall see by moving to Kerr coordinates

The ring singularity

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- The singularity of the metric components at $\Delta = 0$ is a mere coordinate singularity as we shall see by moving to Kerr coordinates
- The singularity at $\rho^2 = 0$ corresponds a curvature singularity as shown by the expression of the Kretschmann scalar:

$$K := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 48\frac{m^2}{\rho^{12}}\left(r^6 - 15r^4a^2\cos^2\theta + 15r^2a^4\cos^4\theta - a^6\cos^6\theta\right)$$

$$\rho^2 := r^2 + a^2\cos^2\theta = 0 \iff r = 0 \text{ and } \theta = \frac{\pi}{2}$$

$\Longrightarrow$ ring singularity $\mathscr{R}$

Ergoregion

Scalar square of the pseudostationary Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_t$:

$$\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{tt} = -1 + \frac{2mr}{r^2 + a^2 \cos^2\theta}$$

$\boldsymbol{\xi}$ timelike $\iff$ $r < r_{\mathscr{E}-}(\theta)$ or $r > r_{\mathscr{E}+}(\theta)$

$$r_{\mathscr{E}\pm}(\theta) := m \pm \sqrt{m^2 - a^2 \cos^2\theta}$$

$$0 \le r_{\mathscr{E}-}(\theta) \le r_- \le m \le r_+ \le r_{\mathscr{E}+}(\theta) \le 2m$$

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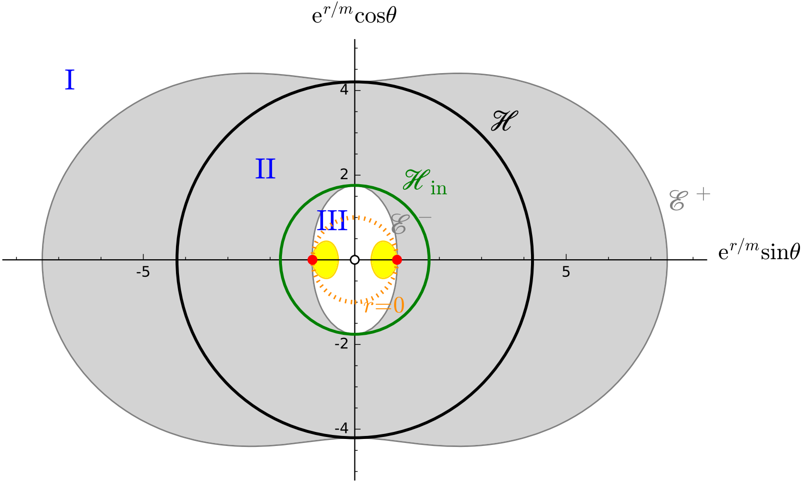
Ergoregion: part $\mathscr{G}$ of $\mathscr{M}$ where $\boldsymbol{\xi}$ is spacelike
Ergosphere: boundary $\mathscr{E}$ of the ergoregion: $r = r_{\mathscr{E}\pm}(\theta)$

$\mathscr{G}$ encompasses all $\mathscr{M}_{\mathrm{II}}$, the part of $\mathscr{M}_{\mathrm{I}}$ where $r < r_{\mathscr{E}+}(\theta)$ and the part of $\mathscr{M}_{\mathrm{III}}$ where $r > r_{\mathscr{E}-}(\theta)$

Remark: at the Schwarzschild limit, $a = 0 \Longrightarrow r_{\mathscr{E}+}(\theta) = 2m$
$\Longrightarrow \mathscr{G}$ = black hole region

Ergoregion



Meridional slice $t = t_0$, $\phi \in \{0, \pi\}$ viewed in O'Neill coordinates
grey: ergoregion; *yellow:* Carter time machine; *red:* ring singularity

Outline

From Boyer-Lindquist to Kerr coordinates

Introduce (3+1 version of) **Kerr coordinates** $(\tilde{t}, r, \theta, \tilde{\varphi})$ by

$$\begin{cases} d\tilde{t} &= dt + \frac{2mr}{\Delta}\, dr \\ d\tilde{\varphi} &= d\varphi + \frac{a}{\Delta}\, dr \end{cases}$$

$$\Longrightarrow \begin{cases} \tilde{t} &= t + \frac{m}{\sqrt{m^2-a^2}} \left(r_+ \ln\left|\frac{r-r_+}{2m}\right| - r_- \ln\left|\frac{r-r_-}{2m}\right| \right) \\ \tilde{\varphi} &= \varphi + \frac{a}{2\sqrt{m^2-a^2}} \ln\left|\frac{r-r_+}{r-r_-}\right| \end{cases}$$

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Kerr coord. reduce to *ingoing Eddington-Finkelstein* coord. when $a \to 0$ ($r_+ \to 2m$, $r_- \to 0$):

$$\begin{cases} \tilde{t} & = & t + 2m \ln\left|\frac{r}{2m} - 1\right| \\ \tilde{\varphi} & = & \varphi \end{cases}$$

Kerr coordinates

Spacetime metric in Kerr coordinates

$$
\begin{aligned}
\mathrm{d}s^2 = & -\left(1-\frac{2mr}{\rho^2}\right)\mathrm{d}\tilde{t}^2 + \frac{4mr}{\rho^2}\mathrm{d}\tilde{t}\,\mathrm{d}r - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{\varphi} \\
& +\left(1+\frac{2mr}{\rho^2}\right)\mathrm{d}r^2 - 2a\left(1+\frac{2mr}{\rho^2}\right)\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\tilde{\varphi} \\
& +\rho^2\mathrm{d}\theta^2 + \left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\tilde{\varphi}^2.
\end{aligned}
$$

Note

- contrary to Boyer-Lindquist ones, the metric components are regular where $\Delta = 0$, i.e. at $r = r_+$ ($\mathscr{H}$) and $r = r_-$ ($\mathscr{H}_{\rm in}$)
- the two Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ coincide with the coordinate vectors associated to $\tilde{t}$ and $\tilde{\varphi}$: $\boxed{\boldsymbol{\xi} = \boldsymbol{\partial}_{\tilde{t}}}$ and $\boxed{\boldsymbol{\eta} = \boldsymbol{\partial}_{\tilde{\varphi}}}$

Outline

Constant-r hypersurfaces

A normal to any $r = \mathrm{const}$ hypersurface is $\boldsymbol{n} := \rho^2 \vec{\boldsymbol{\nabla}} r$, where $\vec{\boldsymbol{\nabla}} r$ is the gradient of r: $\nabla^\alpha r = g^{\alpha\mu}\partial_\mu r = g^{\alpha r} = \left(\frac{2mr}{\rho^2}, \frac{\Delta}{\rho^2}, 0, \frac{a}{\rho^2}\right)$
$\Longrightarrow \boldsymbol{n} = 2mr\, \boldsymbol{\partial}_{\tilde{t}} + \Delta\, \boldsymbol{\partial}_{\tilde{r}} + a\, \boldsymbol{\partial}_{\tilde{\varphi}}$

One has
$\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = g_{\mu\nu} n^\mu n^\nu = g_{\mu\nu}\rho^2 \nabla^\mu r\, n^\nu = \rho^2 \nabla_\nu r\, n^\nu = \rho^2 \partial_\nu r\, n^\nu = \rho^2 n^r$
hence

$$\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = \rho^2 \Delta$$

Constant-r hypersurfaces

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$\Longrightarrow \boldsymbol{n} = 2mr\,\boldsymbol{\partial}_{\tilde{t}} + \Delta\,\boldsymbol{\partial}_{\tilde{r}} + a\,\boldsymbol{\partial}_{\tilde{\varphi}}$

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$\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = g_{\mu\nu} n^\mu n^\nu = g_{\mu\nu}\rho^2 \nabla^\mu r\, n^\nu = \rho^2 \nabla_\nu r\, n^\nu = \rho^2 \partial_\nu r\, n^\nu = \rho^2 n^r$

hence

$$\boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = \rho^2 \Delta$$

Given that $\Delta = (r - r_-)(r - r_+)$, we conclude:

- The hypersurfaces $r = \mathrm{const}$ are timelike in $\mathscr{M}_{\mathrm{I}}$ and $\mathscr{M}_{\mathrm{III}}$
- The hypersurfaces $r = \mathrm{const}$ are spacelike in $\mathscr{M}_{\mathrm{II}}$
- $\mathscr{H}$ (where $r = r_+$) and $\mathscr{H}_{\mathrm{in}}$ (where $r = r_-$) are null hypersurfaces

Killing horizons

The (null) normals to the null hypersurfaces $\mathscr{H}$ and $\mathscr{H}_{\rm in}$ are

$$\boldsymbol{n} = \underbrace{2mr}_{2mr_\pm} \underbrace{\boldsymbol{\partial}_{\tilde{t}}}_{\boldsymbol{\xi}} + \underbrace{\Delta}_{0} \boldsymbol{\partial}_{\tilde{r}} + a \underbrace{\boldsymbol{\partial}_{\tilde{\varphi}}}_{\boldsymbol{\eta}} = 2mr_\pm \boldsymbol{\xi} + a\, \boldsymbol{\eta}$$

On $\mathscr{H}$, let us consider the rescaled null normal $\boldsymbol{\chi} := (2mr_+)^{-1}\boldsymbol{n}$:

$$\boxed{\boldsymbol{\chi} = \boldsymbol{\xi} + \Omega_H\, \boldsymbol{\eta}}$$

with

$$\boxed{\Omega_H := \frac{a}{2mr_+} = \frac{a}{r_+^2 + a^2} = \frac{a}{2m\left(m + \sqrt{m^2 - a^2}\right)}}$$

Killing horizons

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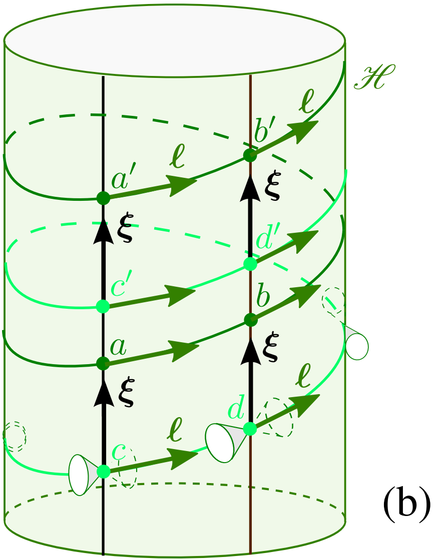
$\boldsymbol{\chi}$ = linear combination with *constant* coefficients of the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ $\Longrightarrow$ $\boldsymbol{\chi}$ is a Killing vector. Hence

> The null hypersurface $\mathscr{H}$ defined by $r = r_+$ is a Killing horizon

Similarly

> The null hypersurface $\mathscr{H}_{\rm in}$ defined by $r = r_-$ is a Killing horizon

Killing horizon $\mathscr{H}$

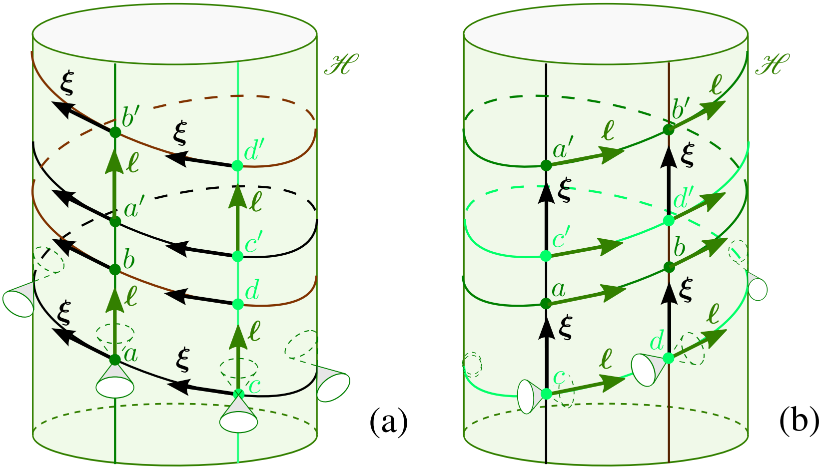


Null normal to $\mathscr{H}$: $\boldsymbol{\chi} = \boldsymbol{\xi} + \Omega_H \, \boldsymbol{\eta}$ (on the picture $\boldsymbol{\ell} \propto \boldsymbol{\chi}$)

$\Longrightarrow$ Ω_H $\sim$ “angular velocity” of $\mathscr{H}$
$\Longrightarrow$ rigid rotation (Ω_H independent of θ)

NB: since $\mathscr{H}$ is inside the ergoregion, $\boldsymbol{\xi}$ is spacelike on $\mathscr{H}$

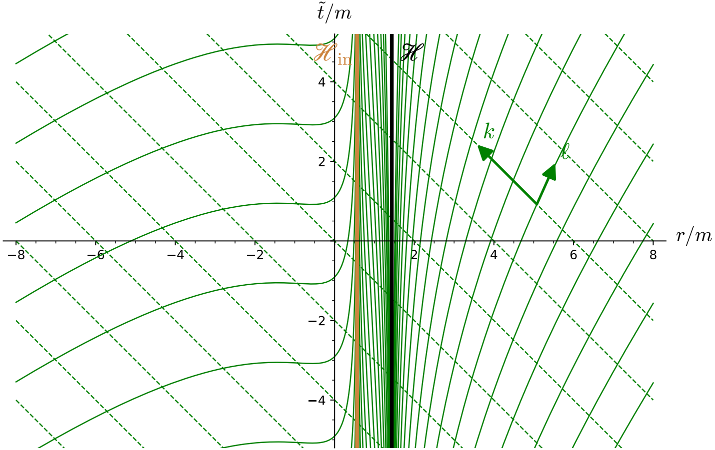
Two views of the horizon $\mathscr{H}$



null geodesic generators drawn vertically

field lines of Killing vector ξ drawn vertically

The Killing horizon $\mathscr{H}$ is an event horizon



← Principal null geodesics for $a/m = 0.9$

Recall: for $r \to +\infty$, Kerr metric $\sim$ Schwarzschild metric
$\Longrightarrow$ same asymptotic structure
$\Longrightarrow$ same $\mathscr{I}^+$

$\mathscr{H}$ is a black hole event horizon

What happens for $a \geq m$?

$$\Delta := r^2 - 2mr + a^2$$

$a = m$: extremal Kerr black hole

$a = m \iff \Delta = (r - m)^2$

$\iff$ double root: $r_+ = r_- = m \iff \mathscr{H}$ and $\mathscr{H}_{\rm in}$ coincide

$\iff$ $\mathscr{H}$ is a **degenerate Killing horizon** (vanishing surface gravity κ, see below)

$a > m$: naked singularity

$a > m \iff \Delta > 0$

$\iff \boldsymbol{g}(\boldsymbol{n}, \boldsymbol{n}) = \rho^2 \Delta > 0 \iff$ all hypersurfaces $r = \mathrm{const}$ are timelike

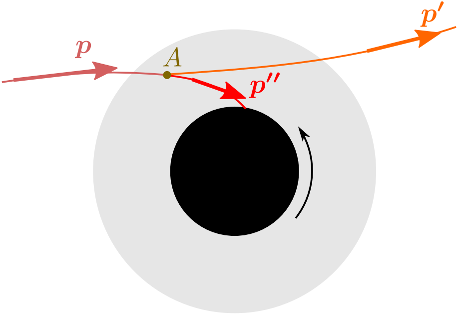
$\iff$ any of them can be crossed in the direction of increasing r

$\iff$ no horizon $\iff$ no black hole

$\iff$ the curvature singularity at $\rho^2 = 0$ is naked

Outline

Penrose process



Particle $\mathscr{P}$ (4-momentum $\boldsymbol{p}$) in free fall from infinity into the ergoregion $\mathscr{G}$. At point $A \in \mathscr{G}$, $\mathscr{P}$ splits (or decays) into

- particle $\mathscr{P}'$ (4-momentum $\boldsymbol{p}'$), which leaves to infinity
- particle $\mathscr{P}''$ (4-momentum $\boldsymbol{p}''$), which falls into the black hole

Energy gain: $\boxed{\Delta E = E_{\rm out} - E_{\rm in}}$

with $E_{\rm in} = -\, \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p})\big|_{\infty}$ and $E_{\rm out} = -\, \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p}')\big|_{\infty}$

since at infinity, $\boldsymbol{\xi} = \boldsymbol{\partial}_t$ is the 4-velocity of the inertial observer at rest with respect to the black hole.

Conserved energy along a geodesic

Geodesic Noether's theorem

Assume

- $(\mathscr{M}, \boldsymbol{g})$ is a spacetime endowed with a 1-parameter symmetry group, generated by the Killing vector $\boldsymbol{\xi}$
- $\mathscr{L}$ is a geodesic of $(\mathscr{M}, \boldsymbol{g})$ with tangent vector field $\boldsymbol{p}$: $\boldsymbol{\nabla}_{\boldsymbol{p}}\, \boldsymbol{p} = 0$

Then the scalar product $E := -\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p})$ is constant along $\mathscr{L}$.

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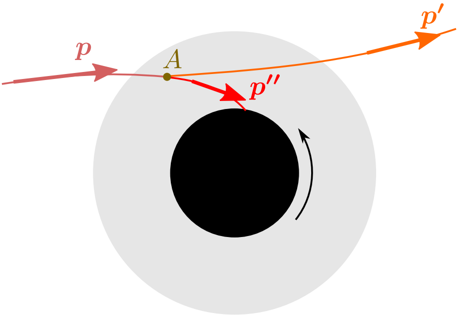
Then the scalar product $E := -\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p})$ is constant along $\mathscr{L}$.

Proof:

$$\begin{aligned} \boldsymbol{\nabla}_{\boldsymbol{p}}\left(\boldsymbol{g}(\boldsymbol{\xi},\boldsymbol{p})\right) &= p^\sigma \nabla_\sigma (g_{\mu\nu}\xi^\mu p^\nu) = p^\sigma \nabla_\sigma (\xi_\nu p^\nu) = \nabla_\sigma \xi_\nu\, p^\sigma p^\nu + \xi_\nu p^\sigma \nabla_\sigma p^\nu \\ &= \frac{1}{2}\underbrace{(\nabla_\sigma \xi_\nu + \nabla_\nu \xi_\sigma)}_{0} p^\sigma p^\nu + \xi_\nu \underbrace{p^\sigma \nabla_\sigma p^\nu}_{0} = 0 \end{aligned}$$

Penrose process

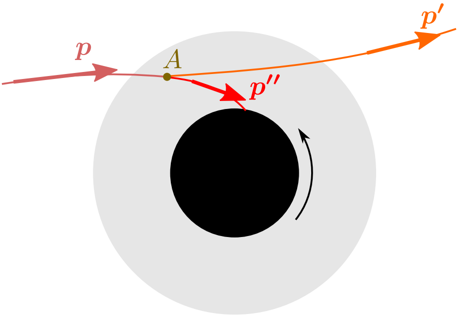


$$\Delta E = -\left. \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p}')\right|_{\infty} + \left. \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p})\right|_{\infty}$$

Geodesic Noether's theorem:

$$\begin{aligned} \Delta E &= -\left. \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p}')\right|_{A} + \left. \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p})\right|_{A} \\ &= \left. \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{p} - \boldsymbol{p}')\right|_{A} \end{aligned}$$

Penrose process



$$\Delta E = -\, g(\xi, p')\big|_\infty + g(\xi, p)\big|_\infty$$

Geodesic Noether's theorem:

$$\begin{aligned} \Delta E &= -\, g(\xi, p')\big|_A + g(\xi, p)\big|_A \\ &= g(\xi, p - p')\big|_A \end{aligned}$$

Conservation of energy-momentum at event A: $p|_A = p'|_A + p''|_A$

$$\Longrightarrow p|_A - p'|_A = p''|_A$$

$$\Longrightarrow \Delta E = g(\xi, p'')\big|_A$$

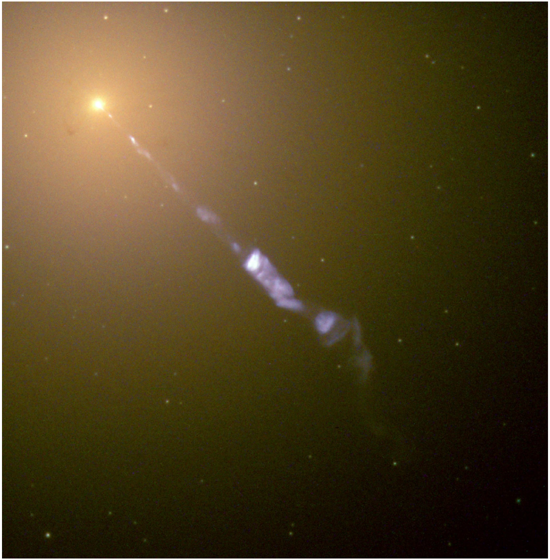
Now

- p'' is a future-directed timelike or null vector
- ξ is a spacelike vector in the ergoregion

$\Longrightarrow$ one may choose some trajectory so that $g(\xi, p'')\big|_A > 0$

$\Longrightarrow$ $\boxed{\Delta E > 0}$, i.e. energy is extracted from the rotating black hole!

Penrose process at work



Jet emitted by the nucleus of the giant elliptic galaxy M87, at the center of Virgo cluster [HST]

$M_{\rm BH} = 3 \times 10^9 \, M_\odot$

$V_{\rm jet} \simeq 0.99 \, c$

Outline

Mass

Total mass of a (pseudo-)stationary spacetime (Komar integral)

$$M = -\frac{1}{8\pi} \int_{\mathscr{S}} \nabla^\mu \xi^\nu \, \epsilon_{\mu\nu\alpha\beta}$$

- $\mathscr{S}$: any closed spacelike 2-surface located in the vacuum region
- $\boldsymbol{\xi}$: stationary Killing vector, normalized to $\boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = -1$ at infinity
- $\boldsymbol{\epsilon}$: volume 4-form associated to $\boldsymbol{g}$ (Levi-Civita tensor)

Physical interpretation: M measurable from the orbital period of a test particle in far circular orbit around the black hole *(Kepler's third law)*

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For a Kerr spacetime of parameters (m, a):

$$M = m$$

Angular momentum

Total angular momentum of an axisymmetric spacetime (Komar integral)

$$J = \frac{1}{16\pi} \int_{\mathscr{S}} \nabla^\mu \eta^\nu \, \epsilon_{\mu\nu\alpha\beta}$$

- $\mathscr{S}$: any closed spacelike 2-surface located in the vacuum region
- $\boldsymbol{\eta}$: axisymmetric Killing vector
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Physical interpretation: J measurable from the precession of a gyroscope orbiting the black hole *(Lense-Thirring effect)*

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$$J = am$$

Black hole area

As a non-expanding horizon, $\mathscr{H}$ has a well-defined (cross-section independent) area A:

$$A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}\theta\, \mathrm{d}\tilde{\varphi}$$

- $\mathscr{S}$: cross-section defined in terms of Kerr coordinates by $\begin{cases} \tilde{t} = \tilde{t}_0 \\ r = r_+ \end{cases}$

  $\Longrightarrow$ coordinates spanning $\mathscr{S}$: $y^a = (\theta, \tilde{\varphi})$
- $q := \det(q_{ab})$, with q_{ab} components of the Riemannian metric $\boldsymbol{q}$ induced on $\mathscr{S}$ by the spacetime metric $\boldsymbol{g}$

Black hole area

Evaluating $\boldsymbol{q}$: set $\mathrm{d}\tilde{t} = 0$, $\mathrm{d}r = 0$, and $r = r_+$ in the expression of $\boldsymbol{g}$ in terms of the Kerr coordinates:

$$\begin{aligned} g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \; & -\left(1 - \frac{2mr}{\rho^2}\right)\mathrm{d}\tilde{t}^2 + \frac{4mr}{\rho^2}\mathrm{d}\tilde{t}\,\mathrm{d}r - \frac{4amr\sin^2\theta}{\rho^2}\,\mathrm{d}\tilde{t}\,\mathrm{d}\tilde{\varphi} \\ & + \left(1 + \frac{2mr}{\rho^2}\right)\mathrm{d}r^2 - 2a\left(1 + \frac{2mr}{\rho^2}\right)\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\tilde{\varphi} \\ & + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2a^2mr\sin^2\theta}{\rho^2}\right)\sin^2\theta\,\mathrm{d}\tilde{\varphi}^2. \end{aligned}$$

and get

$$q_{ab}\,\mathrm{d}y^a\mathrm{d}y^b = (r_+^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 + \left(r_+^2 + a^2 + \frac{2a^2mr_+\sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right)\sin^2\theta\,\mathrm{d}\tilde{\varphi}^2$$

Black hole area

r_+ is a zero of $\Delta := r^2 - 2mr + a^2 \Longrightarrow 2mr_+ = r_+^2 + a^2$
$\Longrightarrow q_{ab}$ can be rewritten as

$$q_{ab}\,\mathrm{d}y^a\mathrm{d}y^b = (r_+^2 + a^2\cos^2\theta)\,\mathrm{d}\theta^2 + \frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\,\sin^2\theta\,\mathrm{d}\tilde{\varphi}^2$$

$$\Longrightarrow q := \det(q_{ab}) = (r_+^2 + a^2)^2\sin^2\theta$$

$$\Longrightarrow A = (r_+^2 + a^2)\underbrace{\int_{\mathscr{S}} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\tilde{\varphi}}_{4\pi}$$

$$\Longrightarrow \boxed{A = 4\pi(r_+^2 + a^2) = 8\pi m r_+}$$

Since $r_+ := m + \sqrt{m^2 - a^2}$, we get

$$A = 8\pi m(m + \sqrt{m^2 - a^2})$$

Black hole surface gravity

Surface gravity: name given to the non-affinity coefficient κ of the null normal $\boldsymbol{\chi} = \boldsymbol{\xi} + \Omega_H \boldsymbol{\eta}$ to the event horizon $\mathscr{H}$ (cf. lecture 1):

$$\boldsymbol{\nabla}_{\boldsymbol{\chi}} \boldsymbol{\chi} \stackrel{\mathscr{H}}{=} \kappa \boldsymbol{\chi}$$

Computation of κ: cf. the SageMath notebook
https://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Kerr_in_Kerr_coord.ipynb

$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$$

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$$\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$$

Remark: despite its name, κ is not the gravity felt by an observer staying a small distance of the horizon: the latter diverges as the distance decreases!

Outline

The no-hair theorem

Doroshkevich, Novikov & Zeldovich (1965), Israel (1967), Carter (1971), Hawking (1972), Robinson (1975)

Within 4-dimensional general relativity, a *stationary* black hole in an otherwise empty universe is necessarily a Kerr-Newman black hole, which is an electro-vacuum solution of Einstein equation described by only 3 parameters:

- the total mass M
- the total specific angular momentum $a = J/M$
- the total electric charge Q

$\Longrightarrow$ *"a black hole has no hair"* (John A. Wheeler)

The no-hair theorem

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Astrophysical black holes have to be electrically neutral:

- $Q = 0$: Kerr solution (1963)
- $Q = 0$ and $a = 0$: Schwarzschild solution (1916)
- ($Q \neq 0$ and $a = 0$: Reissner-Nordström solution (1916, 1918))

The no-hair theorem: a precise mathematical statement

Any spacetime $(\mathscr{M}, \boldsymbol{g})$ that

- is 4-dimensional
- is asymptotically flat
- is pseudo-stationary
- is a solution of the vacuum Einstein equation: $\mathrm{Ric}(\boldsymbol{g}) = 0$
- contains a black hole with a connected regular horizon
- has no closed timelike curve in the domain of outer communications (DOC) (= black hole exterior)
- is analytic

has a DOC that is isometric to the DOC of Kerr spacetime.

The no-hair theorem: a precise mathematical statement

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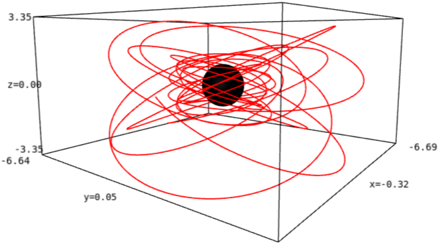
Possible improvements: remove the hypotheses of analyticity and non-existence of closed timelike curves (analyticity removed but only for slow rotation [Alexakis, Ionescu & Klainerman, Duke Math. J. **163**, 2603 (2014)])

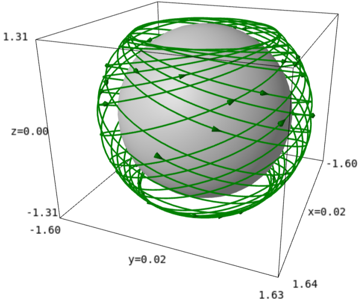
An important topic not discussed here: Kerr geodesics

See Chap. 11 and 12 of the lecture notes for details



timelike geodesic (orbit) around a Kerr BH with $a = 0.998\, m$

$\Longrightarrow$ gravitational waves from extreme mass ratio inspiral (EMRI)



spherical photon orbit around a Kerr BH with $a = 0.95\, m$

$\Longrightarrow$ critical curve on images of the vicinity of a Kerr BH

Examples of images and critical curves

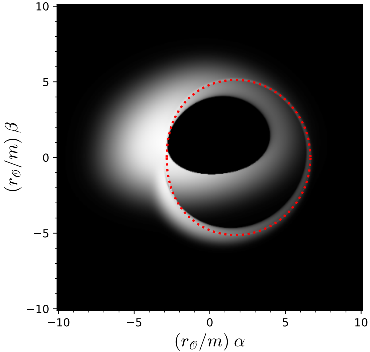
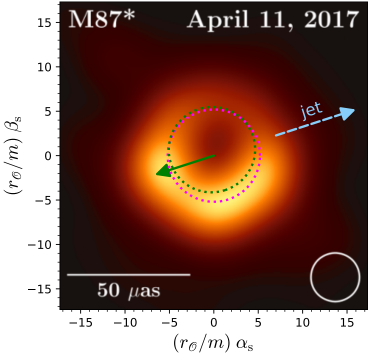


Image of a thick accretion disk around a Kerr BH with $a = 0.95m$ seen from an inclination angle $\theta = 60°$, computed with the open-source ray-tracing code Gyoto [https://gyoto.obspm.fr/] *(Fig. 12.28 of the lecture notes)*



EHT image of M87* [EHT coll., ApJL **875**, L1 (2019)] with 2 critical curves superposed: Schwarzschild BH (magenta dotted) and extremal Kerr BH with inclination $\theta = 163°$ (green dotted) *(Fig. 12.30 of the lecture notes)*