

Basics of black hole physics

4. Black hole dynamics

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
Observatoire de Paris, CNRS, Université PSL, Université de Paris
Meudon, France

<https://luth.obspm.fr/~luthier/gourgoulhon/bh16/chennai/>

School on Black Holes and Gravitational Waves

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Basics of black hole physics

Plan of the lectures

- 1 What is a black hole? (*Monday*)
- 2 Schwarzschild black hole (*Tuesday*)
- 3 Kerr black hole (*Tuesday*)
- 4 Black hole dynamics (*today*)

Home page for the lectures

<https://luth.obspm.fr/~luthier/gourgoulhon/bh16/chennai/>
(slides, lecture notes, SageMath notebooks)

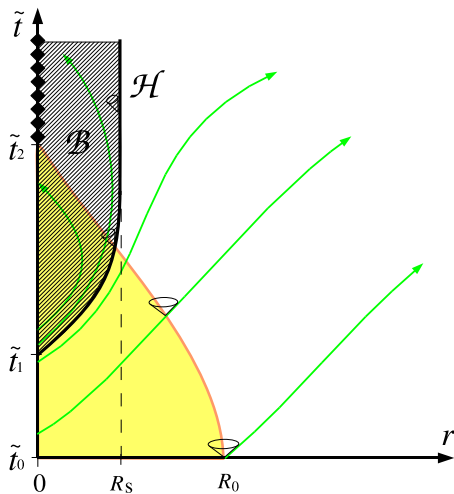
Lecture 4: Black hole dynamics

- 1 Formation of black holes
- 2 First law of black hole dynamics
- 3 Second law of black hole dynamics
- 4 Black hole thermodynamics
- 5 Applications of the second law

Outline

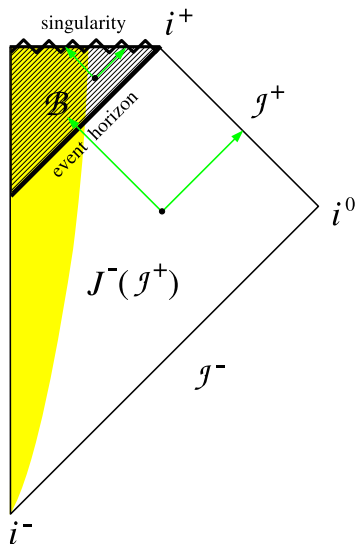
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Gravitational collapse: the astrophysical scenario of black hole formation



Gravitational collapse of a star giving birth to black hole
yellow: matter; orange: stellar surface

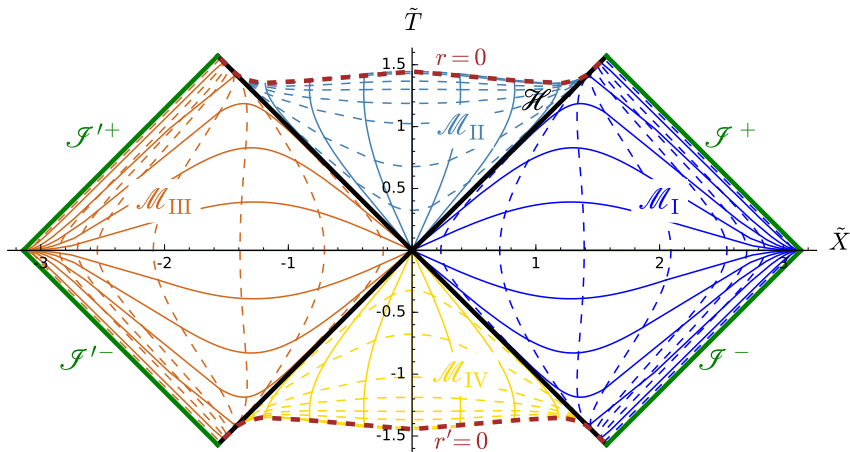
Gravitational collapse: the astrophysical scenario of black hole formation



Carter-Penrose conformal diagram

- \mathcal{I}^+ : future null infinity
- \mathcal{I}^- : past null infinity
- $J^-(\mathcal{I}^+)$: causal past of \mathcal{I}^+
- $\mathcal{B} := \mathcal{M} \setminus (J^-(\mathcal{I}^+) \cap \mathcal{M})$, black hole region
- $\mathcal{H} = \partial\mathcal{B}$, event horizon

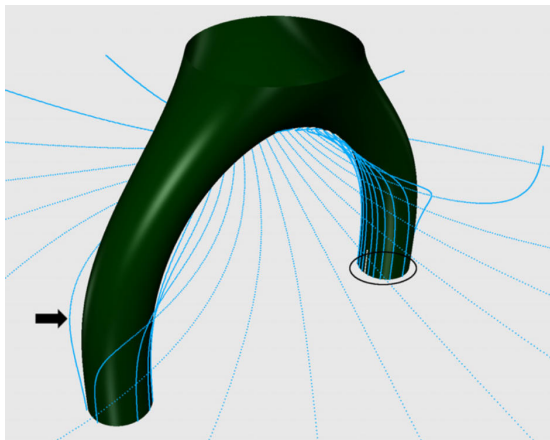
Compare with the “eternal” Schwarzschild black hole



solid: $t = \text{const}$, dashed: $r = \text{const}$
 (cf. **Lecture 2**)

Binary black hole merger

Head-on collision



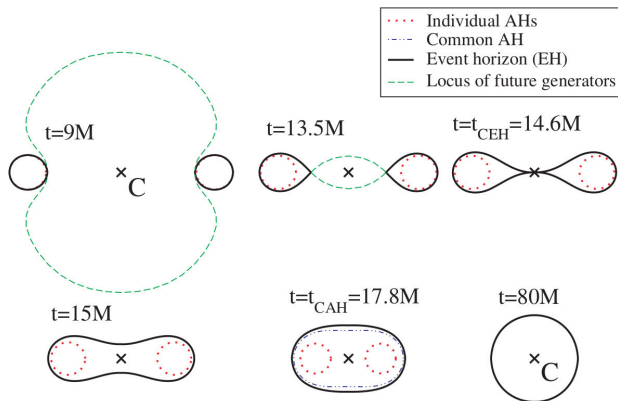
← **event horizon** of a head-on binary black hole merger (computed **a posteriori**)

blue curves: null geodesics that will eventually become null generators of the event horizon.

[Cohen, Pfeiffer & Scheel, CQG 26, 035005 (2009)]

Binary black hole merger

Head-on collision



← cross-sections \mathcal{S}_t
 of the event horizon \mathcal{H}
 of a head-on
 binary black hole
 merger at various
 coordinate times t

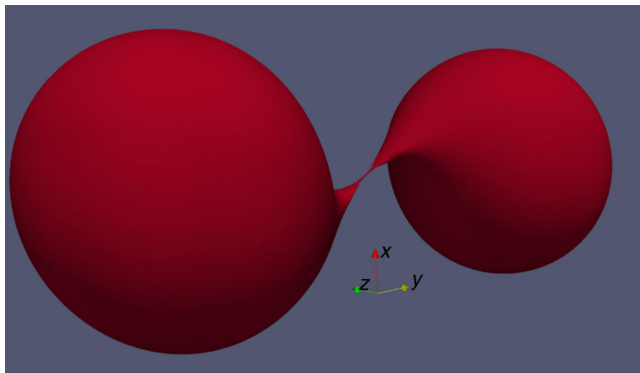
black: \mathcal{S}_t
 green dashed: trace of
 null geodesics that will
 become null
 generators of \mathcal{H}

[Cohen, Pfeiffer & Scheel, CQG 26, 035005 (2009)]

red and blue dashed: apparent horizons (marginally trapped surfaces)

Binary black hole merger

Inspiraling from circular orbit



← First connected **cross-section** of the event horizon of an inspiralling binary black hole merger (slicing by coordinate time t)

(x, y) -axes: orbital plane

[Cohen, Kaplan & Scheel, PRD **85**, 024031 (2012)]

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Small change in a black hole equilibrium

Physical setup: initially isolated Kerr black hole, of parameters (m, a) , perturbed by the arrival of some external body or some gravitational radiation.

After some transitory dynamical regime (e.g. absorption of the external body and emission of gravitational waves), the black hole relaxes to a new equilibrium configuration. According to the **no-hair theorem**, the final state has to be a Kerr black hole, of parameters $(m + \delta m, a + \delta a)$ say.

Question: how do the black hole global properties evolve during the process?

Global properties of a Kerr black hole

As seen in **Lecture 3**, a Kerr black hole of parameters (m, a) has

- mass $M = m$
- angular momentum $J = am$
- area $A = 8\pi(M^2 + \sqrt{M^4 - J^2})$
- angular velocity $\Omega_H = \frac{J}{2M(M^2 + \sqrt{M^4 - J^2})}$
- surface gravity $\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$

Relating the change in M to that in A and J

From $A = 8\pi(M^2 + \sqrt{M^4 - J^2})$, we get:

$$\frac{1}{8\pi} dA = 2M dM + \frac{2M^3}{\sqrt{M^4 - J^2}} dM - \frac{J}{\sqrt{M^4 - J^2}} dJ$$

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$$\implies dM = \frac{1}{8\pi} \underbrace{\frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\kappa} dA + \underbrace{\frac{J}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\Omega_H} dJ$$

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$$\Rightarrow \frac{\sqrt{M^4 - J^2}}{8\pi} dA = 2M(M^2 + \sqrt{M^4 - J^2}) dM - J dJ$$

$$\Rightarrow dM = \frac{1}{8\pi} \underbrace{\frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\kappa} dA + \underbrace{\frac{J}{2M(M^2 + \sqrt{M^4 - J^2})}}_{\Omega_H} dJ$$

Hence

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

On the way to the first law of black hole dynamics...

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

- dM : change in mass \equiv energy
- $\Omega_H dJ$: change in “rotational kinetic energy”
- $\frac{\kappa}{8\pi} dA$: ??

Looks premature to call this relation a *first law of black hole (thermo)dynamics*.

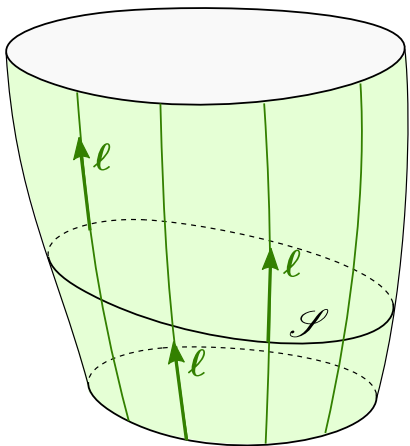
We shall come back to it later...

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Evolution of a cross-section of an event horizon

Framework: generic (dynamical) black hole, event horizon \mathcal{H}



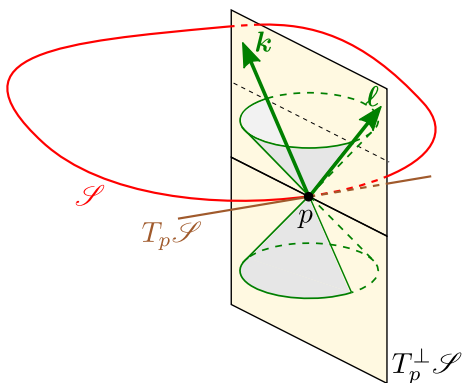
\mathcal{H} is ruled by null geodesic generators (cf. [Lecture 1](#)).

Let ℓ be the **future-directed** null normal vector field associated with an affine parameter λ of these geodesics:

$$\ell = \frac{dx}{d\lambda} \quad \text{and} \quad \nabla_{\ell} \ell = 0$$

Let us consider a cross-section \mathcal{S} of \mathcal{H} and study its **evolution along ℓ** (Lie dragging of \mathcal{S} along ℓ)

Evolution of a cross-section of an event horizon



$$T_p^\perp \mathcal{S} = \text{Span}(\mathbf{k}, \mathbf{l})$$

Since \mathcal{S} is a **spacelike surface**, for all $p \in \mathcal{S}$, the tangent space $T_p \mathcal{S}$ to \mathcal{S} is a spacelike 2-plane and admits an orthogonal complement $T_p^\perp \mathcal{S}$, which is a timelike plane:

$$T_p \mathcal{M} = T_p \mathcal{S} \oplus T_p^\perp \mathcal{S}$$

The intersection of the null cone at p with $T_p^\perp \mathcal{S}$ define 2 null directions orthogonal to \mathcal{S} :

- one is along \mathbf{l} (and thus tangent to \mathcal{H})
- the other one is along a null vector \mathbf{k} , unambiguously defined via the normalization

$$g(\mathbf{k}, \mathbf{l}) = -1$$

Induced metric and orthogonal projector

We introduce the symmetric bilinear form q by

$$q_{\alpha\beta} = g_{\alpha\beta} + \ell_{\alpha}k_{\beta} + k_{\alpha}\ell_{\beta}$$

q is a spacetime extension of the **metric induced by g** on \mathcal{S} :

Proof: if u and v are vectors tangent to \mathcal{S} :

$$q_{\mu\nu}u^{\mu}v^{\nu} = g_{\mu\nu}u^{\mu}v^{\nu} + \underbrace{\ell_{\mu}u^{\mu}}_0 k_{\nu}v^{\nu} + k_{\mu}u^{\mu} \underbrace{\ell_{\nu}v^{\nu}}_0 = g_{\mu\nu}u^{\mu}v^{\nu},$$

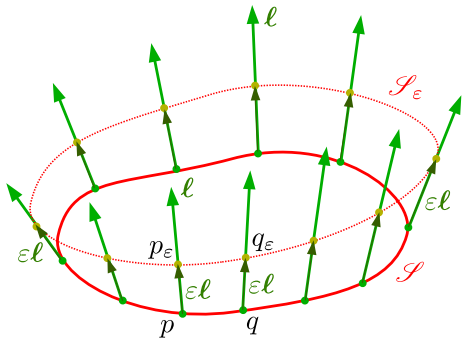
i.e. $q(u, v) = g(u, v)$. □

The **orthogonal projector onto \mathcal{S}** is the type-(1,1) tensor \vec{q} whose components are deduced from those of q by raising the first index:

$$q^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} + \ell^{\alpha}k_{\beta} + k^{\alpha}\ell_{\beta}$$

In particular $q^{\alpha}_{\mu}\ell^{\mu} = 0$ and $q^{\alpha}_{\mu}k^{\mu} = 0$.

Deformation rate tensor



The deformation rate tensor of \mathcal{S} measures the evolution of the metric q of \mathcal{S} along the null normal l , i.e. how the metric of \mathcal{S} varies when \mathcal{S} is Lie-dragged along l . The relevant operator is then the **Lie derivative along l , \mathcal{L}_l** :

$$\Theta := \frac{1}{2} \vec{q}^* \mathcal{L}_l q$$

$$\iff \Theta_{\alpha\beta} = \frac{1}{2} q^\mu{}_\alpha q^\nu{}_\beta \mathcal{L}_l q_{\mu\nu}$$

Expressing the Lie derivative in terms of the covariant derivative, via $\mathcal{L}_l q_{\mu\nu} = l^\sigma \nabla_\sigma q_{\mu\nu} + q_{\sigma\nu} \nabla_\mu l^\sigma + q_{\mu\sigma} \nabla_\nu l^\sigma$, we get

$$\Theta_{\alpha\beta} = q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu l_\nu$$

Expansion and shear tensor

From **Lecture 1**, the **expansion along ℓ** is

$$\theta_{(\ell)} := \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A}$$

Since in adapted coordinates $\delta A = \sqrt{q} dy^1 dy^2$, we get

$$\theta_{(\ell)} = \mathcal{L}_\ell \ln \sqrt{q} = \frac{1}{2} \mathcal{L}_\ell \ln q = \frac{1}{2} q^{ab} \mathcal{L}_\ell q_{ab} = \frac{1}{2} q^{\mu\nu} \mathcal{L}_\ell q_{\mu\nu} = g^{\mu\nu} \Theta_{\mu\nu}$$

Hence the expansion $\theta_{(\ell)}$ is nothing but the trace of the deformation rate tensor Θ .

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The trace-free part of Θ defines the **shear tensor of \mathcal{S}** :

$$\sigma := \Theta - \frac{1}{2} \theta_{(\ell)} q \quad \iff \quad \sigma_{\alpha\beta} = \Theta_{\alpha\beta} - \frac{1}{2} \theta_{(\ell)} q_{\alpha\beta}$$

Null Raychaudhuri equation (1/3)

Starting point: **Ricci identity** (\equiv definition of the Riemann tensor $R^\gamma_{\delta\alpha\beta}$) applied to the vector field ℓ :

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) \ell^\gamma = R^\gamma_{\mu\alpha\beta} \ell^\mu$$

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Contracting over α and γ makes the Ricci tensor $R_{\alpha\beta} := R^\sigma_{\alpha\sigma\beta}$ appear:

$$\nabla_\sigma \nabla_\beta \ell^\sigma - \nabla_\beta \nabla_\sigma \ell^\sigma = R_{\mu\beta} \ell^\mu$$

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Then, contract with ℓ to get a scalar equation:

$$\ell^\nu \nabla_\mu \nabla_\nu \ell^\mu - \ell^\nu \nabla_\nu \nabla_\mu \ell^\mu = R_{\mu\nu} \ell^\mu \ell^\nu$$

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Using $\ell^\nu \nabla_\mu \nabla_\nu \ell^\mu = \nabla_\mu \underbrace{(\ell^\nu \nabla_\nu \ell^\mu)}_0 - \nabla_\mu \ell^\nu \nabla_\nu \ell^\mu$ yields

$$\ell^\nu \nabla_\nu \nabla_\mu \ell^\mu = -\nabla_\mu \ell^\nu \nabla_\nu \ell^\mu - R_{\mu\nu} \ell^\mu \ell^\nu$$

Null Raychaudhuri equation (2/3)

Now, from $\Theta_{\alpha\beta} = q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu l_\nu$, with $q^\alpha{}_\beta = \delta^\alpha{}_\beta + l^\alpha k_\beta + k^\alpha l_\beta$, we get

$$\nabla_\alpha l_\beta = \Theta_{\alpha\beta} - k^\sigma \nabla_\alpha l_\sigma l_\beta - k^\rho k^\sigma \nabla_\rho l_\sigma l_\alpha l_\beta - l_\alpha k^\sigma \nabla_\sigma l_\beta$$

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from which (using $\Theta_{\alpha\mu} l^\mu = 0$, $l_\mu l^\mu = 0$, $l_\mu \nabla_\alpha l^\mu = 0$ and $l^\mu \nabla_\mu l^\alpha = 0$),

$$\nabla_\mu l^\mu = \Theta^\mu{}_\mu = \theta(l)$$

Null Raychaudhuri equation (2/3)

Now, from $\Theta_{\alpha\beta} = q^\mu{}_\alpha q^\nu{}_\beta \nabla_\mu \ell_\nu$, with $q^\alpha{}_\beta = \delta^\alpha{}_\beta + \ell^\alpha k_\beta + k^\alpha \ell_\beta$, we get

$$\nabla_\alpha \ell_\beta = \Theta_{\alpha\beta} - k^\sigma \nabla_\alpha \ell_\sigma \ell_\beta - k^\rho k^\sigma \nabla_\rho \ell_\sigma \ell_\alpha \ell_\beta - \ell_\alpha k^\sigma \nabla_\sigma \ell_\beta$$

from which (using $\Theta_{\alpha\mu} \ell^\mu = 0$, $\ell_\mu \ell^\mu = 0$, $\ell_\mu \nabla_\alpha \ell^\mu = 0$ and $\ell^\mu \nabla_\mu \ell^\alpha = 0$),

$$\nabla_\mu \ell^\mu = \Theta^\mu{}_\mu = \theta(\ell)$$

and

$$\begin{aligned} \nabla_\mu \ell^\nu \nabla_\nu \ell^\mu &= \Theta_{\mu\nu} \Theta^{\mu\nu} = \Theta_{ab} \Theta^{ab} \\ &= \left(\sigma_{ab} + \frac{1}{2} \theta(\ell) q_{ab} \right) \left(\sigma^{ab} + \frac{1}{2} \theta(\ell) q^{ab} \right) \\ &= \sigma_{ab} \sigma^{ab} + \frac{1}{4} (\theta(\ell))^2 \underbrace{q_{ab} q^{ab}}_2 \\ &= \sigma_{ab} \sigma^{ab} + \frac{1}{2} (\theta(\ell))^2 \end{aligned}$$

Null Raychaudhuri equation (3/3)

Hence

$$\ell^\mu \nabla_\mu \theta_{(\ell)} = -\frac{1}{2}(\theta_{(\ell)})^2 - \sigma_{ab}\sigma^{ab} - R_{\mu\nu}\ell^\mu\ell^\nu$$

Null Raychaudhuri equation (3/3)

Hence

$$\ell^\mu \nabla_\mu \theta(\ell) = -\frac{1}{2}(\theta(\ell))^2 - \sigma_{ab}\sigma^{ab} - R_{\mu\nu}\ell^\mu\ell^\nu$$

Finally one invokes **Einstein's equation**:

$$R_{\mu\nu}\ell^\mu\ell^\nu - \frac{1}{2}R \underbrace{g_{\mu\nu}\ell^\mu\ell^\nu}_0 = 8\pi T_{\mu\nu}\ell^\mu\ell^\nu$$

Null Raychaudhuri equation (3/3)

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and use $\ell = \frac{d\mathbf{x}}{d\lambda}$ to rewrite $\ell^\mu \nabla_\mu \theta(\ell) = \frac{d\theta(\ell)}{d\lambda}$ and get

Null Raychaudhuri equation

$$\frac{d\theta(\ell)}{d\lambda} = -\frac{1}{2}(\theta(\ell))^2 - \sigma_{ab}\sigma^{ab} - 8\pi T_{\mu\nu}\ell^\mu\ell^\nu$$

Evolution of the expansion along a null geodesic generator of \mathcal{H}

Null energy condition

Physical assumption: in the vicinity of the event horizon \mathcal{H} , matter and (non-gravitational) fields obey the **null energy condition**:

$$T_{\mu\nu}\ell^\mu\ell^\nu \geq 0 \text{ for any null vector } \ell$$

NB: this is a very mild assumption:

- it is trivially satisfied by vacuum: $T_{\mu\nu} = 0$

- it is satisfied by any electromagnetic field F :

$$T_{\mu\nu}\ell^\mu\ell^\nu = \frac{1}{\mu_0} \left(F_{\sigma\mu}\ell^\mu F^\sigma{}_\nu\ell^\nu - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \underbrace{g_{\mu\nu}\ell^\mu\ell^\nu}_0 \right) = \frac{1}{\mu_0} E_\mu E^\mu,$$

with $E^\alpha := F^\alpha{}_\mu\ell^\mu$ being necessarily spacelike or colinear to ℓ , since it is orthogonal to ℓ thanks to the antisymmetry of F :

$$E_\mu\ell^\mu = F_{\mu\nu}\ell^\nu\ell^\mu = 0; \text{ hence } E_\mu E^\mu \geq 0 \text{ and } T_{\mu\nu}\ell^\mu\ell^\nu \geq 0$$

- it is implied by the **weak energy condition**, which shall be obeyed by any “reasonable” matter: $T_{\mu\nu}u^\mu u^\nu \geq 0$ for any u timelike (positivity of the energy)

Positivity of the expansion on \mathcal{H}

Positive expansion theorem

On a black hole event horizon \mathcal{H} , the expansion along any future-directed null normal ℓ is everywhere positive or zero:

$$\theta_{(\ell)} \geq 0$$

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Proof: if ℓ is only pregeodesic ($\kappa \neq 0$), rescale it $\tilde{\ell} = \alpha\ell$, $\alpha > 0$ to get a geodesic vector field ($\nabla_{\tilde{\ell}}\tilde{\ell} = 0$), then $\theta_{(\tilde{\ell})} = \alpha\theta_{(\ell)} \geq 0 \iff \theta_{(\ell)} \geq 0$.

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Consider the **null Raychaudhuri equation** along with the **null energy condition**:

$$\frac{d\theta_{(\ell)}}{d\lambda} = -\frac{1}{2}(\theta_{(\ell)})^2 \underbrace{-\sigma_{ab}\sigma^{ab}}_{\leq 0} \underbrace{-8\pi T_{\mu\nu}\ell^\mu\ell^\nu}_{\leq 0}$$

where $-\sigma_{ab}\sigma^{ab} \leq 0$ follows from the fact that σ_{ab} is a symmetric matrix in the 2-dimensional vector space $T_p\mathcal{S}$, equipped with the positive definite metric q . It can thus be diagonalized, so that, in a q -orthonormal basis, $\sigma_{ab} = \text{diag}(\sigma, -\sigma)$, with $\sigma \in \mathbb{R}$, so that $\sigma_{ab}\sigma^{ab} = 2\sigma^2 \geq 0$.

Positivity of the expansion on \mathcal{H}

We have then necessarily $\frac{d\theta_{(\ell)}}{d\lambda} \leq -\frac{1}{2}(\theta_{(\ell)})^2$

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Assume that $\theta_{(\ell)} = \theta_0 < 0$ at some point $p \in \mathcal{H}$. Let us choose the affine parameter λ of the null geodesic generator \mathcal{L} through p such that $\lambda = 0$ at p . The above equation implies

$$\forall \lambda \geq 0, \quad \theta_{(\ell)}(\lambda) \leq \bar{\theta}(\lambda)$$

where $\bar{\theta}(\lambda)$ obeys $\frac{d\bar{\theta}}{d\lambda} = -\frac{1}{2}\bar{\theta}^2$ with $\bar{\theta}(0) = \theta_0 \implies \bar{\theta}(\lambda) = \frac{\theta_0}{1 + \theta_0\lambda/2}$

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$\implies \bar{\theta} \rightarrow -\infty$ as $\lambda \rightarrow -2/\theta_0 > 0$

$\implies \theta_{(\ell)} \rightarrow -\infty$ as $\lambda \rightarrow \lambda_*$ with $0 < \lambda_* \leq -2/\theta_0$

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Assume that $\theta_{(\ell)} = \theta_0 < 0$ at some point $p \in \mathcal{H}$. Let us choose the affine parameter λ of the null geodesic generator \mathcal{L} through p such that $\lambda = 0$ at p . The above equation implies

$$\forall \lambda \geq 0, \quad \theta_{(\ell)}(\lambda) \leq \bar{\theta}(\lambda)$$

where $\bar{\theta}(\lambda)$ obeys $\frac{d\bar{\theta}}{d\lambda} = -\frac{1}{2}\bar{\theta}^2$ with $\bar{\theta}(0) = \theta_0 \implies \bar{\theta}(\lambda) = \frac{\theta_0}{1 + \theta_0\lambda/2}$

$$\implies \bar{\theta} \rightarrow -\infty \text{ as } \lambda \rightarrow -2/\theta_0 > 0$$

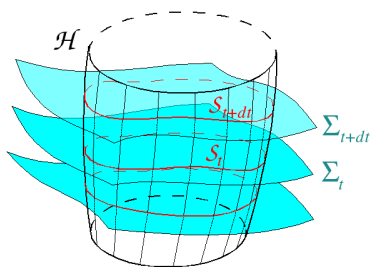
$$\implies \theta_{(\ell)} \rightarrow -\infty \text{ as } \lambda \rightarrow \lambda_* \text{ with } 0 < \lambda_* \leq -2/\theta_0$$

\implies the point $p_* \in \mathcal{L}$ of parameter λ_* is a *focusing point*, i.e. a point where neighbouring null geodesic generators of \mathcal{H} intersect

\implies contradiction with Penrose theorem (see Property 3 of event horizons in [Lecture 1](#)): there must be exactly one null geodesic generator through each point of \mathcal{H} , except at points where null geodesic generators enter \mathcal{H}

(p_* cannot be such point since $\lambda_* > 0$).

Second law of black hole dynamics



Second law (Hawking 1971)

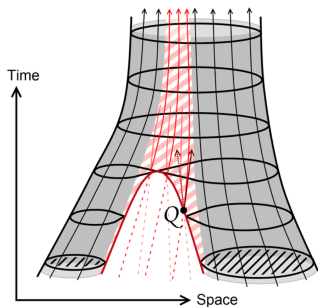
Let (\mathcal{M}, g) be a spacetime containing a black hole of event horizon \mathcal{H} . Assume \exists an open region $\mathcal{V} \supset \overline{\mathcal{M} \cap \mathcal{J}^-(\mathcal{I}^+)}$ that is globally hyperbolic (no naked singularity, no Cauchy horizon).

Consider a foliation of \mathcal{V} by a family of spacelike hypersurfaces $(\Sigma_t)_{t \in \mathbb{R}}$, with t increasing towards the future, such that each Σ_t is a Cauchy hypersurface for \mathcal{V} .

Let $A(t)$ be the area of the cross-section $\mathcal{S}_t = \mathcal{H} \cap \Sigma_t$. Then, assuming Einstein's equation and the null energy condition,

$$\frac{dA}{dt} \geq 0$$

Second law of black hole dynamics



[Hamerly & Chen, PRD **84**,
124015 (2011)]

Proof: Let ℓ be the null normal of \mathcal{H} compatible with the foliation $(\mathcal{S}_t)_{t \in \mathbb{R}}$, i.e. such that $\nabla_{\ell} t = 1$. If there is no null geodesic entering \mathcal{H} between \mathcal{S}_t and \mathcal{S}_{t+dt} , the cross-section \mathcal{S}_{t+dt} is deduced from \mathcal{S}_t by Lie dragging along ℓ by the parameter dt (see **Lecture 1**). By the very definition of $\theta_{(\ell)}$, we have then

$$\frac{dA}{dt} \geq \int_{\mathcal{S}_t} \theta_{(\ell)} \sqrt{q} dy^1 dy^2$$

with equality iff no new null geodesic is entering \mathcal{H} (as the ones depicted in orange in the adjacent figure)

If Einstein's equation and the null energy condition hold, then the result follows from the positive expansion theorem: $\theta_{(\ell)} \geq 0$. □

Outline

- 1 Formation of black holes
- 2 First law of black hole dynamics
- 3 Second law of black hole dynamics
- 4 Black hole thermodynamics**
- 5 Applications of the second law

The first law revisited

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

Second law $\implies A$ can only increase towards the future

$\implies A$ may be identified with some **entropy** and κ with some **temperature**,
to get a TdS term in the first law:

$$\left. \begin{aligned} S &= \alpha A \\ T &= \frac{1}{8\pi\alpha} \kappa \end{aligned} \right\} \implies \frac{\kappa}{8\pi} dA = TdS$$

with α to be determined...

Zeroth law

For a Kerr black hole: $\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$
 $\implies \kappa$ is constant (i.e. it does not depend on θ).

Zeroth law

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 $\implies \kappa$ is constant (i.e. it does not depend on θ).

More generally, one can show (cf. Sec. 3.3.5 of the [lecture notes](#))

Zeroth law of black hole mechanics

Let \mathcal{H} be a Killing horizon and κ the non-affinity coefficient of the null normal coinciding with the Killing vector field on \mathcal{H} . If the matter and the non-gravitational fields obey the null dominant energy condition on \mathcal{H} , κ is uniform over \mathcal{H} :

$$\kappa = \text{const}$$

Null dominant energy condition: $-T^\alpha{}_\mu \ell^\mu$ is future-directed null or timelike for any future-directed null vector ℓ

NB: null dominant energy condition \implies null energy condition invoked in the second law

Third law

For a Kerr black hole: $\kappa = \frac{\sqrt{m^2 - a^2}}{2m(m + \sqrt{m^2 - a^2})}$

Hence $\kappa = 0 \iff a = m$ (extremal Kerr black hole)

Third law

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Is it possible to reach $a = m$ by accretion of matter onto a Kerr black hole with $a < m$? The answer is *no*:

Third law

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Is it possible to reach $a = m$ by accretion of matter onto a Kerr black hole with $a < m$? The answer is *no*:

Third law (Israel 1986)

No continuous process in which the energy-momentum tensor of accreted matter remains bounded and satisfies the weak energy condition in a neighbourhood of the apparent horizon can reduce the surface gravity κ of a black hole to zero within a finite advanced time v (or equivalently Kerr time \tilde{t}).

The four laws of black hole thermodynamics

Zeroth law

The surface gravity κ of a black hole in equilibrium is constant

First law

Two nearby black hole equilibrium configurations are related by

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$$

Second law

The area A of cross-sections of a black hole event horizon can only increase towards the future:

$$\frac{dA}{dt} \geq 0$$

Third law

A nonzero surface gravity κ of a black hole in equilibrium cannot be reduced to zero by accretion of matter.

Set of four laws first formulated in Les Houches in 1972

The Four Laws of Black Hole Mechanics

J. M. Bardeen*

Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking

Institute of Astronomy, University of Cambridge, England

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This work was carried out while the authors were attending the 1972 Les Houches Summer School on Black Holes. The authors would like to thank Larry Smarr, Bryce de Witt and other participants of the school for valuable discussions.

[Commun. Math. Phys., 31, 161 (1973)]

NB: zeroth, first and second law demonstrated in the above article; third law only demonstrated in 1986 by Israel [PRL 57, 397]

Hawking radiation enters the game...

Hawking radiation:

black-body radiation at $T = \frac{\hbar}{2\pi k} \kappa$ (Hawking temperature)

with $k =$ Boltzmann constant

$\frac{\kappa}{8\pi} dA = T dS \implies S = \frac{k}{4} \frac{A}{\ell_P^2}$ (Bekenstein-Hawking entropy)

with $\ell_P = \sqrt{\frac{\hbar G}{c^3}} =$ Planck length $\simeq 1.6 \cdot 10^{-35}$ m

Hawking radiation enters the game...

Hawking radiation:

black-body radiation at $T = \frac{\hbar}{2\pi k} \kappa$ (Hawking temperature)with k = Boltzmann constant $\frac{\kappa}{8\pi} dA = T dS \implies S = \frac{k}{4} \frac{A}{\ell_P^2}$ (Bekenstein-Hawking entropy)with $\ell_P = \sqrt{\frac{\hbar G}{c^3}}$ = Planck length $\simeq 1.6 \cdot 10^{-35}$ mFor a Schwarzschild black hole of mass M : $\kappa = (4M)^{-1}$ and $A = 16\pi M^2$ $\implies T = 6 \cdot 10^{-8} \left(\frac{M_\odot}{M}\right)$ K and $S = 1.1 \cdot 10^{77} \left(\frac{M}{M_\odot}\right)^2 k$!!!

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- 1 Formation of black holes
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Upper bound on energy extracted via Penrose process

Consider some **Penrose process** (cf. **Lecture 3**) extracting energy from a Kerr black hole of initial mass m_i and specific angular momentum a_i , the extraction taking place until the black hole angular momentum has decayed to zero (\implies no longer any ergoregion outside the event horizon). The final state is then a Schwarzschild black hole of mass m_f and the total amount of extracted energy is

$$\Delta E = m_i - m_f$$

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$$\Delta E = m_i - m_f$$

$$\text{Second law} \implies A_f \geq A_i, \text{ i.e. } 2m_f^2 \geq m_i \left(m_i + \sqrt{m_i^2 - a_i^2} \right)$$

ΔE is maximal if m_f is minimal; given the above inequality, this is

$$\begin{aligned} \text{achieved for } a_i = m_i &\implies 2m_f^2 \geq m_i^2 \implies m_f \geq 2^{-1/2} m_i \\ &\implies \Delta E \leq \left(1 - 2^{-1/2} \right) m_i \simeq 0.29 m_i \end{aligned}$$

Upper bound on gravitational radiation from a BH merger

(Hawking, 1971)

Consider a binary black hole merger:

- **initial stage:** two far apart Kerr BH: (m_1, a_1) and (m_2, a_2)
- **final stage:** a single Kerr BH: (m_3, a_3)

The total amount of energy radiated via gravitational waves is

$$\Delta E = m_1 + m_2 - m_3$$

\implies **efficiency of gravitational radiation:**

$$\epsilon := \frac{m_1 + m_2 - m_3}{m_1 + m_2}$$

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Second law $\implies A_3 \geq A_1 + A_2$, i.e.

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq m_1 \left(m_1 + \sqrt{m_1^2 - a_1^2} \right) + m_2 \left(m_2 + \sqrt{m_2^2 - a_2^2} \right)$$

Upper bound on gravitational radiation from a BH merger

(Hawking, 1971)

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ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_1 = m_1$, $a_2 = m_2$ and $a_3 = 0$

$$\implies 2m_3^2 \geq m_1^2 + m_2^2 \implies m_3 \geq \sqrt{(m_1^2 + m_2^2)/2}$$

$$\implies \epsilon \leq 1 - \frac{\sqrt{m_1^2 + m_2^2}}{\sqrt{2}(m_1 + m_2)}$$

The maximum of the r.h.s. is achieved for $m_1 = m_2$ and is $1/2$, hence the upper bound:

$$\epsilon \leq \frac{1}{2}$$

Upper bound on gravitational radiation from a BH merger

Case of initially non-spinning equal-mass BH (Hawking, 1971)

Initially non-spinning equal-mass BH: $a_1 = a_2 = 0$ and $m_1 = m_2$

The second law yields

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq 4m_1^2$$

Again, ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_3 = 0 \implies 2m_3^2 \geq 4m_1^2 \implies m_3 \geq \sqrt{2}m_1$

Hence the upper bound:

$$\epsilon \leq 1 - 2^{-1/2} \simeq 0.29$$

Upper bound on gravitational radiation from a BH merger

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Hence the upper bound:

$$\epsilon \leq 1 - 2^{-1/2} \simeq 0.29$$

The GW efficiency for inspiralling binaries is actually much lower

Inspiralling binary BH merger with $m_1 = m_2$ and $a_1 = a_2 = 0$:
numerical relativity $\implies a_3 = 0.68 m_3$ and $\epsilon = 0.048$

[Scheel et al., PRD **79**, 024003 (2009)]