Time evolution of 3+1 Einstein equations via a constrained scheme

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Free vs. constrained evolution in 3+1 numerical relativity

Einstein equations split into
- dynamical equations $\frac{\partial}{\partial t} K_{ij} = \ldots$
- Hamiltonian constraint $R + K^2 - K_{ij}K^{ij} = 16\pi E$
- momentum constraint $D_j K_i^{\ j} - D_i K = 8\pi J_i$

- **2-D computations** (80's and 90's):
  partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
  fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams et al. (1994)

- **3-D computations** (from mid 90's): almost all based on **free evolution schemes**: BSSN, symmetric hyperbolic formulations, etc...
  $\Longrightarrow$ problem: exponential growth of constraint violating modes
  [see talks by C. Gundlach (constraint-preserving BC), A.P. Gentle (constraints as evolution equations) and H. Pfeiffer (constraint projection)]

Standard issue 1: the constraints usually involve elliptic equations and 3-D elliptic solvers are CPU-time expensive !

Cartesian vs. spherical coordinates in 3+1 numerical relativity

- **1-D and 2-D computations:** massive usage of **spherical coordinates** (r, θ, φ)

- **3-D computations:** almost all based on **Cartesian coordinates** (x, y, z), although spherical coordinates are better suited to study objects with spherical topology (black holes, neutron stars). Two exceptions:
  – Nakamura et al. (1987): evolution of pure gravitational wave spacetimes in spherical coordinates (but with Cartesian components of tensor fields)
  – Stark (1989): attempt to compute 3D stellar collapse in spherical coordinates

Standard issue 2: spherical coordinates are singular at $r = 0$ and $\theta = 0$ or π !

Standard issues 1 and 2 can be overcome

Standard issues 1 and 2 are neither *mathematical* nor *physical*, but *technical* ones
$\Longrightarrow$ they can be overcome with appropriate techniques

Spectral methods allow for

- an automatic treatment of the singularities of spherical coordinates (issue 2)
- fast 3-D elliptic solvers in spherical coordinates: 3-D Poisson equation reduced to a system of 1-D algebraic equations with banded matrices [Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)] (issue 1)

[see talks by H. Dimmelmeier, R. Meinel, J. Novak, and H. Pfeiffer for various examples of usage of spectral methods in numerical relativity]

Dirac gauge

As in BSSN formalism, perform a *conformal decomposition* of the metric γ_{ij} of the spacelike hypersurfaces Σ_t:

$$\gamma_{ij} =: \Psi^4 \, \tilde{\gamma}_{ij} \qquad \text{with} \qquad \tilde{\gamma}^{ij} =: f^{ij} + h^{ij}$$

where f_{ij} is a flat metric on Σ_t, h^{ij} a symmetric tensor and Ψ a scalar field defined by $\Psi := \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}$

Dirac gauge (Dirac, 1959) = **divergence-free** condition on $\tilde{\gamma}^{ij}$: $\boxed{\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0}$
where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.

Compare
- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Dirac gauge: discussion

- introduced by Dirac (1959) in order to fix the coordinates in some Hamiltonian formulation of general relativity; originally defined for Cartesian coordinates only:
$$\frac{\partial}{\partial x^j}\left(\gamma^{1/3}\gamma^{ij}\right) = 0$$
but trivially extended by us to more general type of coordinates (e.g. spherical) thanks to the introduction of the flat metric f_{ij}: $\mathcal{D}_j\left((\gamma/f)^{1/3}\gamma^{ij}\right) = 0$

- fully specifies (up to some boundary conditions) the coordinates in each hypersurface Σ_t, including the initial one $\Rightarrow$ allows for the search for stationary solutions

- leads asymptotically to transverse-traceless (TT) coordinates (same as minimal distortion gauge). Both gauges are analogous to Coulomb gauge in electrodynamics

- turns the Ricci tensor of conformal metric $\tilde{\gamma}_{ij}$ into an elliptic operator for $h^{ij} \Longrightarrow$ the dynamical Einstein equations become a **wave equation** for h^{ij}

- results in a vector elliptic equation for the shift vector β^i

3+1 Einstein equations in maximal slicing + Dirac gauge

[Bonazzola, Gourgoulhon, Grandclément & Novak, gr-qc/0307082 v2]

- 5 elliptic equations (4 constraints + $K = 0$ condition) ($\Delta := \mathcal{D}_k \mathcal{D}^k$ = flat Laplacian):

$$\Delta N = \Psi^4 N \left[4\pi(E+S) + A_{kl}A^{kl} \right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N \quad (N\text{=lapse function})$$

$$\begin{aligned} \Delta(\Psi^2 N) \;&=\; \Psi^6 N \left(4\pi S + \frac{3}{4} A_{kl}A^{kl} \right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2 \Bigg[N \Big(\frac{1}{16} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_l \tilde{\gamma}_{ij} \\ &\quad - \frac{1}{8} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_j \tilde{\gamma}_{il} + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k \ln\Psi \Big) + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N \Bigg]. \end{aligned}$$

$$\begin{aligned} \Delta\beta^i + \frac{1}{3}\mathcal{D}^i \left(\mathcal{D}_j \beta^j \right) \;&=\; 2A^{ij}\mathcal{D}_j N + 16\pi N \Psi^4 J^i - 12 N A^{ij} \mathcal{D}_j \ln\Psi - 2\Delta^i{}_{kl} N A^{kl} \\ &\quad - h^{kl}\mathcal{D}_k\mathcal{D}_l \beta^i - \frac{1}{3} h^{ik} \mathcal{D}_k \mathcal{D}_l \beta^l \end{aligned}$$

3+1 equations in maximal slicing + Dirac gauge (cont'd)

- 2 scalar wave equations for two scalar potentials χ and μ :

$$-\frac{\partial^2\chi}{\partial t^2} + \Delta\chi = S_\chi$$
$$-\frac{\partial^2\mu}{\partial t^2} + \Delta\mu = S_\mu$$

(for expression of S_χ and S_μ see [Bonazzola, Gourgoulhon, Grandclément & Novak, gr-qc/0307082 v2])

The remaining 3 degrees of freedom are fixed by the Dirac gauge:

(i) From the two potentials χ and μ, construct a TT tensor $\bar{h}^{ij}$ according to the formulas (components with respect to a spherical **f**-orthonormal frame)

$$\bar{h}^{rr} = \frac{\chi}{r^2}, \quad \bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial\eta}{\partial\theta} - \frac{1}{\sin\theta}\frac{\partial\mu}{\partial\varphi}\right), \quad \bar{h}^{r\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial\eta}{\partial\varphi} + \frac{\partial\mu}{\partial\theta}\right), \text{ etc...}$$

with $\Delta_{\theta\varphi}\eta = -\partial\chi/\partial r - \chi/r$

Recovering the conformal metric $\tilde{\gamma}_{ij}$ from the TT tensor $\bar{h}^{ij}$

(ii) h^{ij} is uniquely determined by the TT tensor $\bar{h}^{ij}$ as the following divergence-free (Dirac gauge) tensor :

$$h^{ij} = \bar{h}^{ij} + \frac{1}{2}\left(h\, f^{ij} - \mathcal{D}^i\mathcal{D}^j\phi\right) \qquad (1)$$

where $h := f_{ij}h^{ij}$ is the trace of h^{ij} with respect to the flat metric and ϕ is the solution of the Poisson equation $\Delta\phi = h$. The trace h is determined in order to enforce the condition $\det \tilde{\gamma}_{ij} = \det f_{ij}$ (definition of Ψ) by

$$\begin{aligned} h \;=\; & -h^{rr}h^{\theta\theta} - h^{rr}h^{\varphi\varphi} - h^{\theta\theta}h^{\varphi\varphi} + (h^{r\theta})^2 + (h^{r\varphi})^2 + (h^{\theta\varphi})^2 - h^{rr}h^{\theta\theta}h^{\varphi\varphi} \\ & -2h^{r\theta}h^{r\varphi}h^{\theta\varphi} + h^{rr}(h^{\theta\varphi})^2 + h^{\theta\theta}(h^{r\varphi})^2 + h^{\varphi\varphi}(h^{r\theta})^2 \end{aligned} \qquad (2)$$

Equations (1) and (2) constitute a coupled system which can be solved by iterations (starting from $h^{ij} = \bar{h}^{ij}$), at the price of solving the Poisson equation $\Delta\phi = h$ at each step. In practise a few iterations are sufficient to reach machine accuracy.

(iii) Finally $\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$.

Numerical implementation

Numerical code based on the C++ library LORENE (http://www.lorene.obspm.fr) with the following main features:

- **multidomain spectral methods** based on spherical coordinates (r, θ, φ), with compactified external domain ($\Longrightarrow$ spatial infinity included in the computational domain for elliptic equations)

- very efficient **outgoing-wave boundary conditions**, ensuring that all modes with spherical harmonics indices $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing
  [Novak & Bonazzola, J. Comp. Phys. **197**, 186 (2004)]
  (*recall:* Sommerfeld boundary condition works only for $\ell = 0$, which is too low for gravitational waves)

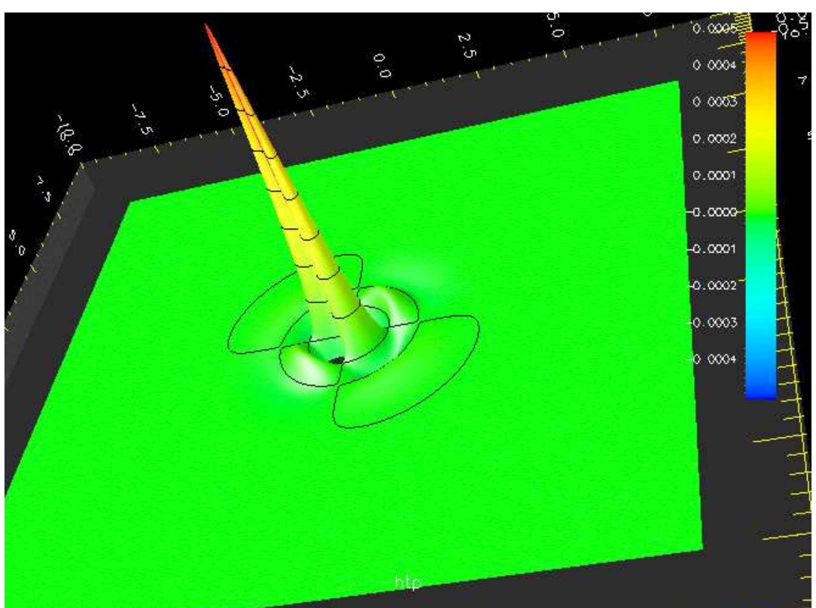
  [see M. Chirvasa's poster for alternative outgoing-wave conditions]

Results on a pure gravitational wave spacetime

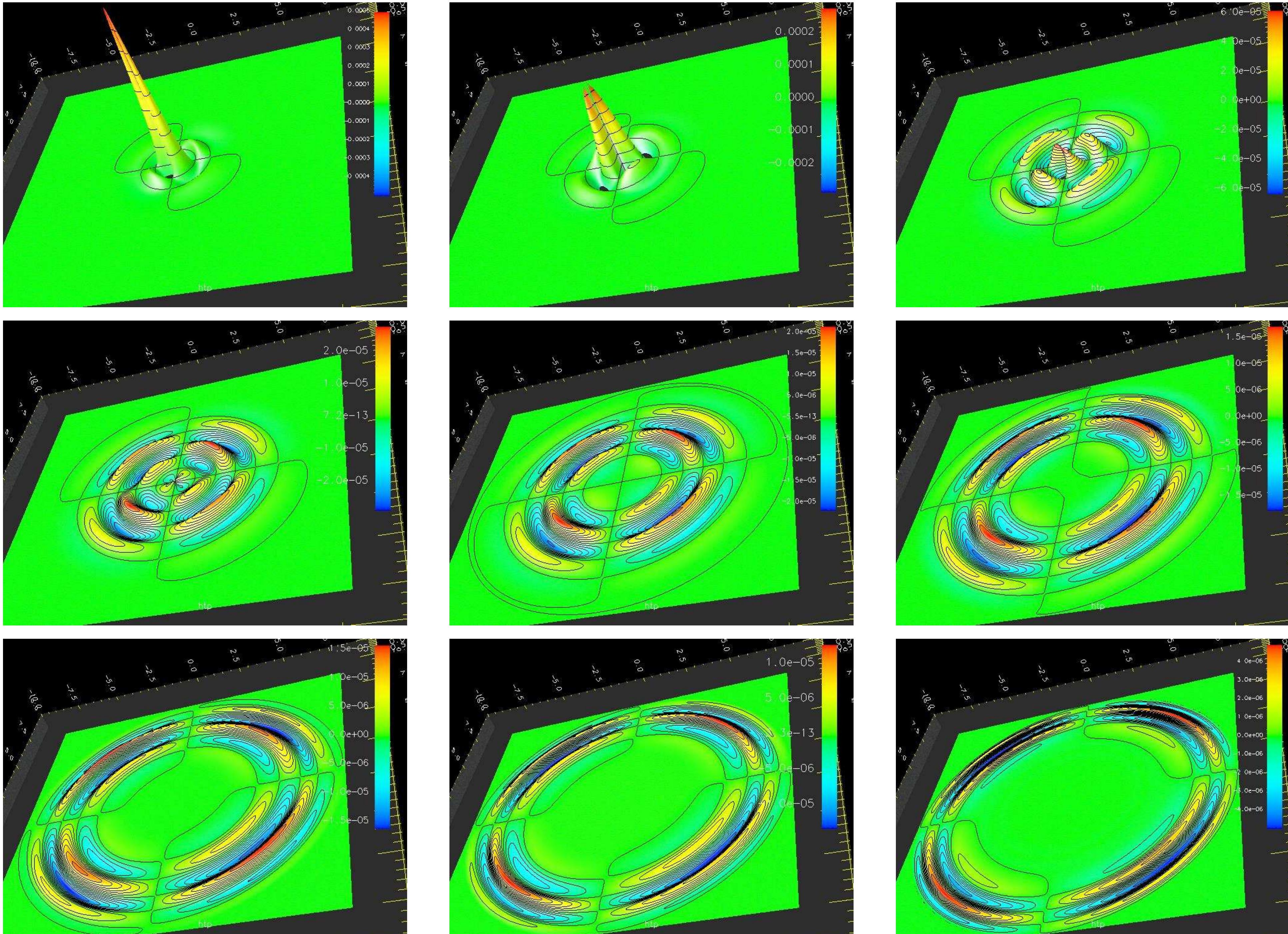
Initial data: similar to [Baumgarte & Shapiro, PRD **59**, 024007 (1998)], namely a momentarily static ($\partial\tilde{\gamma}^{ij}/\partial t = 0$) Teukolsky wave $\ell = 2$, $m = 2$:

$$\begin{cases} \chi(t=0) = \frac{\chi_0}{2} r^2 \exp\left(-\frac{r^2}{r_0^2}\right) \sin^2\theta \, \sin 2\varphi \\ \mu(t=0) = 0 \end{cases} \quad \text{with} \quad \chi_0 = 10^{-3}$$

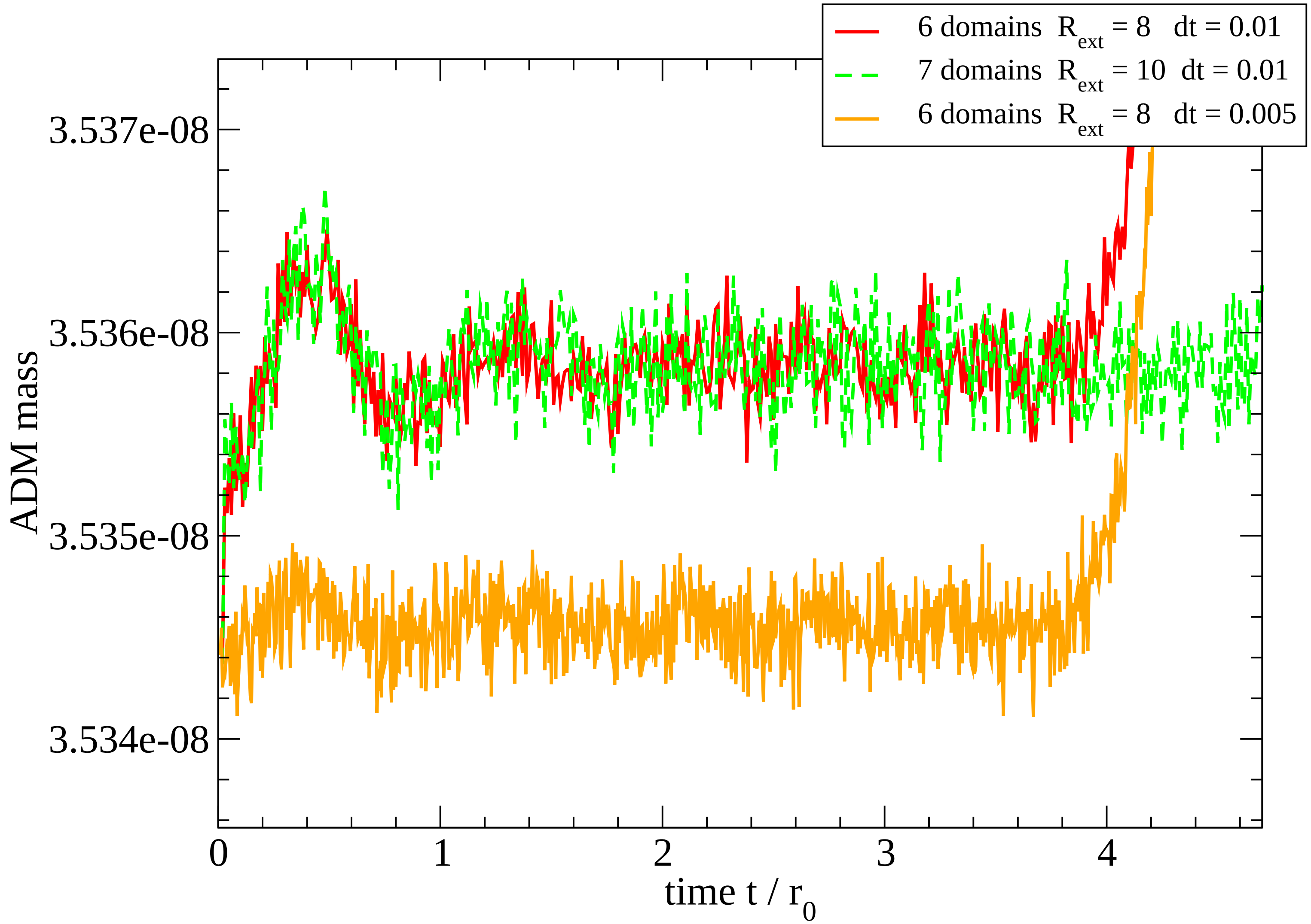
Preparation of the initial data by means of the conformal thin sandwich procedure



Evolution of $h^{\varphi\varphi}$ in the plane $\theta = \frac{\pi}{2}$



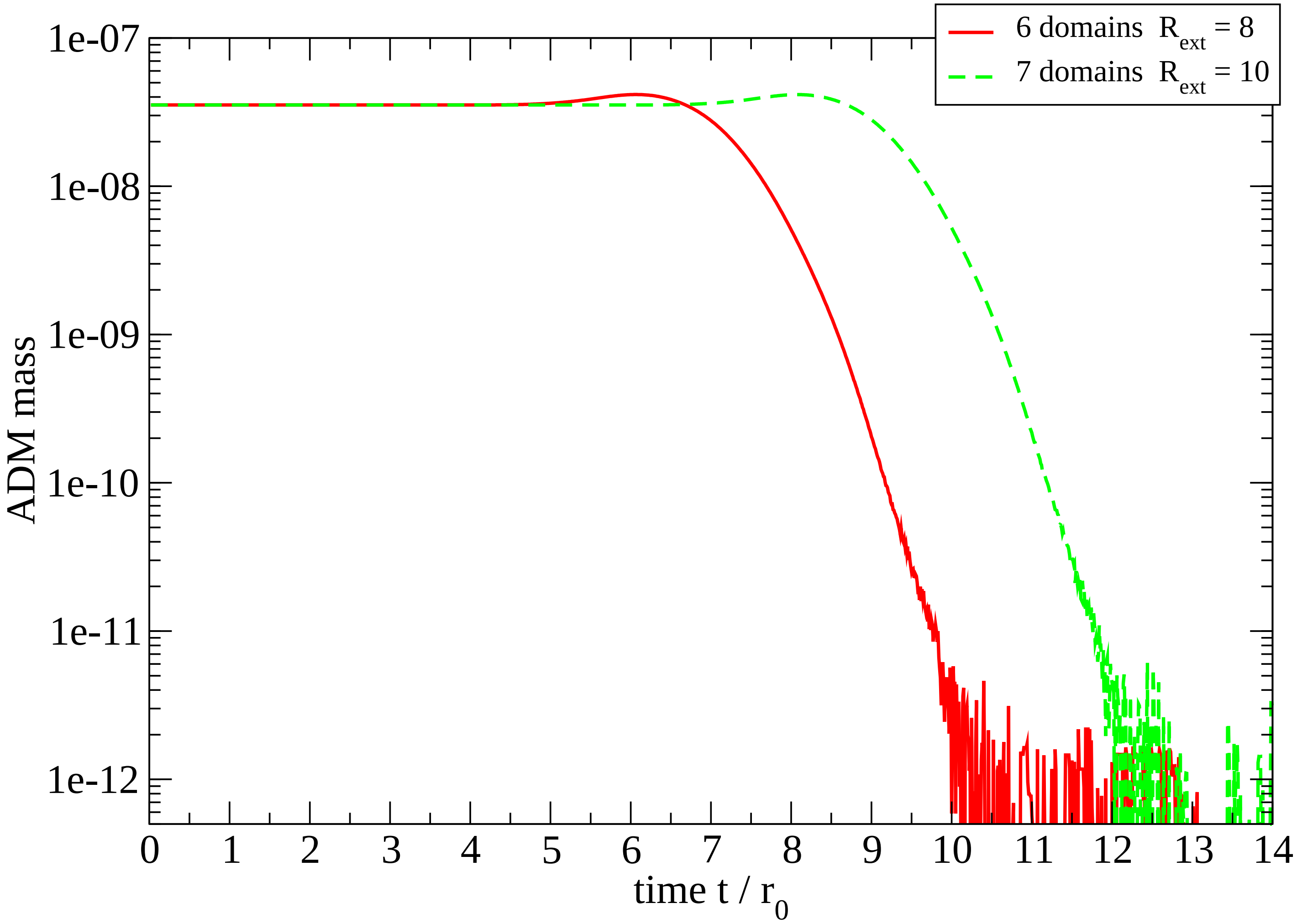
Test: conservation of the ADM mass



Number of coefficients in each domain: $N_r = 17$, $N_\theta = 9$, $N_\varphi = 8$

For $dt = 5\,10^{-3} r_0$, the ADM mass is conserved within a relative error lower than 10^{-4}

Late time evolution of the ADM mass

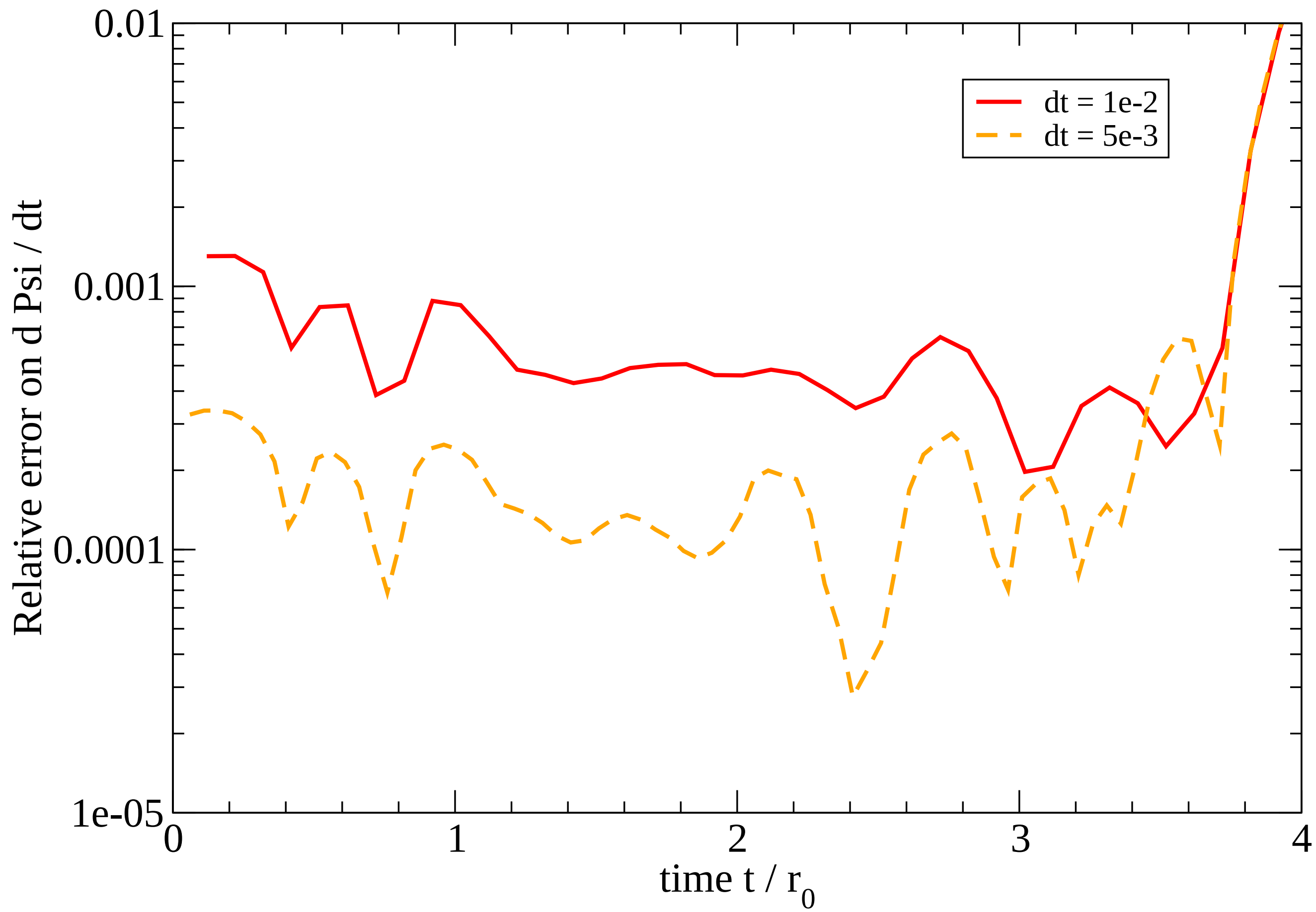


At $t > 10\, r_0$, the wave has completely left the computation domain

Nothing happens until the run is switched off at $t = 300\, r_0$! $\Longrightarrow$ **long term stability**

Another test: check of the $\frac{\partial\Psi}{\partial t}$ relation

The relation $\frac{\partial}{\partial t}\ln\Psi - \beta^k \mathcal{D}_k \ln\Psi = \frac{1}{6}\mathcal{D}_k\beta^k$ (trace of the definition of the extrinsic curvature as the time derivative of the spatial metric) is not enforced in our scheme.
$\Longrightarrow$ This provides an additional test:



Conclusions and future prospects

- Dirac gauge + maximal slicing reduces the Einstein equations into a system of
  – two scalar elliptic equations (including the Hamiltonian constraint)
  – one vector elliptic equations (the momentum constraint)
  – two scalar wave equations (evolving the two dynamical degrees of freedom of the gravitational field)

- The usage of spherical coordinates and spherical components of tensor fields is crucial in reducing the dynamical Einstein equations to two scalar wave equations

- The unimodular character of the conformal metric ($\det \tilde{\gamma}_{ij} = \det f_{ij}$) is ensured in our scheme

- First numerical results show that Dirac gauge + maximal slicing seems a promising choice for stable evolutions of 3+1 Einstein equations and gravitational wave extraction

- It remains to be tested on black hole spacetimes !
  [cf. J.L. Jaramillo's talk for boundary conditions on black hole horizons]