Numerical relativity and sources of gravitational waves

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Outline

Gravitational wave observatories

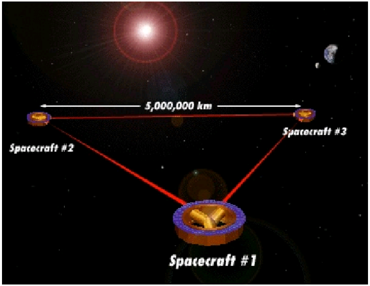
Gravitational wave detectors are coming on line...

VIRGO, Cascina, Italie

$10\ \mathrm{Hz} < f < 10^3\ \mathrm{Hz}$

Other detectors: LIGO, GEO600, TAMA

... or will be launched in the not to distant future (2013)



LISA (ESA/NASA)

$10^{-4}\ \mathrm{Hz} < f < 10^{-1}\ \mathrm{Hz}$

Modelisation of gravitational wave sources

Gravitational waves = new vector for astronomy, complementary to the *photon*

- propagate without noticeable absorption
- are emitted by objects which are poor electromagnetic emitters (e.g. black holes)

Theoretical computations of gravitational wave forms

- are necessary for the detection of the waves (low S/N)
- allow the analysis of the signal and the determination of the physical parameters of the source

Principal sources = compact objects

Dynamics of compact objects is governed by general relativity
$\Longrightarrow$ we must solve the Einstein equation

Einstein equation

Spacetime = $(\mathcal{M}, \boldsymbol{g})$, with

- $\mathcal{M}$ = 4-dimensional real manifold
- $\boldsymbol{g}$ = Lorentzian metric on $\mathcal{M}$, signature $(-,+,+,+)$.

Einstein equation:

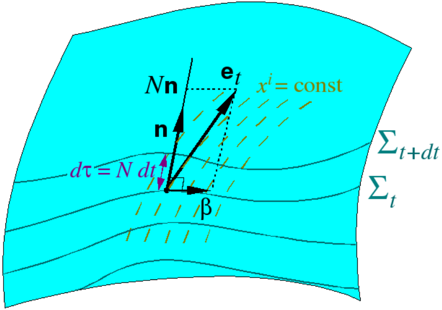
$$\boldsymbol{R} - \frac{1}{2} R \boldsymbol{g} = \frac{8\pi G}{c^4} \boldsymbol{T} \qquad (1)$$

- $\boldsymbol{R}$ = Ricci tensor associated with $\boldsymbol{g}$, "trace" of the Riemann curvature tensor
- $R := \mathrm{tr}\boldsymbol{R}$: Ricci scalar
- $\boldsymbol{T}$ = matter stress-energy tensor

NB: (1) is a *tensorial* equation, not a PDE.

3+1 decomposition of spacetime

Foliation of spacetime by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$; on each hypersurface, pick a coordinate system $(x^i)_{i\in\{1,2,3\}} \Longrightarrow$ $(x^\mu)_{\mu\in\{0,1,2,3\}} = (t, x^1, x^2, x^3) =$ coordinate system on spacetime



$\boldsymbol{n}$: future directed unit normal to Σ_t :
$\boldsymbol{n} = -N\,\mathbf{d}t$, N : lapse function
$\boldsymbol{e}_t = \partial/\partial t$: time vector of the natural basis associated with the coordinates (x^μ)

N : lapse function
$\boldsymbol{\beta}$: shift vector

$$\left.\begin{array}{l} N \\ \boldsymbol{\beta} \end{array}\right\} \boldsymbol{e}_t = N\boldsymbol{n} + \boldsymbol{\beta}$$

Geometry of the hypersurfaces Σ_t:

- induced metric $\boldsymbol{\gamma} = \boldsymbol{g} + \boldsymbol{n} \otimes \boldsymbol{n}$
- extrinsic curvature : $\boldsymbol{K} = -\frac{1}{2}\mathcal{L}_{\boldsymbol{n}}\boldsymbol{\gamma}$

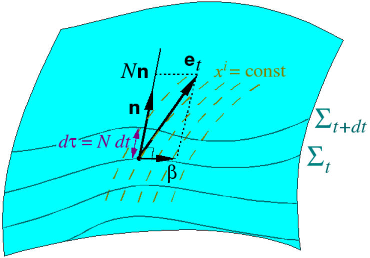
$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -N^2\,dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j\,dt)$$

Choice of coordinates within the 3+1 formalism

$$(x^\mu) = (t, x^i) = (t, x^1, x^2, x^3)$$

Choice of the lapse function N $\Longleftrightarrow$ choice of the slicing (Σ_t)

Choice of the shift vector $\boldsymbol{\beta}$ $\Longleftrightarrow$ choice of the spatial coordinates (x^i) on each hypersurface Σ_t



A well-spread choice of slicing: *maximal slicing*: $K := \mathrm{tr}\,\boldsymbol{K} = 0$

[Lichnerowicz 1944]

3+1 decomposition of Einstein equation

Orthogonal projection of Einstein equation onto Σ_t and along the normal to Σ_t :

- Hamiltonian constraint: $R + K^2 - K_{ij}K^{ij} = 16\pi E$
- Momentum constraint : $D_j K^{ij} - D^i K = 8\pi J^i$
- Dynamical equations :
$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_{\boldsymbol{\beta}} K_{ij} =$$
$$-D_i D_j N + N \left[R_{ij} - 2K_{ik}K^k_{\ j} + KK_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij}) \right]$$

$E := \boldsymbol{T}(\boldsymbol{n},\boldsymbol{n}) = T_{\mu\nu}\, n^\mu n^\nu$, $J_i := -\gamma_i^{\ \mu}\, T_{\mu\nu}\, n^\nu$, $S_{ij} := \gamma_i^{\ \mu}\, \gamma_j^{\ \nu}\, T_{\mu\nu}$, $S := S_i^{\ i}$

D_i : covariant derivative associated with $\boldsymbol{\gamma}$, R_{ij} : Ricci tensor of D_i, $R := R_i^{\ i}$

Kinematical relation between $\boldsymbol{\gamma}$ and $\boldsymbol{K}$: $\frac{\partial \gamma^{ij}}{\partial t} + D^i \beta^j + D^j \beta^i = 2NK^{ij}$

Resolution of Einstein equation $\equiv$ Cauchy problem

Historical context: Cauchy problem of GR

- Darmois (1927), Lichnerowicz (1939): Cauchy problem for *analytic* initial data
- Lichnerowicz (1944): First 3+1 formalism, conformal decomposition of spatial metric
- Fourès-Bruhat (1952): Cauchy problem for C^5 initial data: local existence and uniqueness in harmonic coordinates
- Fourès-Bruhat (1956): 3+1 formalism (moving frame)
- Arnowitt, Deser & Misner (1962): 3+1 formalism (Hamiltonian analysis of GR)
- York (1972): gravitational dynamical degrees of freedom carried by the conformal spatial metric
- Ó Murchadha & York (1974): Conformal transverse-traceless (CTT) method for solving the constraint equations
- Smarr & York (1978): Radiation gauge for numerical relativity: elliptic-hyperbolic system with asymptotic TT behavior
- York (1999): Conformal thin-sandwich (CTS) method for solving the constraint equations

Historical context: Numerical relativity

- Smarr (1977): 2-D (axisymmetric) head-on collision of two black holes: *first numerical solution beyond spherical symmetry of the Cauchy problem for asymptotically flat spacetimes*
- Nakamura (1983), Stark & Piran (1985): 2-D (axisymmetric) gravitational collapse to a black hole
- Bona & Masso (1989), Choquet-Bruhat & York (1995), Kidder, Scheel & Teukolsky (2001), and many others: *(First-order) (symmetric) hyperbolic formulations* of Einstein equations within the 3+1 formalism
- Shibata & Nakamura (1995), Baumgarte & Shapiro (1999): *BSSN formulation:* conformal decomposition of the 3+1 equations and promotion of some connection function as an independent variable
- Shibata (2000): 3-D full computation of binary neutron star merger: *first full GR 3-D solution of the Cauchy problem of astrophysical interest*

Free vs. constrained evolution in 3+1 numerical relativity

Einstein equations split into

$$\begin{cases} \text{dynamical equations} & \frac{\partial}{\partial t} K_{ij} = \ldots \\ \text{Hamiltonian constraint} & R + K^2 - K_{ij}K^{ij} = 16\pi E \\ \text{momentum constraint} & D_j K_i^{\ j} - D_i K = 8\pi J_i \end{cases}$$

- **2-D computations(80's and 90's):**
  - partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
  - fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams et al. (1994)
- **3-D computations (from mid 90's):** Almost all based on free evolution schemes: BSSN, symmetric hyperbolic formulations, etc...
  $\Longrightarrow$ problem: exponential growth of *constraint violating modes*

"Standard issue" 1 :

The constraints usually involve elliptic equations and 3-D elliptic solvers are CPU-time expensive !

Cartesian vs. spherical coordinates in 3+1 numerical relativity

- **1-D and 2-D computations:** massive usage of spherical coordinates (r, θ, φ)
- **3-D computations:** almost all based on Cartesian coordinates (x, y, z), although spherical coordinates are better suited to study objects with spherical topology (black holes, neutron stars). Two exceptions:
  – *Nakamura et al. (1987):* evolution of pure gravitational wave spacetimes in spherical coordinates (but with Cartesian components of tensor fields)
  – *Stark (1989):* attempt to compute 3D stellar collapse in spherical coordinates

"Standard issue" **2** :

Spherical coordinates are singular at $r = 0$ and $\theta = 0$ or π !

"Standard issues" 1 and 2 can be overcome

"Standard issues" 1 and 2 are neither *mathematical* nor *physical*

they are *technical* ones
$\Longrightarrow$ they can be overcome with appropriate techniques

Spectral methods allow for

- an automatic treatment of the singularities of spherical coordinates (issue 2)
- fast 3-D elliptic solvers in spherical coordinates: 3-D Poisson equation reduced to a system of 1-D algebraic equations with banded matrices [Grandclément, Bonazzola, Gourgoulhon & Marck, J. Comp. Phys. **170**, 231 (2001)] (issue 1)

A new scheme for 3+1 numerical relativity

Constrained scheme built upon **maximal slicing** and **Dirac gauge**

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

Conformal metric and dynamics of the gravitational field

Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$

To work with *tensor fields* only, introduce an *extra structure* on Σ_t: a *flat metric* $\boldsymbol{f}$ such that $\frac{\partial f_{ij}}{\partial t} = 0$ and $\gamma_{ij} \sim f_{ij}$ at spatial infinity *(asymptotic flatness)*

Define $\tilde{\gamma}_{ij} := \Psi^{-4} \gamma_{ij}$ or $\gamma_{ij} =: \Psi^4 \tilde{\gamma}_{ij}$ with $\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$, $f := \det f_{ij}$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of γ_{ij} and verifies $\det \tilde{\gamma}_{ij} = f$

Notations: $\tilde{\gamma}^{ij}$: inverse conformal metric : $\tilde{\gamma}_{ik} \tilde{\gamma}^{kj} = \delta_i^{\ j}$
$\tilde{D}_i$: covariant derivative associated with $\tilde{\gamma}_{ij}$, $\tilde{D}^i := \tilde{\gamma}^{ij} \tilde{D}_j$
$\mathcal{D}_i$: covariant derivative associated with f_{ij}, $\mathcal{D}^i := f^{ij} \mathcal{D}_j$

Dirac gauge

Conformal decomposition of the metric γ_{ij} of the spacelike hypersurfaces Σ_t:

$$\gamma_{ij} =: \Psi^4 \, \tilde{\gamma}_{ij} \qquad \text{with} \qquad \tilde{\gamma}^{ij} =: f^{ij} + h^{ij}$$

where f_{ij} is a flat metric on Σ_t, h^{ij} a symmetric tensor and Ψ a scalar field defined by $\Psi := \left(\frac{\det \gamma_{ij}}{\det f_{ij}} \right)^{1/12}$

Dirac gauge (Dirac, 1959) = *divergence-free* condition on $\tilde{\gamma}^{ij}$:

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric f_{ij}.
Compare

- minimal distortion (Smarr & York 1978) : $D_j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left(\partial \tilde{\gamma}^{ij} / \partial t \right) = 0$

Notice: Dirac gauge $\Longleftrightarrow$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

Dirac gauge: discussion

- introduced by Dirac (1959) in order to fix the coordinates in some *Hamiltonian formulation* of general relativity; originally defined for Cartesian coordinates only: $\frac{\partial}{\partial x^j}\left(\gamma^{1/3}\,\gamma^{ij}\right) = 0$

  but trivially extended by us to more general type of coordinates (e.g. spherical) thanks to the introduction of the flat metric f_{ij}:
  $\mathcal{D}_j\left((\gamma/f)^{1/3}\gamma^{ij}\right) = 0$
- fully specifies (up to some boundary conditions) the coordinates in each hypersurface Σ_t, including the initial one $\Rightarrow$ allows for the search for *stationary solutions*
- leads asymptotically to transverse-traceless (TT) coordinates (same as minimal distortion gauge). Both gauges are analogous to *Coulomb gauge* in electrodynamics
- turns the Ricci tensor of conformal metric $\tilde{\gamma}_{ij}$ into an elliptic operator for h^{ij} $\Longrightarrow$ the dynamical Einstein equations become a *wave equation* for h^{ij}
- results in a *vector elliptic equation* for the shift vector β^i

3+1 Einstein equations in maximal slicing + Dirac gauge

[Bonazzola, Gourgoulhon, Grandclément & Novak, PRD **70**, 104007 (2004)]

- 5 elliptic equations (4 constraints + $K = 0$ condition) ($\Delta := \mathcal{D}_k \mathcal{D}^k$):

$$\Delta N = \Psi^4 N \left[4\pi(E+S) + \tilde{A}_{kl} A^{kl}\right] - h^{kl} \mathcal{D}_k \mathcal{D}_l N - 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N$$

$$\begin{aligned} \Delta(\Psi^2 N) &= \Psi^6 N \left(4\pi S + \frac{3}{4} \tilde{A}_{kl} A^{kl}\right) - h^{kl} \mathcal{D}_k \mathcal{D}_l (\Psi^2 N) \\ &\quad + \Psi^2 \left[N \left(\frac{1}{16} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_l \tilde{\gamma}_{ij} - \frac{1}{8} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_j \tilde{\gamma}_{il} \right.\right. \\ &\quad \left.\left. + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k \ln\Psi \right) + 2\tilde{D}_k \ln\Psi \, \tilde{D}^k N \right]. \end{aligned}$$

$$\begin{aligned} \Delta\beta^i + \frac{1}{3} \mathcal{D}^i \left(\mathcal{D}_j \beta^j\right) &= 2A^{ij} \mathcal{D}_j N + 16\pi N \Psi^4 J^i - 12 N A^{ij} \mathcal{D}_j \ln\Psi \\ &\quad - 2\Delta^i{}_{kl} N A^{kl} - h^{kl} \mathcal{D}_k \mathcal{D}_l \beta^i - \frac{1}{3} h^{ik} \mathcal{D}_k \mathcal{D}_l \beta^l \end{aligned}$$

3+1 equations in maximal slicing + Dirac gauge (cont'd)

- 2 scalar wave equations for two scalar potentials χ and μ :

$$-\frac{\partial^2 \chi}{\partial t^2} + \Delta \chi = S_\chi$$
$$-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu = S_\mu$$

The remaining 3 degrees of freedom are fixed by the Dirac gauge:

(i) From the two potentials χ and μ, construct a TT tensor $\bar{h}^{ij}$ according to the formulas (components with respect to a spherical $\boldsymbol{f}$-orthonormal frame)

$$\bar{h}^{rr} = \frac{\chi}{r^2}, \quad \bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \phi}\right), \quad \bar{h}^{r\phi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \eta}{\partial \phi} + \frac{\partial \mu}{\partial \theta}\right), \text{ etc...}$$

with $\Delta_{\theta\phi}\eta = -\partial\chi/\partial r - \chi/r$

Numerical implementation

Numerical code based on the C++ library LORENE (http://www.lorene.obspm.fr) with the following main features:

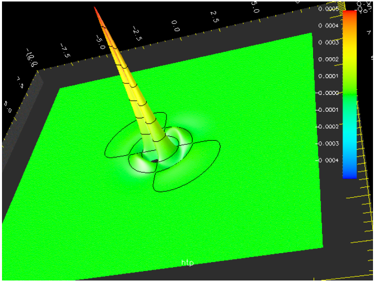
- multidomain spectral methods based on spherical coordinates (r, θ, φ), with compactified external domain ($\Longrightarrow$ spatial infinity included in the computational domain for elliptic equations)
- very efficient outgoing-wave boundary conditions, ensuring that all modes with spherical harmonics indices $\ell = 0$, $\ell = 1$ and $\ell = 2$ are perfectly outgoing [Novak & Bonazzola, J. Comp. Phys. **197**, 186 (2004)]
  (*recall:* Sommerfeld boundary condition works only for $\ell = 0$, which is too low for gravitational waves)

Results on a pure gravitational wave spacetime

Initial data: similar to [Baumgarte & Shapiro, PRD **59**, 024007 (1998)], namely a momentarily static ($\partial\tilde{\gamma}^{ij}/\partial t = 0$) Teukolsky wave $\ell = 2$, $m = 2$:

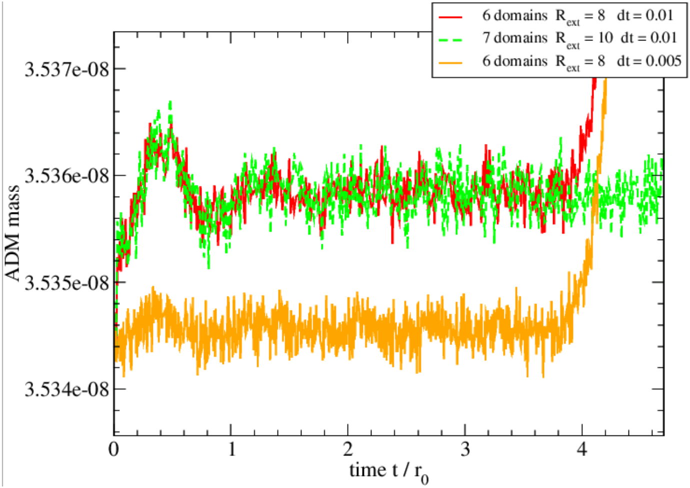
$$\left\{ \begin{array}{lcl} \chi(t=0) & = & \dfrac{\chi_0}{2}\, r^2 \exp\left(-\dfrac{r^2}{r_0^2}\right) \sin^2\theta \, \sin 2\varphi \\ \mu(t=0) & = & 0 \end{array} \right. \qquad \text{with} \quad \chi_0 = 10^{-3}$$

Preparation of the initial data by means of the *conformal thin sandwich* procedure



Evolution of $h^{\phi\phi}$ in the plane $\theta = \frac{\pi}{2}$

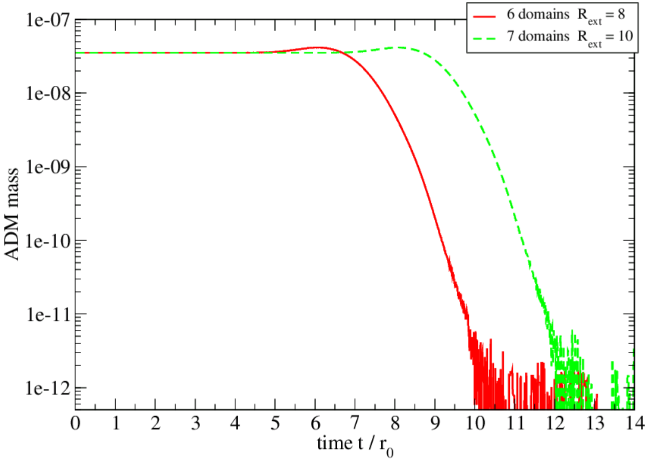
Test: conservation of the ADM mass



Number of coefficients in each domain: $N_r = 17$, $N_\theta = 9$, $N_\varphi = 8$

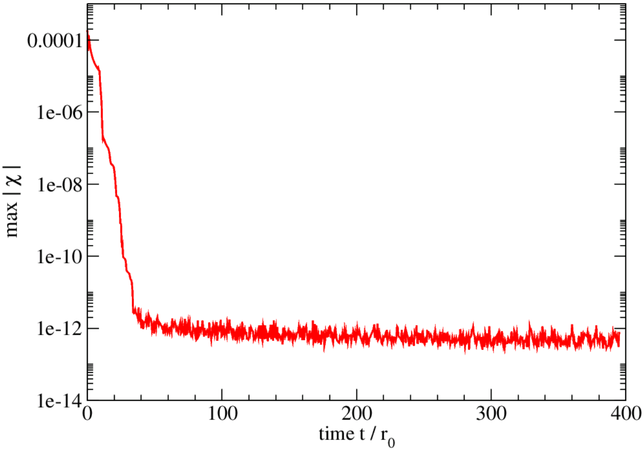
For $dt = 5\,10^{-3} r_0$, the ADM mass is conserved within a relative error lower than 10^{-4}

Late time evolution of the ADM mass



At $t > 10\, r_0$, the wave has completely left the computation domain
$\Longrightarrow$ Minkowski spacetime

Long term stability



Nothing happens until the run is switched off at $t = 400\, r_0$!

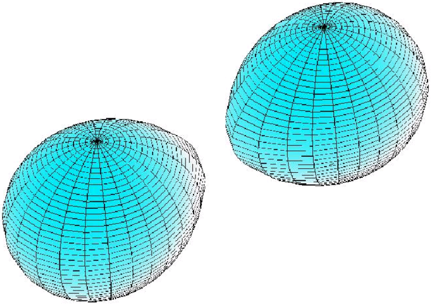
Summary

- Dirac gauge + maximal slicing reduces the Einstein equations into a system of
  – two scalar elliptic equations (including the Hamiltonian constraint)
  – one vector elliptic equations (the momentum constraint)
  – two scalar wave equations (evolving the two dynamical degrees of freedom of the gravitational field)
- The usage of spherical coordinates and spherical components of tensor fields is crucial in reducing the dynamical Einstein equations to two scalar wave equations
- The unimodular character of the conformal metric ($\det \tilde{\gamma}_{ij} = \det f_{ij}$) is ensured in our scheme
- First numerical results show that Dirac gauge + maximal slicing seems a promising choice for stable evolutions of 3+1 Einstein equations and gravitational wave extraction
- It remains to be tested on black hole spacetimes !

Coalescence of binary compact objects

Inspiral and merger of **binary neutron stars** and **binary black holes**:

the most promising source for VIRGO/LIGO/GEO600 and LISA.

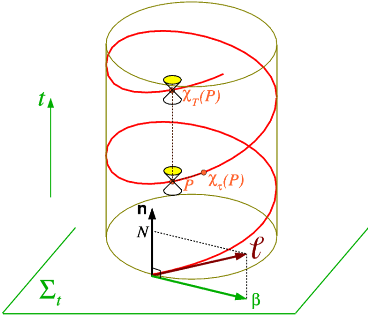


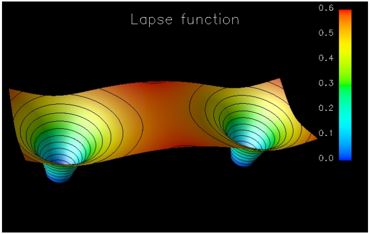
Quasiequilibrium of a binary neutron star system in circular orbit

[Taniguchi, Gourgoulhon & Bonazzola, Phys. Rev. D **64**, 064012 (2001)]

Initial data (Cauchy data)

Quasi-equilibrium sequences of orbiting binary black holes and neutrons stars





Numerical results obtained under the assumption of helical Killing vector

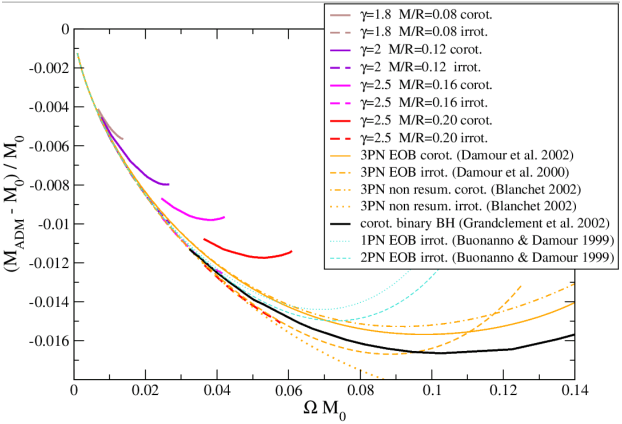
Status:

- **as 4-D spacetimes:** approximate solutions within the *Isenberg-Wilson-Mathews* waveless approximation of GR [Isenberg (1978), Wilson & Mathews (1989)]

- **as 3-D Cauchy data:** exact (for binary NS) or approximate (within 10^{-3}) (for binary BH) solutions of the *constraints*

← [Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

Initial data: quasi-equilibrium sequences of binary NS and BH



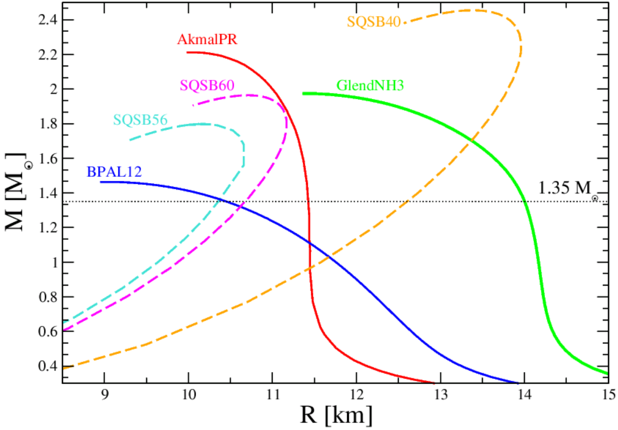
← First good agreement between numerical orbiting binary black holes sequences and post-Newtonian ones

[Grandclément, Gourgoulhon, Bonazzola, PRD **65**, 044021 (2002)]

[Damour, Gourgoulhon & Grandclément PRD **66**, 024007 (2002)]

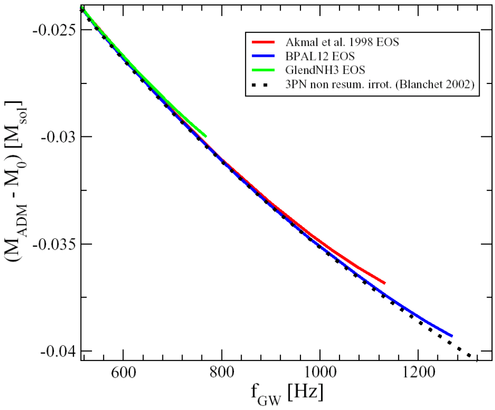
[Taniguchi & Gourgoulhon, PRD **68**, 124025 (2003)]

Determining the nuclear matter EOS from GW observations



Determining the nuclear matter EOS from GW observations

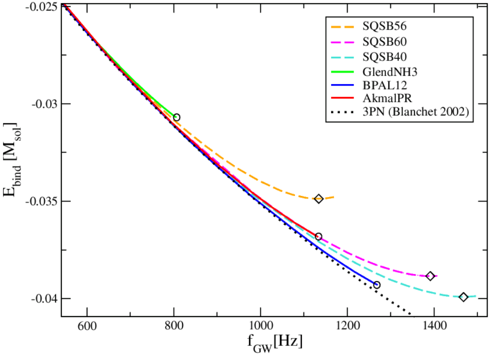
Evolutionary sequences of irrotational binary NS:



[Bejger, Gondek-Rosińska, Gourgoulhon, Haensel, Taniguchi & Zdunik, A&A, in press (preprint: astro-ph/0406234)]

Determining the nuclear matter EOS from GW observations

Evolutionary sequences of irrotational binary strange stars:

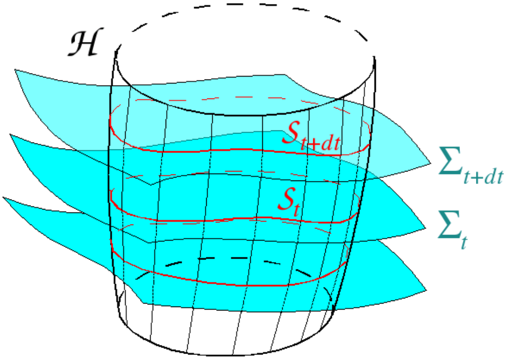


[Limousin, Gondek-Rosińska & Gourgoulhon, PRD, submitted (preprint: gr-qc/0411127)]

[Gondek-Rosińska, Bejger, Bulik, Gourgoulhon, Haensel, Limousin & Zdunik, preprint: gr-qc/0412010)]

Current development: isolated horizons

Using the isolated horizon formalism (Ashtekar et al.) to get boundary conditions for quasiequilibrium binary black spacetimes



[Jaramillo, Gourgoulhon & Mena Marugán, PRD **70**, 124036 (2004)]