NEW FORMULATION OF STOCHASTIC MECHANICS. APPLICATION TO CHAOS

by

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Nottale, L., 1995, in "*Chaos and diffusion in Hamiltonian systems*", Proceedings of the fourth workshop in Astronomy and Astrophysics of Chamonix (France), 7-12 February 1994, Eds. D. Benest et C. Froeschlé (Editions Frontières), pp 173-198.

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résumé

Nous développons une nouvelle méthode pour aborder le problème de l'émergence de structures telle qu'on peut l'observer dans des systèmes fortement chaotiques. Cette méthode s'applique précisément quand les autres méthodes échouent (au delà de l'horizon de prédictibilité), c'est à dire sur les très grandes échelles de temps. Elle consiste à remplacer la description habituelle (en terme de trajectoires classiques déterminées) par une description stochastique, approchée, en terme de familles de chemins non différentiables. Nous obtenons ainsi des équations du type de celles de la mécanique quantique (équation de Schrödinger généralisée) dont les solutions impliquent une structuration spatiale décrite par des pics de densité de probabilité. Après avoir rappelé notre formalisme de base, qui repose sur un double processus de Wiener à coefficient de diffusion constant (ce qui correspond à des trajectoires de dimension fractale 2), nous le généralisons à des dimensions fractales différentes puis à un coefficient de diffusion dépendant des coordonnées mais lentement variable. Nous appliquons finalement cette méthode au problème de la compréhension des différentes structures observées dans le système solaire, telles que la distribution des excentricités, des positions relatives des planètes, de leur masse et du moment angulaire.

Abstract– We develop a new method for tackling the problem of the observed emergence of structures in strongly chaotic systems. The method is applicable precisely when other methods fail, i.e. at very large time-scales. It consists in replacing the usual description using well-defined classical trajectories by a stochastic, approximate description in terms of families of non-differentiable paths. We get quasi-quantum equations that yield spatial structures described by peaks of probability density. After having recalled our basic formalism, which relies on a twin Wiener process with a constant diffusion coefficient (i.e. fractal dimension 2), we generalize it to different fractal dimensions and to the case of a slowly variable diffusion coefficient. We finally apply our method to the problem of understanding various observed structures of the solar system, such as the distribution of eccentricities, planet positions, mass and angular momentum.

1. INTRODUCTION

One of the main open problems of today's physics is that of the nature of fundamental scales and of understanding scale-dependent phenomena. We have suggested (Nottale, 1989, 1992, 1993a) that the failure of present physics when dealing with such questions comes from the lack of a definite frame of thought in which they could be properly asked. Our proposal is then to construct such a framework by generalizing Einstein's principle of relativity to the case of scale transformations. Namely, we redefine space and time *resolutions* as essential variables that characterize the *state of scale* of reference systems (in analogy with *velocity* characterizing their *state of motion*); then we require that the laws of physics apply to all reference systems whatever their state (of motion, which yields standard relativity, but also of scale). This principle is mathematically translated into the requirement of *scale covariance* of the equations of physics under scale transformations (i.e., contractions and dilations of resolutions). Here, covariance is taken in Einstein's meaning, as analysed by Weinberg (1972): the equations of physics must keep not only the same form whatever the coordinate system, but above all must keep *their simplest form*, even in very complicated reference frames.

Now, present physics is founded on the hypothesis that space-time is continuous and differentiable. Hence Einstein's equations of general relativity are the most general simplest equations which are covariant under continuous and at least two times differentiable transformations of the coordinate system (Einstein, 1916). But no experiment nor fundamental principle proves in a definitive way the hypothesis of differentiability of space-time coordinates. On the contrary, Feynman's path integral approach to quantum mechanics (Feynman and Hibbs, 1965) allowed him to demonstrate the opposite: when going to small length- and time-scales, the typical paths of quantum particles are continuous and nondifferentiable, and can be characterized by a fractal dimension $D = 2$.

If nondifferentiability is indeed a universal property of microphysical phenomena at the quantum level, it is also often encountered, though in a more local way, in chaotic phenomena: it is well known that strange attractors show fractal properties (see e.g. Lichtenberg & Lieberman, 1983; Eckmann & Ruelle, 1985) while fractals are themselves, when pushed to their limit, characterized by their nondifferentiability (Mandelbrot, 1982).

Hence it appears that there is a double need to generalize today's physical equations: first to scale laws, second to nondifferentiable phenomena. What is the relation between these two approaches? The answer is given by a theorem due to Lebesgue (see Tricot, 1994): a continuous curve of finite length is nearly everywhere differentiable. This implies another theorem, a direct demonstration of which is given in (Nottale, 1993a) using Non-Standard Analysis methods: a continuous but almost nowhere differentiable curve has an infinite length. In other words, the length of a continuous and nowhere differentiable curve is explicitly dependent on the resolution ε at which it is considered [$l = l(\varepsilon)$], and, further, is divergent [i.e., $l(\varepsilon) \to \infty$ when $\varepsilon \to 0$]. Such a curve is then a fractal curve, in a general acceptance of this concept. This theorem is easily generalizable to any dimension.

Although the preferential domain of application of such methods is microphysics (Nottale and Schneider, 1984; Nottale, 1989, 1993a), we have nevertheless suggested that they can also be applied, as approximations, to developed chaos when observed at very large time scales (Nottale, 1993a,b). Chaos (often defined as extreme sensibility to variation of initial conditions) is relevant in a huge variety of natural systems. Although various methods of analysis have been coined to describe the development of chaos (strange attractors, fractal and information dimensions, entropy, characteristic exponents, catastrophe theory…), all of them have up to now struck against the unpassable barrier of unpredictability at large time scales. However, in many systems where chaos arises, spatial and temporal structures are observed experimentally; such structures are in some few cases found or confirmed in numerical simulations, but very rarely understood or predicted from a fundamental theory.

The method we suggest for tackling this problem is efficient precisely when other methods fail, i.e., on time-scales large compared to the “chaos time” (or inverse Lyapunov exponent τ). We will see that it naturally generates spatial structures, in terms of probability densities. In the present report, we shall first restate and perfect (in Section 2) the stochastic method which was described in (Nottale, 1993 a & b), then suggest some possible generalizations (Section 3). Our approach will be exemplified by its application to the problem of understanding the various observed structures in the Solar System, such as the distribution of planets, of eccentricities, mass, and angular momentum (Section 4).

2. BASIC FORMALISM

We assume the system under consideration to be strongly chaotic, i.e. that the gap between any couple of trajectories diverges exponentially in time, as $\delta x = \delta x_0 \, e^{t/\tau}$, where $1/\tau$ is the Lyapunov exponent. As schematized in Fig. 1, the relative motion of one trajectory with respect to another one, when looked at with a *very long time resolution* (i.e., $\Delta t >> \tau$: right diagram in Fig. 1), becomes non-differentiable at the origin, with different backward and forward slopes, and looks like trajectories arising from a diffusion process or from particle collision. If we now start from a continuum of different values of δx_0, the breaking point occurs anywhere, and the various trajectories become describable by non-differentiable, fractal paths. However one must always keep in mind the fact that this is, strictly, a large time-scale approximation, since when going back to $\Delta t \approx \tau$ (left diagram in Fig. 1), differentiability is recovered.

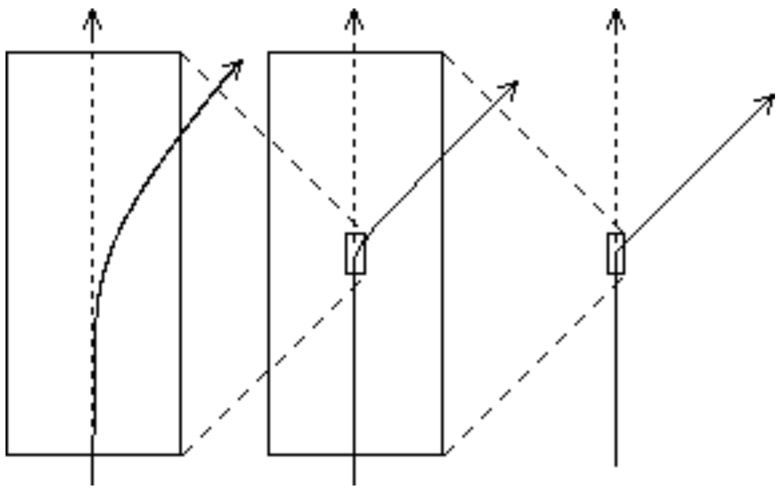

Figure 1. Schematic representation of the relative evolution in space of two initially nearby chaotic trajectories seen at three different time scales (from Nottale, 1993a).

Then the first step of our approach consists in giving up the concept of well-defined trajectory *at large time scales*, and in introducing families of virtual trajectories (Nottale, 1989, 1993a). The real trajectory is one random realization among the infinite number of trajectories of the family. Such families are now characterized by a probability density ρ.

In order to describe them, we assume, once the chaos developed, that the virtual trajectories evolve following a diffusion process, characterized by some diffusion coefficient $\mathcal{D}$. Such a diffusion can be described by a Markov-Wiener process $\xi(t)$, as used in Einstein's theory of Brownian motion.

However such a process is fundamentally irreversible, while our main concern here is hamiltonian, non dissipative systems, the equations of which are fundamentally reversible. As we shall see in Section 4, our first domain of application of this method is the study of the final epoch of the Solar System formation, when the interactions between planetesimals were only of purely gravitational origin. This means that the reversed process ($t \rightarrow -t$) must be equally valid for the description of their temporal evolution. We are then led to introduce, following Nelson (1966, 1985), a twin Wiener (backward and forward) process. Reversibility is recovered from the mixing of the two processes.

Namely, the position vector $\boldsymbol{x}(t)$ is assimilated to a stochastic process which satisfies the following relations (respectively for the forward ($dt > 0$) and backward ($dt < 0$) process)

$$d\boldsymbol{x}(t) = \boldsymbol{b}_+[\boldsymbol{x}(t)]\, dt + d\boldsymbol{\xi}_+(t) = \boldsymbol{b}_-[\boldsymbol{x}(t)]\, dt + d\boldsymbol{\xi}_-(t) \,. \tag{1}$$

In other words, the infinitesimal displacement $d\boldsymbol{x}$ is described for both processes as the sum of a mean, $\langle d\boldsymbol{x} \rangle = \boldsymbol{b}\, dt$, and a fluctuation about this mean, $d\boldsymbol{\xi}$, which is then by definition of zero average: $\langle d\boldsymbol{\xi}_\pm \rangle = 0$.

Consider first the average displacements. As remarked by Nelson (1966, 1985), nondifferentiability implies that the average backward and forward velocities are in general different. So he defines mean forward and backward derivatives, d_+/dt and d_-/dt,

$$\frac{d_\pm}{dt}\boldsymbol{y}(t) = \lim_{\Delta t \to 0\pm} \left\langle \frac{y(t+\Delta t) - y(t)}{\Delta t} \right\rangle \quad , \tag{2}$$

which, once applied to the position vector $\boldsymbol{x}$, yield *forward and backward mean velocities*, $\frac{d_+}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_+$ and $\frac{d_-}{dt}\boldsymbol{x}(t) = \boldsymbol{b}_-$.

Consider now the fluctuations $d\boldsymbol{\xi}_\pm$. If we start from a general fractal behavior, i.e. assume that $\boldsymbol{x}(t)$ is a fractal function, then the general relation between $d\xi_i$ and dt is (see Fig. 2):

$$d\xi_i^{\ D} \approx dt \, . \tag{3}$$

We shall first specialize our discussion to the particular case $D = 2$, which corresponds to standard Brownian motion. In the next section, we shall suggest methods to deal with the more general case $D \neq 2$.

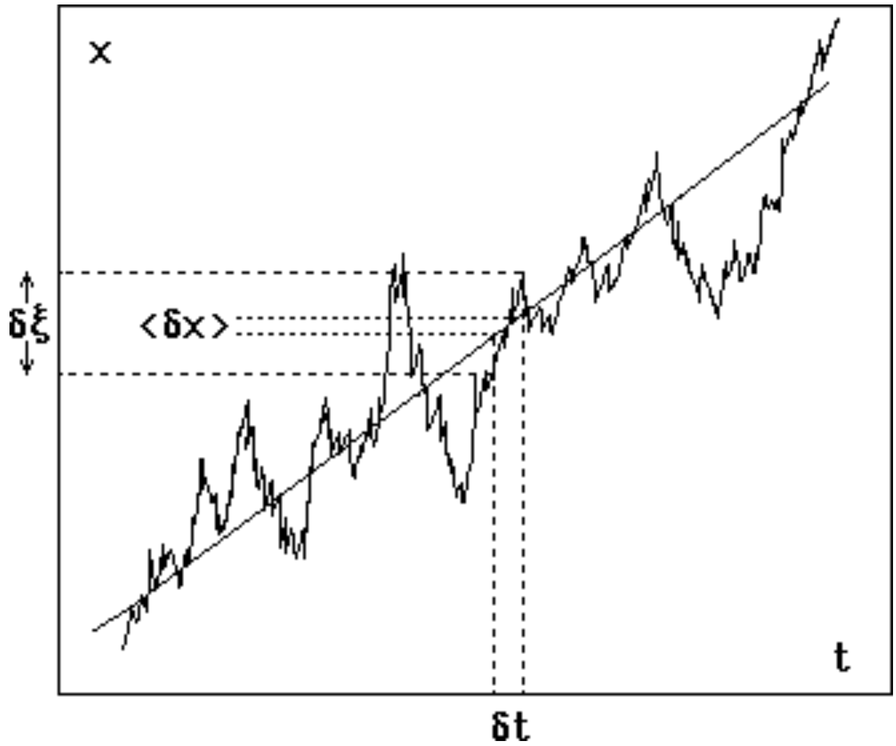

Figure 2. Relation between differential elements on a fractal function. While the average, "classical" $\langle d\boldsymbol{x} \rangle$ is of the same order as the abscissa differential dt, the fluctuation is far larger and depends on the fractal dimension D as: $d\xi \approx dt^{1/D}$.

In the case of trajectories of fractal dimension 2, the $d\boldsymbol{\xi}(t)$'s correspond to a standard Wiener process, since we can describe them as Gaussian, with mean zero, mutually independent and such that (3) becomes

$$< d\xi_{\pm i} \;\; d\xi_{\pm j} > \;\; = \pm\, 2\, \mathcal{D}\, \delta_{ij} \;\; dt \; , \tag{4}$$

$\mathcal{D}$ standing for a "diffusion coefficient".

The non-differentiability and scale divergence of this process is straighforward from the computation of the velocity. Using standard methods, it does not exist, being formally infinite: $\mathrm{v} \approx d\xi/dt \;\; \propto dt^{-1/2} \to \infty$ when $dt \to 0$. This problem is solved by our method which consists in describing physical quantities in terms of explicitly scale dependent functions (Nottale, 1993a, 1994): here for the velocity $\mathrm{v} = \mathrm{v}(t, \delta t)$. Then a fractal velocity writes in general:

$$\mathrm{v}(t, \delta t) \;=\; \mathrm{v}_0(t\,) \left[\, 1 + \left(\frac{\tau_0}{\delta t} \right)^{1-(1/D)} \right] , \tag{5}$$

in which we recognize both the average and fluctuation terms.

While in every present formulations of Nelson's stochastic mechanics, one writes two systems of equations for the forward and backward processes (or for combinations of them) and eventually combine them in the end into a complex equation, we have suggested (Nottale, 1993a, 1994) to work *from the beginning* in terms of complex quantities. So we combine the forward and backward derivatives of Eq. (2) in terms of a complex derivative operator

$$\frac{\not{d}}{dt} \;=\; \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2dt} \quad , \tag{6}$$

which, when applied to the position vector, yields a complex velocity

$$\boldsymbol{\mathcal{V}} \;=\; \frac{\not{d}}{dt}\,\boldsymbol{x}(t) \;=\boldsymbol{V} - i \;\; \boldsymbol{U} \;\;=\;\; \frac{\boldsymbol{b}_+ + \boldsymbol{b}_-}{2} \;\;-\;\; i \;\;\; \frac{\boldsymbol{b}_+ - \boldsymbol{b}_-}{2} \quad . \tag{7}$$

The real part $\boldsymbol{V}$ of the complex velocity $\boldsymbol{\mathcal{V}}$ generalizes the classical velocity, while its imaginary part, $\boldsymbol{U}$, is a new quantity arising from non-differentiability.

Equation (4) now allows us to get a general expression for the complex time derivative $\not{d}/dt$. Consider a function $f(\boldsymbol{x},t)$, and expand its total differential to second order. We get

$$d\,f \;\;=\;\; \frac{\partial f}{\partial t}\; dt \;\; + \boldsymbol{\nabla}\, f \,.\, d\boldsymbol{x} \;\;\; + \;\; \frac{1}{2}\; \frac{\partial^2 f}{\partial x_i\, \partial x_j}\; dx_i \;\; dx_j \quad . \tag{8}$$

We may now compute the forward and backward derivatives of f. In this procedure, the mean value of $<dx_i \;\; dx_j>$ reduces to $<d\xi_{\pm i} \;\; d\xi_{\pm j}>$, so that the last term of Eq. (8) amounts to a Laplacian thanks to Eq. (4). We obtain

$$d_{\pm} f / dt = (\partial/\partial t + \boldsymbol{b}_{\pm} \, . \, \boldsymbol{\nabla} \pm \mathcal{D} \Delta) f \ , \tag{9}$$

so that we can finally give the expression for the complex time derivative operator (Nottale, 1993a):

$$\frac{\hat{d}}{dt} = \frac{\partial}{\partial t} + \boldsymbol{\mathcal{V}} \, . \, \boldsymbol{\nabla} - i \, \mathcal{D} \, \Delta \ . \tag{10}$$

We shall now postulate that the passage from classical (differentiable) mechanics to the new nondifferentiable mechanics that is considered here can be implemented by a unique prescription: Replace the standard time derivative *d/dt* by the new complex operator $\hat{d}/dt$. In other words, this means that $\hat{d}/dt$. will play the role of a *scale-covariant derivative*. This is our main tool for implementing our initial requirement of *scale-covariance*, at least at this level of the analysis. From Eq. (10), a partial covariant derivative can be built:

$$\tilde{\boldsymbol{\nabla}} = \boldsymbol{\nabla} - i \ \mathcal{D} \ \frac{\boldsymbol{\mathcal{V}}}{\boldsymbol{\mathcal{V}}^2} \ \Delta \ . \tag{11}$$

Let us now give the main steps by which one may generalize classical mechanics using this scale-covariance. We assume that any mechanical system can be characterized by a Lagrange function $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\mathcal{V}}, t)$, from which an average stochastic action S is defined:

$$S = \int_{t_1}^{t_2} \mathcal{L}(\boldsymbol{x}, \boldsymbol{\mathcal{V}}, t) \, dt \ . \tag{12}$$

Our Lagrange function and action are *a priori* complex and are obtained from the classical Lagrange function $\mathcal{L}(\boldsymbol{x}, d\boldsymbol{x}/dt, t)$ and classical action S precisely from applying the above prescription $d/dt \rightarrow \hat{d}/dt$. The stationary-action principle, applied on this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations (Nottale, 1993a)

$$\frac{\hat{d}}{dt} \frac{\partial \mathcal{L}}{\partial \mathcal{V}_i} = \frac{\partial \mathcal{L}}{\partial x_i} \ , \tag{13}$$

which are exactly the equations one would have obtained from applying the scale-covariant derivative ($d/dt \rightarrow \hat{d}/dt$) to the classical Euler-Lagrange equations themselves: this result demonstrates the self-consistency of the approach and vindicates the use of complex numbers. Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space in the mean leads to defining a generalized *complex* momentum given by

$$\mathcal{P} = \frac{\partial \mathcal{L}}{\partial \mathcal{V}} \ . \tag{14}$$

If one now considers the action as a functional of the upper limit of integration in Eq. (12), the variation of the action from a trajectory to another nearby trajectory, when combined with Eq. (13), yields a generalization of another well-known result:

$$\boldsymbol{\mathcal{P}} = \nabla \mathcal{S} \ . \tag{15}$$

We shall now specialize and consider Newtonian mechanics. The Lagrange function of a closed system, $L = (1/2)\, m\, v^2 - U$, is generalized as $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\mathcal{V}}, t) = (1/2)\, m\, \mathcal{V}^2 - \mathcal{U}$, where $\mathcal{U}$ denotes a (still classical) scalar potential. Note that the real part of $\mathcal{L}$ becomes $\frac{1}{2}\, m(\boldsymbol{V}^2 - \boldsymbol{U}^2) - \mathcal{U}$, which is the Lagrangian field proposed by Guerra and Morato (1983). The Euler-Lagrange equations keep the form of Newton's fundamental equation of dynamics

$$-\nabla\, \mathcal{U} = m\, \frac{\hat{d}}{dt} \boldsymbol{\mathcal{V}} \ , \tag{16}$$

which is now written in terms of complex variables and operator. The complex momentum $\boldsymbol{\mathcal{P}}$ now reads:

$$\boldsymbol{\mathcal{P}} = m\, \boldsymbol{\mathcal{V}} \ , \tag{17}$$

so that from Eq. (15) we arrive at the conclusion that, in this case, the *complex velocity* $\boldsymbol{\mathcal{V}}$ *is a gradient*, namely the gradient of the complex action:

$$\boldsymbol{\mathcal{V}} = \nabla \mathcal{S} \,/\, m \ . \tag{18}$$

This is an interesting result owing to the fact that in several derivations of Nelson’s stochastic mechanics, the classical velocity $\boldsymbol{V}$ (i.e. the real part of our complex velocity $\boldsymbol{\mathcal{V}}$) is a gradient is *postulated*.

We may now introduce a complex function ψ which *is nothing but another expression for the complex action* $\mathcal{S}$,

$$\psi = e^{i\mathcal{S}/2m\mathcal{D}} \ . \tag{19}$$

It is related to the complex velocity as follows:

$$\boldsymbol{\mathcal{V}} = -2\, i\, \mathcal{D}\, \nabla\, (ln\psi) \ . \tag{20}$$

From this equation and Eq. (17), we obtain:

$$\boldsymbol{\mathcal{P}}\, \psi = -2\, i\, m\, \mathcal{D}\, \nabla\, \psi \ , \tag{21}$$

which is nothing but the *correspondence principle* of quantum mechanics for momentum, but here demonstrated and written in terms of an exact equation.

We have now at our disposal all the mathematical tools needed to write Newton's equation (Eq. 16) in terms of the new quantity ψ. It takes the form

$$\nabla\,\mathcal{U} \;=\; 2\,i\;\mathcal{D}\,m\,\frac{\hat{d}}{dt}\,(\nabla\,ln\psi)\;. \tag{22}$$

Replacing $\hat{d}/dt$ by its expression (Eq. 10) yields:

$$\nabla\,\mathcal{U} = 2\,i\;\mathcal{D}\,m\left[\frac{\partial}{\partial t}\;\nabla\,ln\psi\;-i\;\mathcal{D}\,\Delta(\nabla\,ln\psi)\;-2i\,\mathcal{D}\,(\nabla\,ln\psi\,\bullet\nabla\,)(\nabla\,ln\psi)\right]. \tag{23}$$

Standard calculations on differential operators allows one to simplify this expression thanks to the relation

$$\frac{1}{2}\,\Delta\,(\nabla\,ln\psi) + (\nabla\,ln\psi\,\bullet\nabla\,)(\nabla\,ln\psi)\;=\;\frac{1}{2}\;\nabla\;\frac{\Delta\psi}{\psi}\;, \tag{24}$$

and we obtain

$$\frac{\hat{d}}{dt}\,\mathbf{V}\;=\;-2\;\mathcal{D}\,\nabla\;\;\{\,i\,\frac{\partial}{\partial t}\;ln\psi\;+\;\mathcal{D}\,\frac{\Delta\psi}{\psi}\;\}\;=-\nabla\,\mathcal{U}\,/\,m\;. \tag{25}$$

Integrating this equation finally yields

$$\mathcal{D}^2\,\Delta\psi\;+\;i\;\mathcal{D}\,\frac{\partial}{\partial t}\,\psi\;\;-\;\frac{\mathcal{U}}{2m}\,\psi\;=\;0\;\;, \tag{26}$$

up to an arbitrary phase factor $\alpha(t)$ which may be set to zero by a suitable choice of the phase of ψ . In the very particular case when $\mathcal{D}$ is inversely proportional to mass, $\mathcal{D} = \hbar/2m$, we get the standard form of Schrödinger's equation

$$\frac{\hbar^2}{2m}\;\Delta\psi\;\;+\;i\,\hbar\;\frac{\partial}{\partial t}\,\psi\;\;=\;\;\mathcal{U}\;\psi\;. \tag{27}$$

The hereabove system becomes equivalent to Nelson's stochastic quantum mechanics (Nottale, 1993a), with several advantages, such as the new formulation of the correspondence principle, the fact that the real part of Eq. (13) defines an acceleration which is identical to that *postulated* by Nelson (1966, 1985), and more generally the complete symmetry with classical mechanics, including the definition of the Lagrange function. But the main difference with standard stochastic mechanics is that Nelson's Schrödinger equation is obtained as a mixing of a real Newton equation and of a Fokker-Planck equation for the diffusion process. In our approach, as the reader can check, we have obtained the Schrödinger equation without introducing the probability density and without writing the Kolmogorov equations. Hence this theory is not statistical in its essence, and must be *completed*, as has been the case for quantum mechanics in its own time, by a statistical *interpretation*.

Such a statistical interpretation is simply obtained by setting $\psi\, \psi^{\dagger} = \rho$, and then writing the imaginary part of Eq. (26) in terms of this new variable. One gets:

$$\partial\rho \, / \, \partial t \; + \; \mathrm{div}(\rho \, \boldsymbol{V}) \; = \; 0 \; , \tag{28}$$

which is easily recognized as an equation of continuity. More generally one can demonstrate that ρ verifies a forward and a backward Fokker-Planck equation (Nelson, 1966; Welsh, 1970):

$$\partial\rho \, / \, \partial t \; + \; \mathrm{div}(\rho \, \boldsymbol{b}_{\pm}) \; = \; \pm \; \Delta \, (\mathcal{D}\rho) \; . \tag{29}$$

Note that, from the viewpoint of the standard theory of Markovian processes, these two equations are both *forward* equations. Namely, we have actually considered two processes, a forward process with time running to the future, and another forward process with reversed time. Nelson's new “backward” equation, written with an average velocity different from that of the forward one, “kills” the usual backward Kolmogorov equation (Nottale and Van Waerbeke, 1995). In the case, considered in the present section, when the diffusion coefficient is constant, we can take it out of the Δ sign. The two Fokker-Planck equations can be combined in a single complex equation:

$$\partial\rho \, / \, \partial t \; + \; div(\rho \mathbf{V}) \; = \; -i \;\; \mathcal{D} \, \Delta\rho \;\; . \tag{30}$$

Then we can finally conclude that ρ is the density of probability to find the particle at a given position, and that the complex action S is given in terms of the classical action and of the probability density by the relation:

$$\mathcal{S} \; = \; 2 \, m \, \mathcal{D} \;\; (S - i \;\; \ln\rho^{1/2}) \;\; , \tag{31}$$

i.e., the imaginary part of the generalized complex action is identified with the logarithm of the probability density.

Before considering some possible generalisations of this formalism, then apply it to the problem of the formation of the solar system, we want to stress once again the difference between its application to quantum mechanics and to developed chaos. In the case of quantum mechanics (Nottale, 1993a), our fundamental assumption is that space-time itself is continuous but non-differentiable, then fractal *without any lower limit*. The complete withdrawal of the hypothesis of differentiability is necessary if we want the theory not to be a hidden parameter one and to agree with Bell's theorem and the undeterminism of quantum paths. On the contrary, in the application to classical chaos, we know from the beginning that non-differentiability is only a large time-scale approximation, and that when going back to small time resolution we recover differentiable, predictable classical trajectories. Then it must be clear that, even if the theory yields densities of probability in a quasi-quantum way, the interpretation remains classical.

3. GENERALIZATION TO VARIABLE DIFFUSION COEFFICIENT

3.1. Fractal dimension different from 2

Two generalizations are particularly relevant: the case of a fractal dimension different from 2 (Brownian motion is only a very peculiar case of fractal behavior), and the case of a diffusion coefficient varying with position and time, $\mathcal{D} = \mathcal{D}(x,t)$, since the hypothesis of a constant $\mathcal{D}$ is clearly unjustified in most applications to chaos. The general problem of tackling properly the case $D \neq 2$ lies outside the scope of the present contribution: such processes correspond to fractional Brownian motions, which are known to be non-Markovian, and persistent ($D < 2$) or antipersistent ($D > 2$). We shall consider only the case when the fractal dimension D remains close to 2. Indeed, in this case its deviation from 2 can be approximated in terms of an explicit scale dependence in terms of the time resolution, as first noticed by Mandelbrot and Van Ness (1968). Namely we decompose the $<d\xi^2>$'s, which vary now as $dt^{2/D}$, as a product of a pure Brownian term proportional to dt and a resolution-dependent correction:

$$<d\xi_{\pm i}\ d\xi_{\pm j}> \ = \pm\, 2\, \mathcal{D}_0\, \delta_{ij}\ dt\ \ (\delta t/\tau)^{(2/D)-1} \ , \tag{32}$$

where τ is some characteristic time-scale. Hence the effect of $D \neq 2$ can be dealt with in terms of a generalized, scale-dependent, diffusion coefficient:

$$\mathcal{D} \ = \ \mathcal{D}(\delta t) \ = \mathcal{D}_0\, (\delta t/\tau)^{(2/D)-1} \ . \tag{33}$$

It is easy to verify that the demonstration of Section 2 can be followed without any modification. But now the “wave function” ψ becomes explicitly dependent on scale:

$$\psi \ = \ \exp\, \{\, i\, S\, /2m\, \mathcal{D}(\delta t)\, \}, \tag{34}$$

as well as the generalized Schrödinger equation:

$$\mathcal{D}^2(\delta t)\, \Delta\psi \ + i\ \ \mathcal{D}(\delta t)\ \frac{\partial \psi}{\partial t} \ = \ \frac{\mathcal{U}\psi}{2m} \ , \tag{35}$$

where $\mathcal{D}(\delta t)$ is given by Eq. (33). We have made a first study of the behavior of this equation in (Nottale, 1995): it is relevant in particular in our development of a “Lorentzian scale relativity” (Nottale, 1992, 1993a). Concerning the problem in which we are interested here, the remarkable result is that the form of the equation is conserved, and that the equation with a constant diffusion coefficient (Eq. 26) will remain a good approximation provided the domain of variation of δt is not too large.

3.2. Position-dependent diffusion coefficient

Let us now consider the case of a diffusion coefficient varying with position and time, D = D(x,t), particularly relevant for the application of this approach to chaotic dynamics. Namely we write:

$$<d\xi_{\pm i} \; d\xi_{\pm j}> = \pm\, 2\, \mathcal{D}(x,t)\, \delta_{ij} \; dt \quad . \tag{36}$$

We can now reconsider the various steps of the formalism of Section 2. The complex time derivative operator (Eq. 10) is found to keep its form:

$$\frac{\text{đ}}{dt} \; = \; \frac{\partial}{\partial t} \; + \; \boldsymbol{\mathcal{V}} \, . \, \boldsymbol{\nabla} \; - i \, \mathcal{D}(x,t)\, \Delta \quad . \tag{37}$$

Equations (12) to (18) remain unchanged. This is no longer the case for the following steps, in particular for the definiton of the wave function which depends on $\mathcal{D}$ in Eq. (19). Let us introduce the average value of the diffusion coefficient by writing:

$$\mathcal{D}(x,t) \; = \; <\mathcal{D}> + \; \delta\mathcal{D}(x,t) \quad . \tag{38}$$

We now introduce the complex function ψ from the relation,

$$\psi \; = \; e^{iS/2m<\mathcal{D}>} \quad . \tag{39}$$

Then ψ is related to the complex velocity :

$$\boldsymbol{\mathcal{V}} \; = \; - \, 2 \, i \; <\mathcal{D}> \boldsymbol{\nabla} \, (\ln\psi) \; . \tag{40}$$

The generalized complex Newton's equation now takes the form

$$\boldsymbol{\nabla} \, \mathcal{U} \; = \; 2 \, i \; m <\mathcal{D}> \frac{\text{đ}}{dt} \, (\boldsymbol{\nabla} \ln\psi \,) \; . \tag{41}$$

Remembering that $\mathcal{D}$, which appears in the expression for $\text{đ} / dt$, is now a function of x, we find:

$$\boldsymbol{\nabla}\, \mathcal{U} = \; 2i \; <\mathcal{D}> m \, [\, \frac{\partial}{\partial t} \boldsymbol{\nabla} \ln\psi - i \; \mathcal{D}\, \Delta(\boldsymbol{\nabla} \ln\psi) - 2i <\mathcal{D}> (\boldsymbol{\nabla} \ln\psi \, .\boldsymbol{\nabla} \,)(\boldsymbol{\nabla} \ln\psi) \,], \tag{42}$$

which becomes

$$\frac{\nabla \mathcal{U}}{2m<\mathcal{D}>} = \nabla \left\{ i \frac{\partial}{\partial t} \ln\psi + <\mathcal{D}> \frac{\Delta\psi}{\psi} \right\} + \delta\mathcal{D}(x,t) \nabla (\Delta \ln\psi) . \tag{43}$$

Let us make appear the gradient of $\delta\mathcal{D}$ thanks to the identity $\nabla (\delta\mathcal{D} \,\Delta\ln\psi) = \Delta\ln\psi \,\nabla \delta\mathcal{D} + \delta\mathcal{D} \,\nabla(\Delta\ln\psi)$. Equation (43) may finally be given the form of a generalized Schrödinger equation:

$$\nabla \left\{ \frac{\mathcal{U}}{2m<\mathcal{D}>} - \frac{1}{\psi} \left[\mathcal{D} \Delta\psi + i \frac{\partial\psi}{\partial t} \right] + \delta\mathcal{D} (\nabla \ln\psi)^2 \right\} = -\nabla(\delta\mathcal{D}) \Delta\ln\psi . \tag{44}$$

We let open for future works the study of the general form of this equation. We shall only consider here the special, simplified case when $\nabla(\delta\mathcal{D}) = 0$ or $\nabla(\delta\mathcal{D}) << 1$. In this case, which corresponds to a slowly varying diffusion coefficient in the domain considered, or, at the limit, to a diffusion coefficient depending on time but not on position, the right -hand side of Eq. (44) vanishes, so that it may still be integrated, yielding:

$$\mathcal{D} \Delta\psi + i \frac{\partial\psi}{\partial t} = \left[\frac{\mathcal{U}}{2m<\mathcal{D}>} + a + \delta\mathcal{D} (\nabla \ln\psi)^2 \right] \psi , \tag{45}$$

where a is a constant of integration. Assuming that $\delta\mathcal{D}/\mathcal{D}$ remains <<1, the effect of the term $\delta\mathcal{D} \,\psi\, (\nabla \ln\psi)^2$ which is in addition to the standard Schrödinger equation and the effect of $\mathcal{D}$ being a function of x and t can be treated perturbatively. One finally gets the equation

$$<\mathcal{D}>^2 \Delta\psi + i <\mathcal{D}> \frac{\partial\psi}{\partial t} = \left[\frac{\mathcal{U}(x,t)}{2m} - <\mathcal{D}> [\Delta \ln\psi]_0 \,\delta\mathcal{D}(x,t) \right] \psi . \tag{46}$$

Hence the effect of the new terms amounts to adding a new term to the potential in the standard Schrödinger equation. Note that such a behavior could be of interest in the perspective of a future development of a field theory based on the concept of scale relativity and fractal space-time. Indeed, fluctuations in the fractal space-time geometry are expected to imply fluctuations $\delta\mathcal{D}(x,t)$ of the diffusion coefficient (in a way which remains to be described). Then even in the absence of an artificially added external potential $\mathcal{U}$ in Eq. (46), these fluctuations will imply the appearance of an internal potential term coming from the geometry of space-time itself. (More generally, we expect the “diffusion coefficient” to become a tensor in such a space-time approach). The development of such ideas lies beyond the scope of the present contribution and will be carried on elsewhere

Let us conclude this section by a brief comment. In the particular cases considered above, the statistical interpretation of the wave function ψ in terms of $\rho = \psi\psi^\dagger$ giving the probability of presence of the particle remains correct, since the imaginary part of this generalized Schrödinger equation remains the equation of continuity. We recall indeed that we do not need to write the Fokker-Planck equations in our derivation of the Schrödinger equation and of its generalized form. This may no longer be the case for the general equation, since we took for our definition of ψ in Eq. (39) the simplest possible generalization, which may not be the adequate one. A more complete treatment would certainly need coming back to the basic stochastic approach, and to writing Kolmogorov equations containing new terms due to the presence of $\mathcal{D}(x,t)$ into the Laplacian (Eq. 29). Such an improved method will be presented in a forthcoming work.

4. APPLICATION TO CELESTIAL MECHANICS

The existence of structuration in the Solar System has been recognized for long, at first by Kepler himself. The Titius-Bode “law” (see e.g. Nieto, 1972) is the first empirical attempt at describing these regularities, and was followed by several other proposals (see e.g. Neuhäuser and Feitzinger, 1986; Souriau, 1989). Most attempts at understanding the distribution of planets were based on theories of formation of the Solar System (see e.g. Brahic, 1982). However, there has been increasing evidence, in recent years, that chaos may play a leading role in celestial mechanics (Hénon and Heiles, 1964; Petit and Hénon, 1986; Wisdom, 1987; Sussman and Wisdom, 1988; Laskar,1989, 1990).

Hills (1970) has noticed that it is likely that an epoch of strong dynamical encounters occurred before the planets relaxed into stable orbits. It is also remarkable that numerical simulations (Conway and Elsner, 1988) which have considered various initial conditions for planetary systems, find that systems placed in initially arbitrary distributions are generally chaotic, while when initially placed in Titius-Bode-like laws (increasing planetary separations), they are very stable systems.

It has been recently argued by Graner & Dubrulle (1994) that the various avatars of Titius-Bode law share the property of scale invariance, and then that the "law", if real, would be only a consequence of the scale and rotational invariance of the initial protoplanetary disk. Since these are natural symmetries of such disks, Graner and Dubrulle conclude that Titius-Bode-like laws cannot serve as diagnostic for the validity of a model or theory of planet formation. We would agree with their point only in case this was the only test of such a theory. The distribution of planet distances is one among the various structures observed in the Solar System that one could try to reproduce in a theory of its formation (including possibly the distribution of eccentricities, of mass, of angular momentum, of element abundances, etc..), and as such it must be included into the test. Note that, in particular, the distribution of angular momentum, which is mainly carried by Jupiter (62 %) and Saturn (25 %) is an important unsolved problem for these theories.

With these considerations in mind, we shall investigate the possibility that, at the end of the formation of the Solar System, when planetesimals (see e.g. Wetherill, 1990) had become large enough for their motion to be only of purely gravitational origin, an epoch of strong chaos occurred (to which the above formalism can be applied), and that the present rather stable state of the system resulted precisely from structuration originating from chaos itself (Nottale, 1993a,b).

Consider a gravitational system described by a Newtonian potential $\mathcal{U}$ such that $\Delta\mathcal{U} = 4\pi G\rho$, and assumed to be subjected to developed chaos: assuming that one can define an average diffusion coefficient, Eq. (26) applies (as a first approximation model) to this problem. We have also seen in Section 3 that even when accounting for a slowly variable diffusion coefficient the equation still kept the same form. Then consider a test planet (or planetesimal) orbiting in the field of the Sun, $\mathcal{U} = -GmM/r$, and describe the collective, chaotic effect of all the other planetesimals by a Brownian-like motion, as given by the double Wiener fluctuation of the above formamism. The specialization of Eq. (26) to the case of stationary motion with conservative energy $E \equiv 2i\mathcal{D}m\partial/\partial t$ yields

$$\mathcal{D}\, \Delta\psi \;+\; \left[\, \frac{E}{2m\,\mathcal{D}} \;+\; \frac{G\,M}{2\,\mathcal{D}\,r} \,\right] \psi \;=\; 0 \;. \tag{47}$$

The equivalence principle suggests that $\mathcal{D}$ is now independent of m. This equation is similar to the Schrödinger equation for the hydrogen atom, up to the substitution $\hbar/2m \rightarrow \mathcal{D}$, $e^2 \rightarrow GmM$, so that the natural unit of length (which corresponds to the Bohr radius) is:

$$a_0 = 4\,\mathcal{D}^2/GM \;. \tag{48}$$

We thus find that the energies of planets scale as $E_n = -GmM^2/8\mathcal{D}^2n^2$, $n =$ 1, 2, 3, ..., and that the probability densities of their distances to the Sun are confined to definite regions given by the square of the well-known radial wave functions of the hydrogen atom. We also expect angular momenta to scale as $L = 2m\mathcal{D}l$, with $l = 0$, 1, ..., $n-1$: this means that, unlike in quantum mechanics, E/m and L/m are “quantized” rather than E and L . (One must be cautious that here the “quantization” does not take as strict a meaning as in quantum mechanics: since the trajectories become classic again at small time-scales, it must be understood as indicating the occurence of preferential values).

The average distance to the Sun and the eccentricity e are given, in terms of the two quantum numbers n and l , by the following relations (see e.g. Messiah, 1959):

$$a_{nl} \;=\; \left\{ \frac{3}{2}\, n^2 \;-\; \frac{1}{2}\; l\; (l\; +1) \right\} a_0 \,, \tag{49a}$$

$$e^2 = 1 - \frac{l\ (l+1)}{n\,(n-1)}\,. \tag{49b}$$

Let us now briefly compare these predictions to the observed structures in the Solar System. Note that the difference of physical and chemical composition of the inner and outer solar systems suggests to us that they can be treated as two different systems, i.e., that we expect two different diffusion coefficients for them. (See Section 4.3 (iii) for a possible justification of this point). The main results are summarized hereafter (see Fig. 3).

4.1. Distribution of eccentricities of planets

The observed orbits of the planets in the solar system are quasi-circular. Even the largest eccentricities (Pluto, $e^2 = 0.065$; Mercury, $e^2 = 0.042$) actually correspond to small values of e^2. Such a result is clearly a prediction of our theory: Indeed, Eq. (49b) implies that, after the purely circular state $l = n - 1$, the first non circular state, $l = n - 2$, yields eccentricities larger that 0.58 for $n \leq 6$ (which is the range observed for n in the solar system, see below). Such a large value would imply orbit crossing between planets and strong chaos and cannot correspond to a stable configuration on large time scales. Then only the quasi-circular orbits remain admissible solutions.

4.2. Distribution of planet distances

We may now compare the observed values of semi-major axes of the planets to our prediction (Eq. 49a) with $l = n - 1$: $\sqrt{a} = n\,(1+1/2n)^{1/2}\,\sqrt{a_0}$, for the inner and outer systems respectively. Note that *the ordinate at origin is predicted to be zero*. This prediction is very well verified for the two systems: we find $a_{\rm int}(0) = 2 \times 10^{-4}$ A.U. and $a_{\rm ext}(0) = 4 \times 10^{-3}$ A.U.

Mercury, Venus, Earth and Mars take respectively ranks $n = 3$, 4, 5 and 6 in the inner system. The average slope is $(\sqrt{a_0})_{\rm int} = 0.195 \pm 0.0022$. The orbits $n = 1$ corresponds to a distance so close to the Sun (0.05 A.U.), that its emptiness may be easily understood, while the possibility that $n = 2$ hosts a still undiscovered small planet is not excluded.

The central peak of the asteroid belt (2.7 A.U.) agrees remarkably well with $n = 8$ of the inner system, and the main peak (3.15 A.U.) with $n = 9$. Including them yields $(\sqrt{a_0})_{\rm int} = 0.195 \pm 0.0017$. This result may help understanding the fact that there is no large planet there: the zone where the belt lies, even though it corresponds to maxima of probability density for the inner system, also corresponds to a *minimum* in the outer system. The region between Mars and Jupiter is where the two systems overlap. The emptiness of the orbits $n = 7$ and $n = 10$ is easily understandable, since they coincide with the resonances 1:4 and 2:3 with Jupiter, where small time-scale dynamical chaos is expected to occur (Wisdom, 1987).

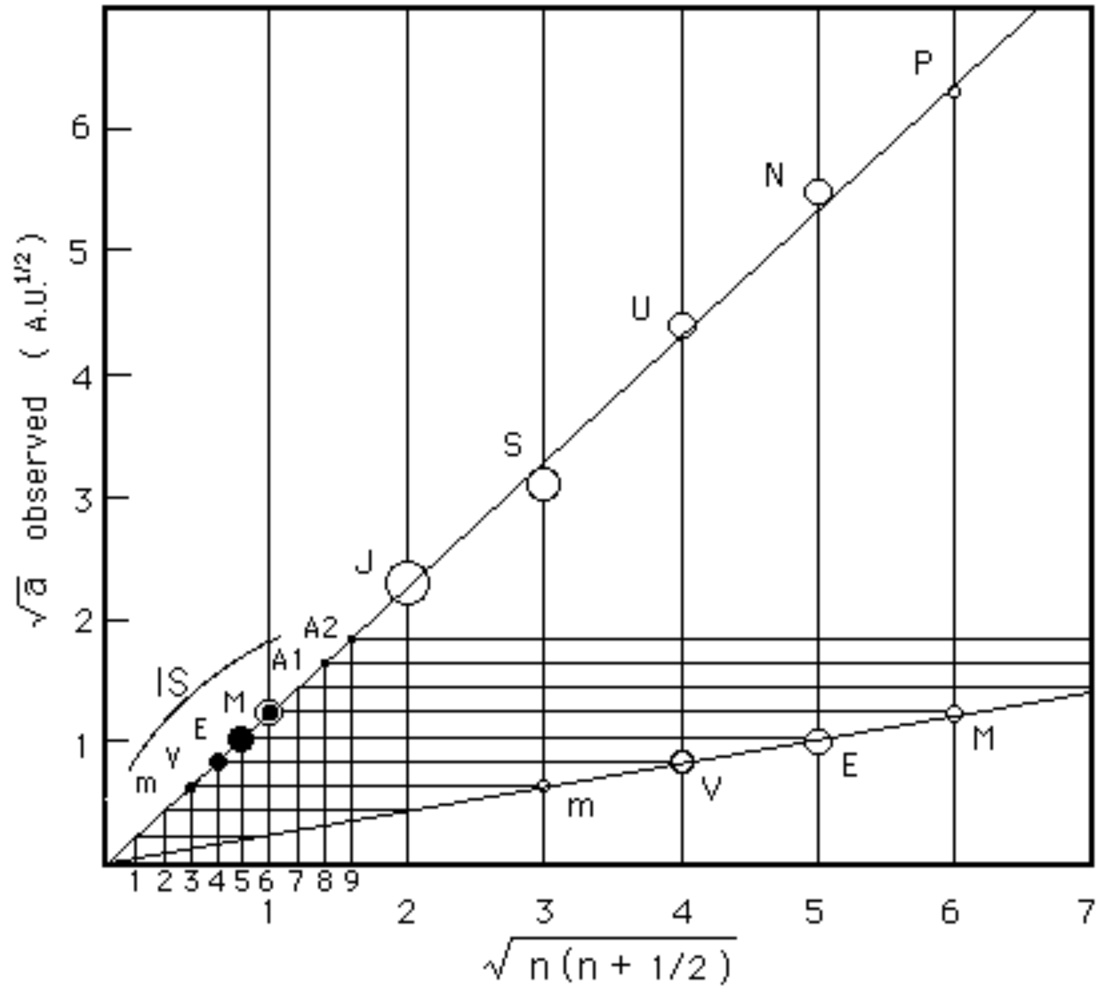

Figure 3. Comparison of the observed average distances of planets to the Sun with our prediction (see text). The abscissa is labelled by the value of n, but is given by $\sqrt{n(n+1/2)}$. A1 and A2 are for the two main peaks in the distribution of asteroids in the asteroid belt. The size of the symbols indicates the planet relative masses in a qualitative way. “IS” stands for the whole inner solar system, which corresponds to “orbital” $n = 1$ of the outer system.

Jupiter, Saturn, Uranus, Neptune and Pluto rank $n = 2, 3, 4, 5, 6$ in the outer system (see Fig. 3). The average slope is $(\sqrt{a_0})_{\text{ext}} = 1.014 \pm 0.016$. The average distance of the inner solar system in very good agreement with $n = 1$ of the *outer* system (see below our suggestion that it corresponds to a secondary process of fragmentation): including it yields an improved slope $(\sqrt{a_0})_{\text{ext}} = 1.014 \pm 0.012$. Note also the agreement of Neptune and especially Pluto with the outer relation (recall that they did not fit the original Titius-Bode law).

4.3 Distribution of mass in the solar system

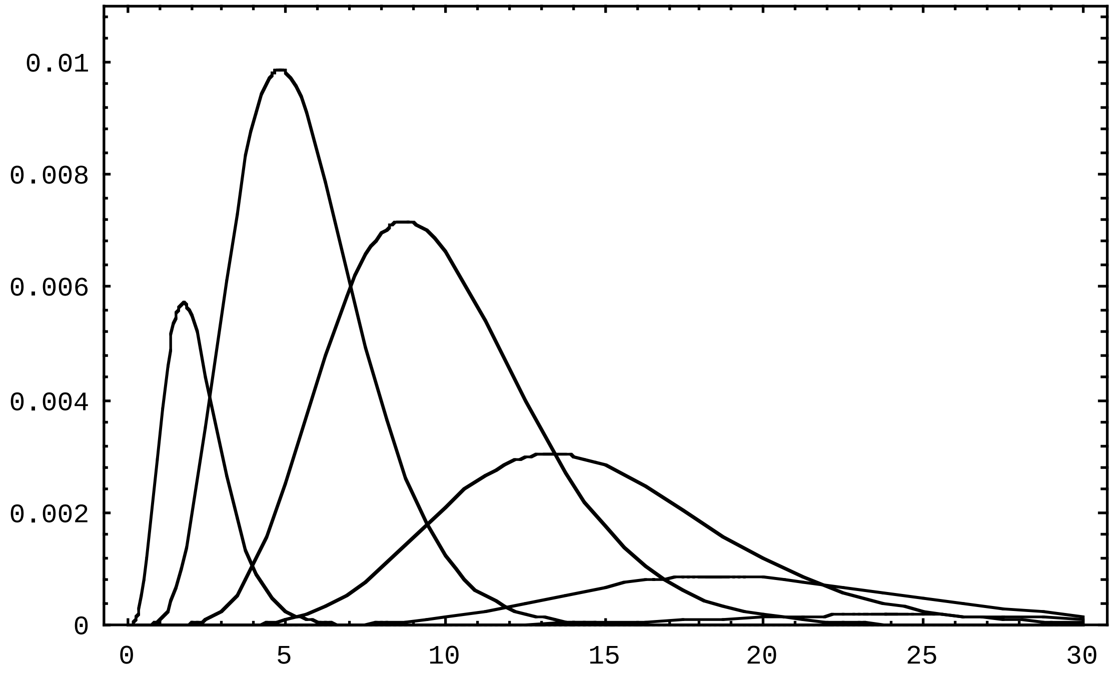

Figure 4. Possible mechanism of fragmentation for the mass distribution in the solar system (see text).

Not only the distribution of planet positions, but the distribution of mass itself is not at random in the solar system. Consider first the outer system, in which the average inner system is counted as one. We see the mass increase, reach a peak with Jupiter, then decrease up to Pluton (this decrease may continue with the possible ultra-plutonian small bodies). Now consider the inner solar system: the mass distribution follows the same shape, with an increase for Mercury to Earth, then a decrease up to the asteroids. Such a mass distribution is in agreement, at least in its great lines, with the laws of probability density derived from Eq. (47), which write for the various values of n (circular orbits, $l = \mathrm{l} - 1$, and $\int P(r)dr = 1$):

$$P(r) \propto \frac{1}{2n!} \left(\frac{2}{na}\right)^{2n+1} r^{2n} \; e^{-2r/na} \quad . \tag{50}$$

This suggests to us a possible mechanism for the mass distribution in the solar system. The first step would be a distribution of planetesimals according to the fundamental state ($n_0 = 1$) of Eq. (50), which is in qualitative agreement with the global mass distribution. Then a first process of fragmentation would occur, once again according to Eqs (47) and (50). The peak of probability density will give rise to the formation of the most massive planet in the system, i.e. Jupiter, which fixes the unit in Eq. (50) and for all other length scales. The remaining planetesimals would then make the other planets of the outer system (see Fig. 4), with distances increasing in terms of a new index n_1. However, although far from the sun the planetesimals accrete in only one planet in each orbital, tidal effects imply for the fundamental one ($n_1 = 1$) a new fragmentation process in terms of a third "quantum number", n_2. This "orbital" is then identified with the whole inner solar system. The advantage of such a process is that it relates the scales of the inner and outer systems and then reduces the number of free parameters to only one. Indeed the ratio of distances between the peak of orbital $n_1 = 2$ (Jupiter) and the peak of orbital $n_1 = 1$ (which is identified with the planet of largest mass in the inner solar system, i.e. the Earth) is expected to be $a_J/a_E \approx 5$, in good agreement with the observed value 5.2.

4.4 Distribution of angular momentum

Our finding that L/m is quantized rather than L allows us to suggest a solution to the problem of the distribution of angular momentum among planets (Jupiter 60%, Saturn 25%). Indeed, since the quantum number n remains small (≤ 6), the distribution of angular momentum is expected to mainly mirror that of mass: then most angular momentum must be carried on by the largest planets, as observed.

5. DISCUSSION AND CONCLUSION

In spite of the success already obtained by the method presented here, progress still needs to be made in order to improve its applicability, in particular to the solar system problem. The first need of improvement concerns the flattening of the proto-planetary disk, since our method remains, up to now, mainly tri-dimensional. Concerning the diffusion coefficient, even if we have demonstrated the stability of our solution under slow variations of it, it is nevertheless clear that a better treatment would consist in writing coupled equations in which the diffusion coefficient would be itself dependent of the probability density. Such a generalized method will be presented in a future work.

One of the difficulty of theories of the solar system formation and structures is, up to now, its uniqueness: we do not know whether an observed "law" is a peculiar configuration of our own system, or whether it is shared by all planetary systems in the universe. But we can expect such other systems to be discovered in the forthcoming years, and new informations to be obtained about the very distant solar system (Kuyper's belt, Oort cometary cloud...). In this regard our theory is a falsifiable one, since it makes definite predictions about such observations of the near future: observables such as the distribution of eccentricities, mass, angular momentum, the preferred positions of planets and asteroids, or possibly the ratio of distance of the largest gazeous planet and the largest telluric one, are expected in our framework to be universal structures shared by any planetary system.

Before concluding, we emphasize once again the fact that, even though our equations are quantum mechanical-like equations, their interpretation is very different from that of quantum mechanics. We have conjectured elsewhere (Nottale, 1989, 1991, 1993a, 1994) that the origin of the quantum properties which are irreducible to the classical behaviour was the nondifferentiability (and fractal structure) of the quantum space-time. On the other hand, the application of the above formalism to classical chaotic systems relies on the *approximation* of nondifferentiability, which is no longer valid on small time-scales, for which one recovers predictable and differentiable trajectories. Then, while this formalism is able to provide us with peaks of probabilities, and then to help us understanding the emergence of structures from chaos, the probabilities nevertheless remain classical.

To conclude, we think that it is the demonstration of the ability of our new method to solve several different problems involving stongly chaotic dynamics that will be the best proof of its physical meaning: we shall indeed present in forthcoming works its application to new problems such as the structuration of double galaxies and large scale structures in the universe.

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