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Quadratic relativistic invariant and metric form in quantum mechanics

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Abstract. The Klein–Gordon equation is recovered in the framework of the theory of scale-relativity, first in the absence, then in the presence of an electromagnetic field. In this framework, spacetime at quantum scales is characterized by non-differentiability and continuity, which involves the introduction of explicit resolution-dependent fractal coordinates. Such a description leads to the notion of scale-covariance and its corresponding tool, a *scale-covariant*; derivative operator $\eth/ds$. Due to it, the Klein–Gordon equation is written as an equation of free motion and interpreted as a geodesic equation in fractal spacetime. However, we obtain a new form for the corresponding relativistic invariant, which differs from that of special and general relativity. Characterizing quantum mechanics in the present approach, it is *not simply quadratic* in terms of velocities, but contains an extra term of divergence, which is intrinsically present in its expression. Moreover, in spite of the scale-covariance statements of the present theory, we find an extra term of current in addition to the Lorentz force, within the equations of motion with electromagnetic field written in this framework. Finally, we introduce another tool—a 'symmetric product'—from the requirement of recovering the usual form of the Leibniz rule written with the operator $\eth/ds$. This tool allows us to write most equations in this framework in their usual classical form; in particular the simple rules of differentiation, the equations of motion with field and also our new relativistic invariant.

1. Introduction

One of the aims of scale-relativity theory [1] is to give a new approach to quantum mechanics, in which quantum behaviour could arise from a geometric description of spacetime at quantum scales. Scale-relativity itself is associated with an extension of the principle of relativity to scale transformations. More generally, it is an approach which would apply to several physical domains which involve scale laws, such as cosmology and chaotic systems [1–4]. The theory proceeds along the following lines.

First, one generalizes Einstein's principle of relativity to scale transformations. That is, one redefines spacetime resolutions as characterizing the state of scale of reference systems, in the same way as velocity characterizes their state of motion. One then requires that the laws of physics apply whatever the state of the reference system (of motion and of scale). This would be mathematically achieved by a principle of scale-covariance, which we shall discuss at the end of this paper and which requires that the equations of physics keep their simplest form under transformations of resolutions [1].

The geometrical framework for implementing Einstein's general relativity is Riemannian, curved spacetime. Similarly, scale-relativity may be given a geometric meaning, in terms of a non-differentiable, fractal spacetime [1]. The notion of 'fractal spacetime' has been also introduced by Ord [5], who exhibited an explicit correspondence between many quantum equations and specific relations arising from a description of spacetime having fractal

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properties. Note that the word 'fractal' is used in the present approach in its general meaning [6] of a set that shows structures at all scales and so becomes explicitly scale-dependent: in such a framework, resolutions are considered to be intrinsic characteristics of spacetime itself. Moreover, the absence of smooth trajectories in fractal spacetime requires that the notion of point-particle be modified. As has already been indicated in the work of Ord [5], the modification suggested [1, 8] links wave particle duality to fractal structure on trajectories in such spacetime.

Some features of non-relativistic quantum mechanics have already been recovered in such a framework, being described as a manifestation of the non-differentiability and 'fractality' of space [1, 5, 8]. Extending and correcting the results obtained in [9], we shall show in this paper that one can also obtain similar results in the relativistic case. Namely, the free particle Klein–Gordon equation can be recovered from the equations of free motion 'in a non-differentiable, fractal spacetime', interpreted as geodesic equations of such spacetime. Associated with it, we find the form of the relativistic invariant, characterizing quantum mechanics in our present interpretation. We also consider the electromagnetic case where we show that the corresponding equations of motion—leading to the Klein–Gordon equation with an electromagnetic four-potential—contain an extra term of current in addition to the usual term of Lorentz force. We conclude this paper with a discussion about scale-covariance, which could be implemented in this framework by the use of a '*scale-covariant derivative*' $\eth/\mathrm{d}s$. Thanks to it, we should be able to deduce equations which are scale-covariant and then, equations of quantum mechanics. We find that the prescription postulated in [1–4, 9], which consists of replacing the total derivative $\mathrm{d}/\mathrm{d}s$ in classical equations by $\eth/\mathrm{d}s$, has actually to be extended and cannot be used directly in all cases. Introducing a 'symmetric product' $f \circ \eth g$, we show that such an extension is possible and that we can write most of the equations of this framework in their usual classical form.

2. Schrödinger equation: a '*geodesic equation*' in fractal space

Let us first recall how, following [1, 2], the notion of geodesics in a non-differentiable space is introduced. The basic assumption of the scale-relativistic approach, consists of identifying the quantum properties of the particles and the concept of quantum particles itself, with the fractal properties of spacetime at very 'small scales'. As we shall see, general relativity is taken as a model at almost all levels of the theory. First, one uses the notions of spacetime geometry, geodesics and finally, the idea of general covariance. Second, one tries to construct corresponding mathematical tools to achieve these concepts, by analogy with the general relativistic tools, which are those of a Riemannian and differentiable geometry. Concerning the notion of geodesics, two main problems were encountered in this approach. (i) Fractal trajectories have an infinite length. The first one is then to define a geodesic, which must be a minimal curve. (ii) Moreover, geodesic equations are written in terms of differential equations. How can we write such equations, if the spacetime coordinates are not derivable? The description proposed in [1, 2], is supposed to solve both problems at the same time. As a first step, Nottale and Schneider [10] showed that we could define (in fact, for a particular sample) an intrinsic curvilinear coordinate on a fractal curve by using non-standard analysis (NSA). This allowed one to work with infinitesimal and infinite quantities and this coordinate can be naturally renormalized to a finite value: that is, the length of a fractal curve can be written as the produce of a finite and of an infinite part (see later). The finite part is equivalent to taking the D-measure of the curve (where D is its fractal dimension) instead of its standard length (of topological dimension $D_{\mathrm{T}} = 1$). Actually, even if the length of each curve among a set of fractal curves is infinite, the ratio of the length of two curves remains finite provided their fractal dimension is the same. Then, the lengths of different curves can be compared and a minimal curve can be defined, so that the concept of geodesics keeps its meaning.

The second step concerns the equivalence between this description and the explicit introducing resolutions in the description. Following [1, 2, 4], as a direct consequence of a Lebesgues theorem, which states that a curve of finite length is almost everywhere differentiable, we show that the length of an *almost nowhere differentiable* curve is a *function of resolutions*, $\mathcal{L} = \mathcal{L}(\varepsilon)$ and *diverges toward the small scales*, i.e. $\mathcal{L}(\varepsilon) \to \infty$ when $\varepsilon \to 0$.

In the case of a coordinate measured along a fractal curve with constant fractal dimension D, this length can be written as

$$\mathcal{L}(\tau, \varepsilon) = L(\tau)(\lambda/\varepsilon)^{\delta}$$

where $\delta = D - 1$. Here ε is a length-resolution and τ is a parameter along the curve. One recovers the non-standard description by taking an infinitesimal value for ε. In this case $\varepsilon^{-\delta}$ is infinite, while $L(\tau)\lambda^{\delta}$, which can be identified with the D-measure of the fractal coordinate, remains finite. The optimization process leading to the definition of geodesics could then be performed on $L(\tau)$.

More generally, one can introduce an upper transition scale λ toward scale independence by writing the length variation as [1, 2]

$$\mathcal{L}(\tau, \varepsilon) = L(\tau)[1 + (\lambda/\varepsilon)^{\delta}].$$

In general, the transition may itself be a function of $\tau, \lambda = \lambda(\tau)$. Such an expression is a solution of the simplest scale differential equation one can write for the variation of $\mathcal{L}$ with resolution, i.e. a first-order, renormalization group-like equation $\partial\mathcal{L}/\partial \ln\varepsilon = \beta(\mathcal{L}) = a + b\mathcal{L} + \cdots$. The transition scale λ then appears as a constant of integration.

This decomposition may also concern an elementary displacement on the fractal curve. Projecting on the three space axes, we write this as

$$\mathrm{d}X^i = \mathrm{d}x^i + \mathrm{d}\xi^i = v^i(t)\,\mathrm{d}t + \zeta^i\lambda^{1-1/D}|c\,\mathrm{d}t|^{1/D}$$

where $\mathrm{d}\xi^i$ describes the fractal behaviour, $\langle\zeta^i\rangle = 0$, $\langle\zeta^i\zeta^j\rangle = \delta^{ij}$. The length scale λ, as well as the velocity of light c, appears for dimensional reasons. Let ξ be a fractal function of t. Then, the expression of the variation $\delta\xi$ with respect to the resolution δt is given by

$$\left(\frac{\delta\xi}{\lambda}\right)^D = \frac{|\delta t|}{\lambda/c} \Rightarrow \delta\xi = \lambda^{1-1/D}|c\delta t|^{1/D}.$$

So, provided the identification $\delta t \equiv \mathrm{d}t$, we write for each coordinate axis

$$\mathrm{d}\xi^i = \zeta^i\lambda^{1-1/D}|c\,\mathrm{d}t|^{1/D}$$

such that

$$\langle\,\mathrm{d}\xi^i\rangle = 0 \text{ and } \langle\,\mathrm{d}\xi^i\,\mathrm{d}\xi^j\rangle = \delta^{ij}\lambda^{2-2/D}|c\,\mathrm{d}t|^{2/D}.$$

This leads to the second term of the right-hand side of the expression for $\mathrm{d}X^i$. As has been shown by Abbott and Wise [11], the special case when $D = 2$ corresponds precisely to the fractal dimension of typical quantum mechanical paths, described by Feynman and Hibbs as curves which are *continuous*, but *non-differentiable* (see, p 177, [12]). In the present approach, we assume that if paths show the properties to be '*highly irregular on a fine scale*' and '*non-differentiable*' (as we can read in [12]), it is because space(-time) itself has, at quantum scales these properties in an intrinsic manner.

We have, however, to be careful that the previous relation is not the direct projection of the expression for $\mathcal{L}(\tau, \varepsilon)$, after differentiation. Here x^i is a differentiable mean coordinate, which corresponds to the usual 'classical' coordinate, while the full variable X^i is non-differentiable, since $\mathrm{d}X^i/\mathrm{d}t \propto \mathrm{d}t^{1/D-1}$, which is divergent when $\mathrm{d}t \to 0$ for $D > 1$.

Now, the goal which we search is *not* to describe a given deterministic fractal trajectory, but instead to understand the trajectories as being the geodesics of a fractal space. Following [1, 2], we assume that the non-differentiability of space would imply that there will be an infinity of geodesics coming in to any point and another infinity coming out from it. Two sets of geodesics are introduced, assuming that the differential time reflection symmetry $\mathrm{d}t \to -\mathrm{d}t$ is broken by non-differentiability. Then we are led to replace the individual mean velocity $\mathfrak{v}^i$ by two velocity fields $\mathfrak{v}^i_+\{x(t), t\}$ and $\mathfrak{v}^i_-\{x(t), t\}$.

One of the important new features of scale relativity compared with similar approaches such as stochastic mechanics [13] consists of introducing a complex velocity field, which mixes the forward and the backward fields as [1]

$$\mathcal{V}^i \equiv \left(\frac{\mathfrak{v}^i_+ + \mathfrak{v}^i_-}{2}\right) - \mathrm{i}\left(\frac{\mathfrak{v}^i_+ - \mathfrak{v}^i_-}{2}\right) \equiv V^i - \mathrm{i}U^i.$$

This choice is motivated by the constraint to recover a real field V and a vanishing imaginary field in the classical limit $\mathfrak{v}_+ = \mathfrak{v}_-$.

The last step is to construct a '*scale-covariant derivative*', which describes the effects on physical quantities of the new geometric structure of space. The variation of a field $f\{x(t), t\}$ during a time interval $\mathrm{d}t$ is given by $\eth f$ [1, 2], where

$$\frac{\eth}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathcal{V}\cdot\nabla - \mathrm{i}\frac{c\lambda}{2}\Delta$$

in the special case when the fractal dimension is $D = 2$, which indeed plays a critical role in such an approach [2, 4, 5, 11]. We are now able to write the equations of free motion 'in fractal space'. One starts from the equations of geodesics in Euclidean space, i.e. the equations of inertial motion, $\mathrm{d}\boldsymbol{V}/\mathrm{d}t = 0$ and one uses $\eth/\mathrm{d}t$ as a '*scale-covariant derivative*'. One obtains the free equations [1]

$$\frac{\eth\mathcal{V}}{\mathrm{d}t} = 0 \Leftrightarrow \frac{\partial\mathcal{V}}{\partial t} + \mathcal{V}\cdot\nabla\mathcal{V} - \mathrm{i}\frac{c\lambda}{2}\Delta\mathcal{V} = 0.$$

We may also use the Euler–Lagrange equations. Given a complex Lagrange function in its usual form: $\mathbb{L}(x, \mathcal{V}) = (1/2)m\mathcal{V}^2$, we write the Euler–Lagrange equations as in classical mechanics. We then obtain precisely the previous form, $\eth\mathcal{V}/\mathrm{d}t = 0$. The same can be said in both cases, in the presence of a potential $\phi(x)$. Our equations of motion then take the usual form of the Newton law: $m\eth\mathcal{V}/\mathrm{d}t = -\nabla\phi$ and the corresponding Lagrange function becomes $\mathbb{L}(x, \mathcal{V}) = (1/2)m\mathcal{V}^2 - \phi$.

This equation can finally be transformed by introducing the wavefunction as another expression for the action, following the familiar ansatz

$$\psi \equiv \mathrm{e}^{\mathrm{i}\mathcal{S}/mc\lambda}.$$

Therefore, the complex velocity field is related to the wavefunction ψ by the relation $\mathcal{V} = -\mathrm{i}c\lambda\nabla(\ln\psi)$. The replacement of $\mathcal{V}$ in the equations of motion with potential by this expression yields an equation which, once integrated, is the Schrödinger equation [1, 2]

$$\frac{(c\lambda)^2}{2}\Delta\psi + \mathrm{i}c\lambda\frac{\partial\psi}{\partial t} - \frac{\phi}{m}\psi = 0$$

with $mc\lambda = \hbar$.

The Born interpretation of quantum mechanics would also be consistent with such an approach. A probability density is defined from the fluid of geodesics, while the imaginary part of the geodesic-Schrödinger equation writes $\partial\rho/\partial t + \nabla\cdot(\rho\boldsymbol{V}) = 0$, where $\rho = \psi\psi^\dagger$ and where $\boldsymbol{V}$ is the real part of $\mathcal{V}$. This is the continuity equation (which is now in this approach a part of the equations of the dynamics, more precisely the imaginary part of the Hamilton–Jacobi equation given in appendix A), so that $\psi\psi^\dagger$ is easily interpreted as that probability density. This has recently been evidenced by numerical simulations due to Hermann [14], who obtained solutions to the Schrödinger equation without using it, by constructing trajectories directly from these equations of elementary displacements and by Ord and Deakin [15] who obtained a similar result using a binary symmetric random walk.

Finally, let us mention the following important point, which we shall detail in appendix A. If the Schrödinger equation without external potential ($\phi = 0$) is written in this approach as free equations $\mathfrak{d}\mathcal{V}/\mathrm{d}t = 0$, we can show [20–22, 26] that the complex quantity which corresponds to the 'energy' in the free case, *is not simply quadratic in velocities* $\mathcal{V}$. Indeed, its new form is

$$\mathcal{E}_{\mathrm{free}} \equiv \mathcal{E} - \phi = \frac{1}{2}m\mathcal{V}^2 - \mathrm{i}m\frac{c\lambda}{2}\nabla\cdot\mathcal{V}.$$

One can show that the divergence term is present intrinsically in this expression and *has not to be seen as an external potential*, which would be added to a basic quadratic form corresponding to the kinetic energy in this framework.

3. Klein–Gordon equation and 'quadratic' relativistic invariant

Most elements of the approach summarized previously, as described in [1, 2, 4], would remain correct in the motion-relativistic case, with the time differential element $\mathrm{d}t$ replaced by the proper time differential, $\mathrm{d}s$. Now not only space, but the full spacetime continuum, is considered to be non-differentiable, then fractal. Consider a small increment $\mathrm{d}X^\mu$ of non-differentiable four-coordinates along one of the geodesics of the fractal spacetime. We assume that, because of non-differentiability, there will be an infinity of fractal geodesics between any couple of events. This suggests jumping to a statistical description. As in the non-relativistic case, we can decompose $\mathrm{d}X^\mu$ in terms of a mean, $\langle \mathrm{d}X^\mu\rangle = \mathrm{d}x^\mu = \mathfrak{v}^\mu\,\mathrm{d}s$ and a fluctuation respective to the mean, $\mathrm{d}\xi^\mu$ (such that $\langle \mathrm{d}\xi^\mu\rangle = 0$ by definition).

As for the non-relativistic case, the supposed non-differentiable nature of spacetime would imply a more fundamental consequence, namely, the breaking reflection invariance at the infinitesimal level. If one reverses the sign of the proper time differential element, the mean velocity $\mathfrak{v}_+$ becomes $\mathfrak{v}_-$ and there is no reason for these two velocities to be equal, in contrast to what happens in the classical, differentiable case. However, both choices of a reversed proper time element ($-\mathrm{d}s$) and the initial choice ($\mathrm{d}s$), have to be considered as equivalent for the dynamic stochastic description (see the discussion in [19]). Therefore, we assume that we have to consider both the forward ($\mathrm{d}s > 0$) and backward ($\mathrm{d}s < 0$) processes on the same footing. Then the information needed to describe the system is doubled with respect to the classical, differentiable description. This fundamental two-valuedness can be accounted for by using complex numbers. The complex process given below by equation (4), takes into account the equivalence of these two processes, as a whole, and recovers the fundamental property of microscopic reversibility.

One is then led to write the elementary displacement along a geodesic of fractal dimension two, respectively, for the forward (+) and backward (−) processes, in the form

$$\mathrm{d}X^\mu_\pm = \mathrm{d}_\pm x^\mu + \mathrm{d}\xi^\mu_\pm = \mathfrak{v}^\mu_\pm\,\mathrm{d}s + \lambda^{1/2}\zeta^\mu_\pm\,\mathrm{d}s^{1/2} \tag{1}$$

with $\mathrm{d}_\pm x^\mu = \mathfrak{v}^\mu_\pm\,\mathrm{d}s$ and $\mathrm{d}\xi^\mu_\pm = \lambda^{1/2}\zeta^\mu_\pm\,\mathrm{d}s^{1/2}$. In these expressions, $\zeta^\mu_\pm$ is a dimensionless fluctuation and the length-scale λ must be introduced for dimensional argument. One defines, following Nelson [13], mean forward and backward derivatives, $\mathrm{d}_+/\mathrm{d}s$ and $\mathrm{d}_-/\mathrm{d}s$:

$$\frac{\mathrm{d}_\pm}{\mathrm{d}s}y(s) \equiv \lim_{\delta s\to 0\pm}\left\langle\frac{y(s+\delta s) - y(s)}{\delta s}\right\rangle. \tag{2}$$

In stochastic mechanics, the expectation is taken on a previously defined probability density. Here, though it has yet to be rigorously defined, the expectation would concern the infinite set of geodesics. Once applied to x^μ, they yield the *forward and backward mean four-velocities*

$$\frac{\mathrm{d}_+}{\mathrm{d}s}x^\mu(s) = \mathfrak{v}_+^\mu \qquad \frac{\mathrm{d}_-}{\mathrm{d}s}x^\mu(s) = \mathfrak{v}_-^\mu. \tag{3}$$

As in the non-relativistic case [1], the forward and backward derivatives (3) can be combined in terms of a complex derivative operator [9]

$$\frac{\eth}{\mathrm{d}s} \equiv \frac{(\mathrm{d}_+ + \mathrm{d}_-) - \mathrm{i}(\mathrm{d}_+ - \mathrm{d}_-)}{2\,\mathrm{d}s}. \tag{4}$$

When applied to the position vector, it yields a complex four-velocity

$$\mathcal{V}^\mu \equiv \frac{\eth x^\mu}{\mathrm{d}s} = \frac{\mathfrak{v}_+^\mu + \mathfrak{v}_-^\mu}{2} - \mathrm{i}\frac{\mathfrak{v}_+^\mu - \mathfrak{v}_-^\mu}{2} = V^\mu - \mathrm{i}U^\mu. \tag{5}$$

Let us now jump to the stochastic interpretation of the theory. This leads us to consider the question of the definition of a Lorentz-covariant diffusion in spacetime. Forward and backward fluctuations, $\mathrm{d}\xi_\pm^\mu(s)$, are defined, which are Gaussian with mean zero, mutually independent and such that

$$\langle\,\mathrm{d}\xi_\pm^\mu\,\mathrm{d}\xi_\pm^\nu\rangle = \mp\lambda\eta^{\mu\nu}\,\mathrm{d}s. \tag{6}$$

In this paper, we choose a $(+,-,-,-)$ signature. It is not the purpose of this article to examine the difficulties which appear when we consider stochastic processes in the relativistic case, but as has been pointed out by Hakim [16], we cannot naively transpose to the relativistic case the usual classical definitions of stochastic processes. Several authors have considered the extension of Nelson stochastic mechanics to the relativistic case and have introduced some specific notions to achieve as fairly as possible this extension. We may mention [17], where Dohrn and Guerra introduce two metrics; a 'Brownian metric' $\eta^{\mu\nu}$, which is positive definite and which is present in the expectation $\langle\,\mathrm{d}W^\mu\,\mathrm{d}W^\nu\rangle \propto \eta^{\mu\nu}\,\mathrm{d}\tau$ of the stochastic process $\mathrm{d}W^\mu$; and a 'kinetic metric' $g_{\mu\nu}$, which is used to write the kinetic term $(1/2)g_{\mu\nu}\mathfrak{v}_+^\mu\mathfrak{v}_-^\nu$ in a stochastic Lagrangian. The compatibility condition between these two metrics reads $g_{\mu\nu}\eta^{\mu\rho}\eta^{\nu\lambda} = g^{\rho\lambda}$. We also refer the reader to the work of Zastawniak [18] and finally to Serva [19], who gives up Markov processes and considers a covariant process that belongs to a larger class, known as 'Bernstein processes'. We do not claim here to end the discussion about relativistic stochastic processes, but we may keep in mind that the role played by the Laplacian operator in the non-relativistic case, is now played by the Dalembertian operator in the relativistic equations. So, we can show that the two forward and backward differentials of a function $f(x,s)$ are written, assuming a Minkowskian metric $(+,-,-,-)$ for classical spacetime:

$$\frac{\mathrm{d}_\pm f}{\mathrm{d}s} = \left(\frac{\partial}{\partial s} + \mathfrak{v}_\pm^\mu\partial_\mu \mp \frac{\lambda}{2}\partial^\mu\partial_\mu\right) f. \tag{7}$$

In the following, we only consider s-stationary functions, i.e. functions that do not explicitly depend on the proper time s. In this case the time derivative complex operator finally reduces to [9]

$$\frac{\eth}{\mathrm{d}s} = \mathcal{V}^\mu\partial_\mu + \mathrm{i}\frac{\lambda}{2}\partial^\mu\partial_\mu. \tag{8}$$

Note the correction of sign with respect to [9]. The + sign of the Dalembertian comes from the choice of a metric signature $(+,-,-,-)$ for the classical spacetime.

Following [4, 9], the operator (8) would play the role, in the relativistic case of motion, of a '*scale-covariant derivative*'. Therefore, we assume that the equations of motion of a free relativistic quantum particle, may be written as

$$\frac{\eth\mathcal{V}^\alpha}{\mathrm{d}s} = 0 \tag{9}$$

which are interpreted as the equations of free motion 'in fractal spacetime', or as geodesic equations. Replacing now the derivative operator in (9) by its expression (8), then using the expression for the complex four-velocity related to the wavefunction by $\mathcal{V}^\nu = \mathrm{i}\lambda\partial^\nu \ln\psi$, one obtains from (9) the free Klein–Gordon equation

$$\partial^\nu \left\{ \lambda^2 \frac{\partial^\mu \partial_\mu \psi}{\psi} \right\} = 0. \tag{10}$$

The demonstration proceeds by defining a complex action $\mathcal{S}$, as the non-relativistic case:

$$\psi = \mathrm{e}^{\mathrm{i}\mathcal{S}/mc\lambda}. \tag{11}$$

Then, we link $\mathcal{S}$ to the complex four-velocity $\mathcal{V}_\mu$ as in classical mechanics by

$$-\partial_\mu \mathcal{S} = mc\mathcal{V}_\mu. \tag{12}$$

Using the following derivation formulae

$$\frac{\partial^\mu \psi}{\psi}\partial_\mu\left[\frac{\partial^\nu\psi}{\psi}\right] = \frac{\partial^\mu\psi}{\psi}\partial^\nu\left[\frac{\partial_\mu\psi}{\psi}\right] = \frac{1}{2}\partial^\nu\left[\frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi}\right]$$
$$\partial_\mu\left[\frac{\partial^\mu\psi}{\psi}\right] + \frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi} = \frac{\partial^\mu\partial_\mu\psi}{\psi} \tag{13}$$

we obtain equation (10). So, the Klein–Gordon equation

$$-\hbar^2\partial^\mu\partial_\mu\psi = m^2c^2\psi \tag{14}$$

becomes an 'integral of motion' of the free particle, once λ is identified with the Compton length of the particle

$$\lambda = \hbar/mc. \tag{15}$$

The quantum behaviour described by this equation and the probabilistic interpretation given to ψ would be reduced here to the description of a 'free fall' in fractal spacetime, in analogy with Einstein general relativity where a particle subjected to the effects of gravitation is described as being in free fall in a curved spacetime. However, we shall find some important differences with general relativity, in particular in the counterpart of the usual quadratic invariant, as well as in the electromagnetic case.

Without electromagnetic field, we can see the equations of motion (9) as 'scale-covariant', since the relativistic quantum equation written in terms of the complex derivative operator $\mathfrak{d}/\mathrm{d}s$ has the *same form* as the equations of a relativistic macroscopic free particle, written with the usual derivative $\mathrm{d}/\mathrm{d}s$.

We now arrive at the main new result of this paper. Contrary to the hope expressed in [9], the quadratic relativistic invariant of special and general relativity $V^\mu V_\mu = 1$ is not conserved in the present description of quantum mechanics. Indeed, we can show [20–22, 26] that the relativistic *invariant* associated with the free equations of motion (9) written with the operator (8) now takes the form

$$\mathcal{V}^\mu\mathcal{V}_\mu + \mathrm{i}\lambda\partial^\mu\mathcal{V}_\mu = 1. \tag{16}$$

As the usual quadratic invariant $g^{\mu\nu}V_\mu V_\nu = 1$ is directly related to the metric itself $\mathrm{d}s^2 = g^{\mu\nu}\,\mathrm{d}x_\mu\,\mathrm{d}x_\nu$, the expression (16), which is *not purely quadratic* in terms of velocities, could *a priori* define a 'metric form' which characterizes quantum mechanics in the present picture. Following the aims of this approach, we would say that its new form reflects the internal structures of the spacetime at quantum scales. However, we recall that the principle of scale-covariance, which *a priori* requires that equations and relevant quantities of physics keep their simplest form at all scales, has been stated as a postulate in the introduction. So, we cannot justify the change of the form of some equations by the geometry of space, after having postulated that they have to keep their usual form, when written in scale relativity theory. From our invariant (16), we can also write the relativistic Hamilton–Jacobi equation in this framework. In the absence of an electromagnetic field A^μ, $\mathcal{V}^\mu$ and $\mathcal{S}$ are related by equation (12), so that (16) becomes

$$\partial^\mu \mathcal{S} \partial_\mu \mathcal{S} - \mathrm{i} mc\lambda \partial^\mu \partial_\mu \mathcal{S} = m^2 c^2 \tag{17}$$

which is the form taken by the Hamilton–Jacobi equation in the present picture. Using this parametrization, it directly yields the Klein–Gordon equation.

4. Klein–Gordon equation with electromagnetic field

Let us now consider a spinless particle in an electromagnetic field. In the presence of an electromagnetic field, (16) still applies, but with $\mathcal{V}^\mu$ now given by

$$mc\mathcal{V}^\mu = -\left(\partial^\mu \mathcal{S} + \frac{e}{c} A^\mu\right) \tag{18}$$

so that we obtain the Hamilton–Jacobi equation [20–22, 26]

$$\left(\partial^\mu \mathcal{S} + \frac{e}{c} A^\mu\right)\left(\partial_\mu \mathcal{S} + \frac{e}{c} A_\mu\right) - \mathrm{i} mc\lambda \partial^\mu \left(\partial_\mu \mathcal{S} + \frac{e}{c} A_\mu\right) = m^2 c^2 \tag{19}$$

which is the electromagnetic counterpart of (17). In a consistent way, equation (19) is equivalent to

$$\left(\mathrm{i} mc\lambda \partial^\mu - \frac{e}{c} A^\mu\right)\left(\mathrm{i} mc\lambda \partial_\mu \psi - \frac{e}{c} A_\mu \psi\right) = m^2 c^2 \psi \tag{20}$$

in the electromagnetic case. We recognize here the free and electromagnetic Klein–Gordon equations, once the identification $\lambda = \hbar/mc$ is made.

Let us now try to write the corresponding complex equations of motion written with the operator (8). Taking the partial derivative of (16), we find

$$\left(\mathcal{V}^\mu + \mathrm{i}\frac{\lambda}{2} \partial^\mu\right) \partial_\alpha \mathcal{V}_\mu = 0. \tag{21}$$

Without electromagnetic field, due to the fact that $\mathcal{V}^\mu$ is a gradient, we have $\partial_\alpha \mathcal{V}_\mu = \partial_\mu \mathcal{V}_\alpha$. Therefore, we recover from (21) the free equations written at the beginning of this section

$$\frac{\eth \mathcal{V}_\alpha}{\mathrm{d}s} = 0. \tag{22}$$

In the presence of an electromagnetic field, we can derive from the expression for the complex four-momentum the following relation

$$mc(\partial_\alpha \mathcal{V}_\mu - \partial_\mu \mathcal{V}_\alpha) = -\frac{e}{c} F_{\alpha\mu}. \tag{23}$$

Substituting $\partial_\alpha \mathcal{V}_\mu$ in (21), we find the complex equations of motion [20–22, 26]

$$mc \frac{\eth \mathcal{V}_\alpha}{\mathrm{d}s} = \frac{e}{c} \mathcal{V}^\mu F_{\alpha\mu} + \mathrm{i}\frac{\lambda}{2}\frac{e}{c} \partial^\mu F_{\alpha\mu}. \tag{24}$$

We recognize here a complex equation that generalizes the fundamental relation of classical relativistic electrodynamics. Its real part is identical to the equation proposed by Serva [19] and Zastawniak [18] (see also Ord [23]), since the real part of our complex acceleration is Nelson stochastic acceleration [1, 2]. However, its imaginary part contains an additional imaginary term of current [20–22], which was *a priori* not expected precisely because of scale-covariance postulated in [1–4, 9]. We have to mention the work of de la Pena-Auerbach and Cetto [27], where the non-relativistic counterpart of our current term has been found in the framework of the stochastic approach independently from our work. Because of the presence of this extra term, these equations of motion (24) are not deduced from their classical—i.e. macroscopic—counterparts: $mc(\mathrm{d}V_\alpha/\mathrm{d}s) = (e/c)(\mathrm{d}x^\mu/\mathrm{d}s)F_{\alpha\mu}$, by applying directly the substitution $\mathrm{d}/\mathrm{d}s \to \eth/\mathrm{d}s$ within these equations, as it is required in [1–4, 9]. In this sense, their *form has changed* and a strict scale-covariance is lost. We shall show in the next section that we can recover the usual expected form by introducing a new tool.

5. Leibniz rule, symmetric product and 'scale-covariance'

We will now show that it is possible to introduce an additional tool, which allows us to write the equations of motion with electromagnetic field in their usual classical form and go further in implementation of scale-covariance. This tool is in fact more than a simple notation. It may be seen as arising from the requirement of a fundamental property of the derivative, namely the Leibniz rule. Indeed both complex derivative operators, the non-relativistic and the relativistic one (8), contain a partial derivative of second order in the presence of the Laplacian and Dalembertian operator, respectively. The presence of these operators does not allow us to write the rules of differentiation in their usual form [21, 22, 26]. Let us consider here only the relativistic case. If we consider two test functions $f(x^\alpha)$ and $g(x^\alpha)$—such that their second-order derivatives exist—and a composite function $f\{g(x^\alpha)\}$, the use of (8), applied to fg and $f\{g\}$, yields the two 'rules' of differentiation

$$\frac{\eth(fg)}{\mathrm{d}s} = f\frac{\eth g}{\mathrm{d}s} + g\frac{\eth f}{\mathrm{d}s} + \mathrm{i}\lambda(\partial^\mu f)(\partial_\mu g) \tag{25}$$

and

$$\frac{\eth f\{g(x^\alpha)\}}{\mathrm{d}s} = f'\{g\}\frac{\eth g}{\mathrm{d}s} + \mathrm{i}\frac{\lambda}{2}(\partial^\mu f'\{g\})(\partial_\mu g). \tag{26}$$

One can actually show that formula (25) can be related to the canonical commutation relations of quantum mechanics [21]. So, we will introduce [21, 22, 26] a specific notation which allows us to write the formula (25) with the usual form of the Leibniz rule. A possible notation is

$$f \circ \eth g = f\eth g + \mathrm{i}\frac{\lambda}{2}\,\mathrm{d}s(\partial^\mu f)(\partial_\mu g). \tag{27}$$

The previous expression is inspired from the 'symmetric product' introduced in the formalism of the stochastic integral, in particular to re-express the Fisk–Stratonovich integral (see Ikeda and Watanabe [24]). Thanks to (27), relations (25) and (26) now take their usual form

$$\frac{\eth(fg)}{\mathrm{d}s} = f\circ\frac{\eth g}{\mathrm{d}s} + g\circ\frac{\eth f}{\mathrm{d}s} \tag{28}$$

and

$$\frac{\eth f\{g(x^\alpha)\}}{\mathrm{d}s} = f'\{g\}\circ\frac{\eth g}{\mathrm{d}s}. \tag{29}$$

Moreover, we are now able to deduce the right-hand side of our complex equations of motion with field (24), by writing

$$\begin{aligned} F_{\alpha\mu}\circ\frac{\eth x^\mu}{\mathrm{d}s} &= F_{\alpha\mu}\frac{\eth x^\mu}{\mathrm{d}s} + \mathrm{i}\frac{\lambda}{2}(\partial_\rho F_{\alpha\mu})(\partial^\rho x^\mu) = F_{\alpha\mu}\frac{\eth x^\mu}{\mathrm{d}s} + \mathrm{i}\frac{\lambda}{2}(\partial_\rho F_{\alpha\mu})\eta^{\rho\mu} \\ &= F_{\alpha\mu}\frac{\eth x^\mu}{\mathrm{d}s} + \mathrm{i}\frac{\lambda}{2}\partial^\mu F_{\alpha\mu}. \end{aligned} \tag{30}$$

So, equations (24) now take the '*scale-covariant*' form [21, 22, 26]

$$mc\frac{\eth \mathcal{V}_\alpha}{\mathrm{d}s} = \frac{e}{c}F_{\alpha\mu}\circ\frac{\eth x^\mu}{\mathrm{d}s}. \tag{31}$$

Let us now examine another important case. In classical relativistic mechanics, the action S satisfies

$$S = -mc \int \mathrm{d}s$$

which yields

$$\mathrm{d}S = -mc\,\mathrm{d}s \Leftrightarrow \frac{\mathrm{d}S}{\mathrm{d}s} = -mc. \tag{32}$$

However, we shall show that the counterpart of equation (32) written in the present picture takes a more complicated form. Indeed, because of relation (16), we have

$$\frac{\eth\mathcal{S}}{\mathrm{d}s} \neq -mc. \tag{33}$$

Using expressions (8) and (12), we get instead

$$\frac{\eth\mathcal{S}}{\mathrm{d}s} = \mathcal{V}^\mu \partial_\mu \mathcal{S} + \mathrm{i}\frac{\lambda}{2}\partial^\mu\partial_\mu\mathcal{S} = -mc\left(\mathcal{V}^\mu\mathcal{V}_\mu + \mathrm{i}\frac{\lambda}{2}\partial^\mu\mathcal{V}_\mu\right) \tag{34}$$

which becomes, due to (16)

$$\frac{\eth\mathcal{S}}{\mathrm{d}s} = -mc\left(\frac{1+\mathcal{V}^\mu\mathcal{V}_\mu}{2}\right). \tag{35}$$

Actually, this expression may be deduced from the symmetric product itself (27). Indeed, remarking that we can write the fundamental interval as $\mathrm{d}s = V^\mu\,\mathrm{d}x_\mu$, let us formally write the differential of the complex action as

$$\eth\mathcal{S} = -mc\mathcal{V}^\mu \circ \eth x_\mu. \tag{36}$$

Expanding expression (27), we get

$$\begin{aligned} -\frac{1}{mc}\eth\mathcal{S} &= \mathcal{V}^\mu\eth x_\mu + \mathrm{i}\frac{\lambda}{2}\,\mathrm{d}s(\partial^\rho\mathcal{V}^\mu)(\partial_\rho x_\mu) = \mathcal{V}^\mu\eth x_\mu + \mathrm{i}\frac{\lambda}{2}\,\mathrm{d}s(\partial^\rho\mathcal{V}^\mu)\eta_{\rho\mu} \\ &= \mathcal{V}^\mu\eth x_\mu + \mathrm{i}\frac{\lambda}{2}\,\mathrm{d}s\,\partial^\mu\mathcal{V}_\mu \end{aligned} \tag{37}$$

which may be rewritten as

$$-\frac{1}{mc}\frac{\eth\mathcal{S}}{\mathrm{d}s} = \mathcal{V}^\mu\mathcal{V}_\mu + \mathrm{i}\frac{\lambda}{2}\partial^\mu\mathcal{V}_\mu \tag{38}$$

which is equation (34). Now, let us recall that, in relativistic mechanics, the differential of the action $\mathrm{d}S$ is proportional to the interval $\mathrm{d}S$. Starting from the real part of equation (35), one can show [26] that we have to consider the conformal transformation

$$\mathrm{d}\sigma = (1+Q)\,\mathrm{d}s. \tag{39}$$

The quantity Q denotes the relativistic version of the 'quantum potential' defined as

$$Q \equiv \lambda^2\frac{\partial^\mu\partial_\mu\rho^{1/2}}{\rho^{1/2}} \tag{40}$$

which contains the density probability $\rho = \psi\psi^*$.

Finally, let us consider the case of our relativistic invariant (16). As we have seen, the usual quadratic form $V^\mu V_\mu = 1$ is lost. The extra divergence term $\mathrm{i}\lambda\partial^\mu\mathcal{V}_\mu$ that we found is present in an intrinsic manner in its expression. We shall show that we can deduce this term from a property of the symmetric product. Let us set $\dot{f} \equiv \eth f/\mathrm{d}s$ and $\dot{g} \equiv \eth g/\mathrm{d}s$. Because $\partial^\mu(\dot{f})\partial_\mu g \neq \partial^\mu(\dot{g})\partial_\mu f$, we have

$$\dot{f}\circ\dot{g} \neq \dot{g}\circ\dot{f}.$$

Consider now the two products

$$\dot{f} \circ_+ \dot{g} \equiv \dot{f}\dot{g} + \mathrm{i}\frac{\lambda}{2}\{\partial^\mu(\dot{f})\partial_\mu g + \partial^\mu(\dot{g})\partial_\mu f\} \tag{41a}$$

and

$$\dot{f} \circ_- \dot{g} \equiv \dot{f}\dot{g} + \mathrm{i}\frac{\lambda}{2}\{\partial^\mu(\dot{f})\partial_\mu g - \partial^\mu(\dot{g})\partial_\mu f\}. \tag{41b}$$

They satisfy, respectively, the commutation and anti-commutation relations

$$\dot{f} \circ_+ \dot{g} + \dot{f}\dot{g} = \dot{g} \circ_+ \dot{f} + \dot{g}\dot{f}$$

and

$$\dot{f} \circ_- \dot{g} - \dot{f}\dot{g} = -(\dot{g} \circ_- \dot{f} - \dot{g}\dot{f}). \tag{42}$$

Now, the symmetric product (27) (concerning $\dot{f} \equiv \eth f/\mathrm{d}s$ and $\dot{g} \equiv \eth g/\mathrm{d}s$) may be rewritten using these commutative and anti-commutative products (41) as

$$\dot{f} \circ \dot{g} \equiv \frac{1}{2}\{\dot{f} \circ_+ \dot{g} + \dot{f} \circ_- \dot{g}\}. \tag{43}$$

For $f = g \equiv x^u$, we have $\dot{f} = \dot{g} \equiv \mathcal{V}^\mu$ and formulae (41) give

$$\mathcal{V}^\mu \circ_- \mathcal{V}_\mu = \mathcal{V}^\mu\mathcal{V}_\mu \tag{44a}$$

and

$$\mathcal{V}^\mu \circ_+ \mathcal{V}_\mu = \mathcal{V}^\mu\mathcal{V}_\mu + \mathrm{i}\lambda\partial^\mu\mathcal{V}_\mu. \tag{44b}$$

We recognize in this last relation (44*b*) our expected relativistic invariant (16). So, in order to deduce *a priori* scale-covariant laws from classical equations of physics, it has been proposed in [1–4, 9] to follow, as a postulate, the prescription where total derivatives $\mathrm{d}/\mathrm{d}s$ would be replaced by the '*scale-covariant derivative*' $\eth/\mathrm{d}s$. However, we have seen that it was not possible to apply directly this prescription in all cases, in particular for the simple Leibniz rule written with (8) (equation (25)) and for the electromagnetic equations of motion (24). Nevertheless, at least from formulae (28) to (44) [21, 22, 26], we can see that it is in fact possible to extend this prescription by replacing the products $f\,\mathrm{d}g$ by our symmetric product (27) $f \circ \eth g$. Of course, the same can be said for the non-relativistic case.

Thus, using other formal generalizations in addition to the symmetric product, arising for instance from stochastic calculus, we may expect to obtain the formal tools, which really implement the scale-covariance, as tensorial calculus does for the covariance of motion.

6. Conclusion

In summary, the Klein–Gordon equation has been obtained as a re-expression of free equations of motion in spacetime, which would be characterized by non-differentiability and continuity and then called 'fractal spacetime'. Such a concept involves the introduction of explicitly resolution-dependent fractal coordinates. Moreover, as a consequence of this geometric description and of the extension to scale transformations of all notions used in relativity, two kinds of constraints would be introduced.

(i) The first constraint concerns the resolution transformations, which would be written in terms of 'Galilean', then 'Lorentzian transformations' (see [1, 7]).

(ii) The second constraint concerns the *form* of the equations of physics. The 'geometric' description which is proposed here, leads to the introduction of specific tools, which would enable us to write equations of physics in a scale-covariant form: the first tool is the derivative operator (8) that allows us to write the geodesic equations in the form of a free particle equations, $\eth\mathcal{V}^\alpha/\mathrm{d}s = 0$; the second is the symmetric product (27), which has been introduced to recover the Leibniz rule written with (8) and which allows us to write most equations of this framework with their usual classical form (for instance, equations (28), (29), (31), (36) and (44*b*)).

Before concluding this paper, we stress the fact that the scale relativity theory, even though it shares common features with Nelson stochastic mechanics differs from it in the following points. In contrast to Nelson, one obtains the Schrödinger equation (in the non-relativistic case [1]) and the Klein–Gordon equation in the present work, without using Kolmogorov or Fokker–Planck diffusion equations. This is a crucial point, since it is now known that the predictions of stochastic mechanics disagree with that of standard quantum mechanics in the case of multitime correlations [25] and that the disagreement precisely comes in great part from the diffusion, Brownian motion interpretation of the theory, via the Fokker–Planck equations.

Let us now underline two important points. First, by analogy with general covariance in general relativity, it has been *postulated* in all Nottale papers from [1] that the passage from 'classical' mechanics (the differentiable case) to a '*new non-differentiable mechanics*'—which would lead to quantum mechanics—could be '*implemented by a unique prescription*' (see, [1, 2, 4, 9]), which consists in replacing the standard total derivative $\mathrm{d}/\mathrm{d}t$ (and $\mathrm{d}/\mathrm{d}s$) by the new complex operator $\eth/\mathrm{d}t$ (and $\eth/\mathrm{d}s$). In fact, it has been shown by the author [20–22, 26], that this prescription is actually too simple and leads, if we apply it strictly in all quantities and equations of physics, to equations which are inconsistent, and which do not lead—except for the equations of motion without electromagnetic field—to their corresponding quantum counterparts (see, for instance, equation (24)). Moreover, as we see in equation (25), the operator $\eth/\mathrm{d}s$ *does not fulfil the Leibniz rule*, which defines the derivation. Therefore, we can say that the '*scale-covariant derivative*' is not a derivation.

Second, the usual quadratic form $V^\mu V_\mu = 1$ corresponding to the relativistic geodesic equation is lost in an intrinsic manner and is now replaced by the new invariant $\mathcal{V}^\mu\mathcal{V}_\mu + \mathrm{i}\lambda\partial^\mu\mathcal{V}_\mu = 1$. Recalling that $\lambda = \hbar/mc$ is the Compton length of the particle and following the expression of the complex action (11) and the velocity (12), the two terms on the left-hand side of (16) contain terms which are of the same order in $\hbar$. Therefore, we cannot neglect the extra divergence term without destroying the whole structure of our invariant.

A more detailed treatment of these results, with particular emphasis on the new invariant (16) corresponding to a new energy formula in the non-relativistic case, is given in [21, 22, 26]. We also find some explicit links between the operator $\eth/\mathrm{d}t$ and quantum mechanical relations as canonical commutation relations $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$. Moreover, we deduce from (16) a possible form of the Dirac equation in this framework and give a Riemannian version of equations (16) and (17). The intrinsic projective properties of expression (16), which are mentioned in appendix B, are studied in [26].

Appendix A. The energy 'at quantum scales', in the non-relativistic case

In this appendix, we will emphasize that the quantity which shall play the role of 'energy' in the present approach has not the expected quadratic form. In order to do this, let us recall some basic results of classical and analytic mechanics. In the non-relativistic case, the equations of motion for a free particle are given by the inertial equations

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = 0.$$

The corresponding energy is then given by the *quadratic* expression $E_{\text{free}} = (1/2)mV^2$, which corresponds to the kinetic energy $T \equiv E - \phi$ of the particle. In the presence of an external potential $\phi(x)$, the equations of motion become $m\,\mathrm{d}\boldsymbol{V}/\mathrm{d}t = -\nabla\phi$ and the total energy is written

$$E = \frac{1}{2}mV^2 + \phi.$$

This quantity satisfies (i) the conservation equation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

and (ii) the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} = -H$$

where $H \equiv P^2/(2m) + \phi$ and $P \equiv mV$. So, what is the complex quantity which satisfies these two equations written in the framework of scale-relativity? Actually, we can show [20–22, 26] that the generic form of the Hamilton function $H = P \cdot V - L$, becomes an irrelevant definition in this theory. Indeed, if we look after the Hamilton function $\mathcal{H}$, which satisfies the complex Hamilton–Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} = -\mathcal{H} \tag{A.1}$$

which corresponds to the equations of motion $\mathfrak{d}\mathcal{P}/\mathrm{d}t = -\nabla\phi$ after differentiation, we find

$$\mathcal{H} = \frac{\mathcal{P}^2}{2m} - \mathrm{i}\frac{c\lambda}{2}\nabla \cdot \mathcal{P} + \phi. \tag{A.2}$$

Then, equation (A.1) yields the Schrödinger equation in the presence of the external potential ϕ. If we now look after the complex quantity $\mathcal{E}$, which satisfies the ‘conservation equation’

$$\frac{\mathfrak{d}\mathcal{E}}{\mathrm{d}t} = 0 \tag{A.3}$$

we find

$$\mathcal{E} = \frac{1}{2}m\mathcal{V}^2 - \mathrm{i}\frac{mc\lambda}{2}\nabla \cdot \mathcal{V} + \phi. \tag{A.4}$$

In a consistent way, we have $\mathcal{H} = \mathcal{E}$, provided $\mathcal{P} = m\mathcal{V}$. For the non-relativistic case as well as for the relativistic case, the important point is as follows. In classical (i.e. non-quantum) mechanics, we can write for each case a quadratic invariant which corresponds to the free equations of motion. Indeed, for the Galilean and special relativistic cases, we have the equivalences

$$\frac{\mathrm{d}V^i}{\mathrm{d}t} = 0 \Leftrightarrow V^2 = \mathrm{C}^{\text{st}} \qquad \frac{\mathrm{d}V_\alpha}{\mathrm{d}s} = 0 \Leftrightarrow V^\mu V_\mu = 1.$$

What about the general relativistic case? The equations of motion for a test particle in a gravitational field are given by the geodesic equations in Riemannian spacetime, called equations of ‘free fall’:

$$\frac{DV^\alpha}{\mathrm{d}s} = \frac{\mathrm{d}V^\alpha}{\mathrm{d}s} + \Gamma^\alpha_{\mu\nu}V^\mu V^\nu = 0.$$

The indexed object $\Gamma^{\alpha}_{\mu\nu}$ is the linear connection arising from the notion of parallel transport in Riemannian geometry. In general relativity, the effects of the gravitation are described as the manifestation of the spacetime curvature. In this sense, *gravitation disappears as an external force* in the equations of motion, which become free equations. What is the corresponding quadratic form? This is that of special relativity, but now written with the Riemannian metric: $g_{\mu\nu}V^{\mu}V^{\nu} = 1$. Then, what is the Hamilton–Jacobi equation which corresponds to the test particle in free fall in gravitational field? One has to give the same answer: that of special relativity now written with the Riemannian metric: $g_{\mu\nu}\partial^{\mu}S\partial^{\nu}S = m^2c^2$. In a consistent way with the status of the equations of motion, *no external potential* is present in addition to the term quadratic in the partial derivatives of the action. In scale-relativity theory, we assume that 'quantum behaviour' arises from fractal properties of spacetime at quantum scales, by analogy with general relativity. Moreover, we claim to be able to write the Schrödinger (and the Klein–Gordon) equation as a 'geodesic equation' in fractal space(-time). In spite of these assumptions, the 'classical' equivalences giving the correspondence *free equations of motion/quadratic energy* are lost. Indeed, following equations (A.4) and (16), we have

$$\frac{\eth\mathcal{V}^i}{\mathrm{d}t} = 0 \Leftrightarrow \mathcal{V}^2 - \mathrm{i}mc\lambda\nabla\cdot\mathcal{V} = \mathrm{C}^{\mathrm{st}} \tag{A.5}$$

and

$$\frac{\eth\mathcal{V}^{\alpha}}{\mathrm{d}s} = 0 \Leftrightarrow \mathcal{V}^{\mu}\mathcal{V}_{\mu} + \mathrm{i}\lambda\partial^{\mu}\mathcal{V}_{\mu} = 1. \tag{A.6}$$

In a consistent way with these equations, the quadratic term alone, $\mathcal{V}^2$ or $\mathcal{V}^{\mu}\mathcal{V}_{\mu}$, leads to equations which are wrong, and are therefore irrelevant. The conclusion is that the form in '$\mathcal{V}^2 + \mathrm{i}\lambda\nabla\cdot\mathcal{V}$', corresponds to the intrinsic form of the 'energy' for the free case, associated with the free equations of motion 'in fractal spacetime'.

Appendix B. Projective properties of the *non-quadratic* invariants

In order to point out that the new form of the invariants is the intrinsic form of the energy we consider the free case in this framework, let us exhibit a very interesting and relevant property of these invariants. For several reasons mentioned in [26], we may want to consider *homographic transformations* on velocities $\mathcal{V}$. After some calculation, we can check that the homographic transformation

$$\tilde{\mathcal{V}} \equiv \frac{\mathcal{V} + \mathcal{W}}{\mathcal{V}\mathcal{W}/\varepsilon + 1} \tag{B.1}$$

leads to the equivalence

$$\tilde{\mathcal{V}}^2 - \mathrm{i}\lambda c\partial_x\tilde{\mathcal{V}} = \varepsilon \Leftrightarrow \mathcal{V}^2 - \mathrm{i}\lambda c\partial_x\mathcal{V} = \varepsilon \tag{B.2}$$

where $\mathcal{E} \equiv (1/2)m\varepsilon$ [26]. The interesting point is the interpretation of this property in terms of *hyperbolic geometry*. We cannot develop this point here, but we have to know that homography represents what we call a 'motion' in hyperbolic space (and more generally in non-Euclidean and projective spaces), i.e. an isometry, which preserves the distance built from the cross-ratio.

In order to characterize and understand the profound meaning of our transformation, we have to keep in mind two points.

(i) The first concerns the geometric locus which remains invariant under homographic or projective transformation. We call this locus the *absolute* of the transformation. We can show that the nature of the absolute defines a *geometry*. For instance, if the absolute is a real conic, then the interior of it gives us a representation of the two-dimensional hyperbolic geometry. The Poincaré representation is the particular case where the absolute is represented by the *real axis* (a real conic or circle with an infinite radius) and the hyperbolic plane by the upper half-plane of the whole Cauchy plane, which represents the complex line. The 'motions' are then given by complex homographies. We can show that their coefficients have to be real to leave the real axis invariant and that they lead to a positive determinant in order to preserve each half-plane. Therefore, by choosing $\mathcal{W} = W$ and $\mathcal{E} = E$ real, we may consider equation (B.1) as a *motion in the hyperbolic plane for the Poincaré representaion*.

(ii) The second point is that a homography is completely characterized by its fixed points z_0, which verify $\tilde{z}_0 = z_0$. In our case, we can check that $\tilde{\mathcal{V}}_0 = \mathcal{V}_0$ implies

$$E = \tfrac{1}{2} m \mathcal{V}_0^2. \tag{B.3}$$

As we see, our energy formula recovers its *classical form*, i.e. *its usual quadratic form at the fixed points*. If $E > 0$, $\mathcal{V}_0 \equiv V_0$ is real. However, as we saw in the introduction, the imaginary part of $\mathcal{V}$ corresponds to its 'quantum part', which is non-zero for $\mathfrak{v}_+ \neq \mathfrak{v}_-$, i.e. in the non-differentiable case. Therefore, we can consider equation (B.3), as the *expression of energy at the classical limit*. The interpretation which is proposed in [26], is to make a correspondence between the limit that represents *classical mechanics* for quantum mechanics and the geometric limit which represents the *horizon* for the hyperbolic plane, in particular for the upper-half plane Poincaré model.

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