Nottale, L., 1994, invited conference in "*Relativity in General*", (Proceedings of Spanish Relativity Meeting, Salas, 1993), Eds. J. Diaz Alonso and M. Lorente Paramo (Editions Frontières), pp. 121-132.

Scale Relativity: First Steps toward a Field Theory.

Laurent Nottale

CNRS, DAEC, Observatoire de Paris-Meudon, F-92195, Meudon Cedex, France

Abstract. After having briefly recalled the principles and methods on which the theory of scale relativity is founded, we apply our scale-covariant methods to the problem of recovering relativistic quantum mechanics. The withdrawal of the hypothesis of differentiability of space-time leads to the demonstration of the divergence of space-time coordinates in terms of resolutions, then to the definition of scale-covariant derivatives that transform the free particle classical equation into the Klein-Gordon equation, including electromagnetic potentials. Our suggestion of a generalization of the dilation laws to a Lorentzian form at high energies and its consequence for the generation of elementary particle masses is reminded. We conclude by new theoretical predictions, concerning in particular the *W/Z* boson mass ratio, which we expect to be $\sqrt{(10/13)}$.

1. Introduction.

Present physics is founded on the hypothesis that space-time is continuous and differentiable. Hence Einstein's equations of general relativity are the most general simplest equations which are covariant under continuous and at least two times differentiable transformations of the coordinate system [1]. But no experiment nor fundamental principle proves in a definitive way the hypothesis of differentiability of space-time coordinates. On the contrary, Feynman's path integral approach to quantum mechanics [2,3] allowed him to demonstrate the opposite: when going to small length- and time-scales, the typical paths of quantum particles are continuous and nondifferentiable, and can be characterized by (what we call now, after Mandelbrot [4]) a fractal dimension $D = 2$.

In the present paper, we shall recall the present state of our attempt to construct a theory based on the withdrawal of the hypothesis of differentiability of space-time [5-7], then give some hints about its future development, including in particular the incorporation of fields in its framework.

The main consequence of continuity and nondifferentiability is scale divergence [7,8]. One can indeed demonstrate that the length of a continuous and nowhere differentiable curve is explicitly dependent on the resolution ε at which it is considered [$\mathcal{L} = \mathcal{L}(\varepsilon)$], and, further, that it is divergent [$\mathcal{L}(\varepsilon)\rightarrow\infty$ when $\varepsilon \rightarrow 0$], i.e., in other words, that this curve is fractal (in a general meaning) [7-9]. This theorem is easily generalizable to any dimension.

We then introduce explicitly the resolutions in the main physical quantities, and, as a consequence, in the fundamental equations of physics. This means that a physical quantity ϕ, usually expressed in terms of space-time variables x, *i.e.* $\phi = \phi(x)$, must be now described as depending also on resolutions, $\phi = \phi(x,\varepsilon)$.

The standard function $\phi(x)$ is identified with $\phi(x,0)$, and the nondifferentiability means that $\partial\phi(x)/\partial x = \partial\phi(x,0)/\partial x$ *does not exist*. This is a really crucial problem, since this non-existence would prevent us from writing differential equations.

Our proposal amounts to realize that, even if $\partial\phi(x,0)/\partial x$ no longer exists, $\partial\phi(x, \varepsilon)/\partial x$ can be defined for any non zero ε. But ϕ now depending also on the resolution variables ε, we need to complete the usual differential equations which are written in terms of displacements (laws of motion), by new differential equations implying derivatives like $\partial\phi(x,\varepsilon)/\partial ln\varepsilon$, and which will describe the laws of scale.

Owing to the fact that, as do velocities, resolutions can be defined only in a relative way (only a ratio of length- or time-scale has a physical meaning), we have proposed that Einstein's principle of relativity, which in its present acceptation is applied to motion laws [1], should also be applied to scale laws. Namely, defining resolutions as characterizing the *state of scale* of a reference system, the *principle of scale relativity* states that the laws of nature should apply to all reference systems, whatever their state of scale. Its mathematical translation is *scale covariance*, i.e. the invariance of the form of the equations of physics under transformations of *resolutions* (dilations and contractions). Its geometrical achievement is the concept of fractal space-time [5-7,10].

2. Scale Covariance and Quantum mechanics.

2.1. Nondifferentiability and Renormalization Group Equations.

As recalled in the introduction, continuous nondifferentiability implies scale-divergence. We shall now see that when one wants to implement scale relativity, one is naturally led to renormalization-group like equations and to fractal behavior [8]. Consider an essential physical quantity ϕ which is assumed to depend not only on usual variables x, but also on resolution ε. If ϕ is indeed the quantity which determines the physics, we expect the variation of ϕ under an infinitesimal scale transformation $dln\varepsilon$ to depend only on ϕ itself. This reads:

$$\frac{\partial\phi(x,\varepsilon)}{\partial ln\varepsilon} = \beta(\phi) \ , \tag{1}$$

which is typical of a renormalisation group equation, β being a "Callan-Symanzik" function. Once again looking for the simplest possible form of such an equation, we expand $\beta(\phi)$ in powers of ϕ and obtain to first order the linear equation $\partial\phi(x,\varepsilon)/\partial ln\varepsilon = a - \delta\ \phi$. Note that this Taylor expansion can always been done, since one may always renormalize ϕ by dividing it by some large value ϕ_1, ($\phi \to \phi' = \phi/\phi_1$) in such a way that the new variable ϕ' remains << 1 in the domain of interest. The solution of Eq. (2) is

$$\phi(x,\varepsilon) = \phi_o(x)\ \{1 + \zeta(x)\ (\frac{\lambda}{\varepsilon})^{\delta}\}\ , \tag{2}$$

where $\lambda^{\delta}\zeta(x)$ is an integration "constant" (i.e. constant in terms of ε) and $\phi_o = a/\delta$. The meaning of this result is that the simplest scale-dependent equations are solved in terms of the combination of *a standard fractal (power-law) behavior* for small scales (respectively large scales, depending on the sign of b) and of *a transition to scale-independence* at large scales (respectively small scales).

Such a behavior is precisely characteristic of microphysics (respectively cosmology): it yields both quantum phenomena (i.e., Schrödinger's equation and correspondence principle) and the quantum-classical transition, as demonstrated for the non-relativistic case in Refs. [6-8]. We shall now develop the consequences of this result in the (motion)-relativistic case, and show that one can demonstrate the Klein-Gordon equation from such grounds.

2.2. "Quantum Covariance" and the Klein-Gordon Equation.

Assume that space-time is continuous but nondifferentiable. As a consequence, there will be an infinity of fractal geodesics between any couple of points in such a space-time. In a theory based on these concepts, predictions can be made only using the infinite family of geodesics, and so become of a probabilistic nature, though particles can be considered as having followed one particular (but undetermined) geodesic of the family [6,7]. Let us consider a small increment dX^i of nondifferentiable 4-coordinates along one of the geodesics. We know from the above considerations that such a geodesic is scale-dependent at small length-scales, while scale-independence is recovered at large length-time-scales. Let us decompose dX^i in terms of its mean, $<dX^i> = dx^i$ and a fluctuation around the mean $d\xi^i$ (such that $<d\xi^i> = 0$ by definition):

$$dX^i = dx^i + d\xi^i \quad . \tag{3}$$

Let us now introduce a fractal invariant (proper time) $\mathcal{S}$ along the fractal trajectory, and a classical proper time s. Consider the case when the trajectory is a fractal curve of dimension $D = 1 + \delta$. The two proper times are related as given in Eq.(2):

$$\mathcal{S} = s \left[1 + \zeta \left(\frac{\lambda}{\delta\mathcal{S}}\right)^{\delta}\right] , \tag{4}$$

where ζ is such that $<\zeta^2> = 1$. The meaning of the fundamental scale λ, which appears as an integration constant in Eq.(2), can be specified: the comparison with quantum mechanics shows that it must be identified with the Compton length of the particle, $\lambda = \hbar/2mc$. Let us set $\zeta s = \sigma$ and place ourselves in the small scale regime, $\delta\mathcal{S} << \lambda$. Taking the derivative of Eq.(4) yields $d\mathcal{S} = d\sigma\,(\lambda/\delta\mathcal{S})^{\delta}$. We can identify the differential $d\mathcal{S}$ with the resolution $\delta\mathcal{S}$ (this can be made rigorously by using a non-standard analysis description, see Refs. [5,7]) and this yields:

$$d\mathcal{S} = (\lambda^{\delta} d\sigma)^{1/D} \quad . \tag{5}$$

In the particular case $D = 2$, which plays a central role in our approach, this reads in the mean:

$$<d\mathcal{S}^2> = \lambda\, ds \quad , \tag{6}$$

so that, introducing classical and "fractal" velocities v^i and u^i, such that $<(u^i)^2> = 1$ and $<u^i> = 0$, we may write Eq.(3) in the form:

$$dX^i = v^i\, ds + \lambda^{1/2} u^i\, ds^{1/2}. \tag{7}$$

Let us now account for the fact that there are an infinity of fractal geodesics between any couple of events. This forces us, if we want to make predictions, to jump to a statistical representation.

Eq.(7) means that, when $D = 2$, the fluctuation is such that $<d\xi^2> \propto ds$, and one recognizes here the basic law of a diffusion Markov-Wiener process. But the principle of microscopic reversibility (cf. Feynman in [3]) means that one must also consider the reverse process, obtained by reversing the sign of the classical proper time. It was realized by Nelson [12] that, because of nondifferentiability, the mean derivatives of a given function were a priori different for the forward and the backward processes. So we define, following Nelson [12,13], mean forward and backward derivatives, d_+/ds and d_-/ds:

$$\frac{d_\pm}{ds} y(s) = lim_{\delta s \to 0\pm} < \frac{y(s + \delta s) - y(s)}{\delta s} > \tag{8}$$

which, once applied to x^i, yield *forward and backward mean 4-velocities,*

$$\frac{d_+}{ds} x^i(s) = v^i_+ \quad ; \quad \frac{d_-}{ds} x^i(s) = v^i_- \quad . \tag{9}$$

The forward and backward derivatives of (9) can be combined in a complex derivative operator

$$\frac{\mathcal{d}}{ds} = \frac{(d_+ + d_-) - i\,(d_+ - d_-)}{2ds} \quad , \tag{10}$$

[7,8], which, when applied to the position vector, yields a complex velocity

$$\mathcal{V}^i = \frac{\mathcal{d}}{ds} x^i = V^i - i\; U^i = \frac{v^i_+ + v^i_-}{2} - i\; \frac{v^i_+ - v^i_-}{2} \quad . \tag{11}$$

The real part V^i of the complex velocity $\mathcal{V}^i$ generalizes the classical velocity, while its imaginary part, U^i, is a new quantity arising from the non-differentiability.

Let us now consider the question of the definition of a Lorentz-covariant diffusion in space-time. This problem has been addressed by several authors in the framework of a relativistic generalization of Nelson’s stochastic mechanics [12,13]. Forward and backward fluctuations, $d\xi^i_\pm(s)$, are defined,which are Gaussian with mean zero, mutually independent and such that

$$<d\xi^i_\pm \; d\xi^j_\pm> = \pm\, \lambda\, \eta^{ij} \; ds \quad . \tag{12}$$

The identification with a diffusion process allows us to relate λ to a diffusion coefficient: we get $2\,\mathcal{D} = \lambda\, c$, so that $\mathcal{D} = \hbar/2m$, which is precisely the value postulated by Nelson. But the difficulty comes from the fact that such a diffusion makes sense only in $\boldsymbol{R}^4$, i.e. the “metric” η^{ij} should be positive definite, if one wants to interpret the continuity equation satisfied by the probability density (see hereafter) as a Kolmogorov equation. Several proposals have been made.

Dohrn and Guerra [14] introduce the above “Brownian metric” and a kinetic metric g_{ij}, and obtain a compatibility condition between them which reads $g_{ij}\, \eta^{ik}\, \eta^{jl} = g_{kl}$. An equivalent method was developed by Zastawniak [15], who introduces, in addition to the covariant forward and backward drifts v^i_+ and v^i_- (be careful that our notations are different from his) new forward and backward drifts b^i_+ and b^i_- in terms of which the Fokker-Planck equations that one can derive from the Klein-Gordon equation become Kolmogorov equations for a standard Markov-Wiener diffusion in $\boldsymbol{R}^4$. Serva [16] gives up Markov processes and considers a covariant process which belongs to a larger class, known as "Bernstein processes".

All these proposals are equivalent, and amount to transforming a Laplacian operator in $\boldsymbol{R}^4$ into a Dalembertian. Namely, the two forward and backward differentials of a function $f(x,s)$ read (we assume a Minkowskian metric for classical space-time):

$$d_{\pm}f/ds \;=\; (\partial/\partial s \;+ v^i_{\pm} \cdot \partial_i \;\pm\; \tfrac{1}{2}\;\lambda\;\;\partial^i\,\partial_i\,)\,f \quad . \tag{13}$$

In what follows, we shall only consider functions that are not explicitly dependent on the proper time *s*. In this case our time derivative operator writes:

$$\frac{\mathcal{d}}{ds} \;=\; (\mathcal{V}^k \;-\; \tfrac{1}{2}\;i\,\lambda\;\;\partial^k\,)\;\partial_k \quad . \tag{14}$$

We shall now generalize to the relativistic case the demonstration [7,8] that the passage from classical (differentiable) mechanics to quantum mechanics can be implemented by a *unique* prescription: *Replace the standard time derivative d/dt by the new complex operator* $\mathcal{d}/dt$, that plays the role of a "quantum-covariant derivative".

Let us first note that (14) can itself be derived from the introduction of a partial quantum-covariant derivative $\mathcal{d}_k = \mathcal{d}/dx^k$:

$$\mathcal{d}_k \;=\; \partial_k \;-\tfrac{1}{2}\;i\;\;\lambda\;\frac{\mathcal{V}_k}{\mathcal{V}^2}\;\partial^j\,\partial_j \quad , \tag{15}$$

where $\mathcal{V}^2 = \mathcal{V}_k\mathcal{V}^k$. It is easy to check that $\mathcal{d}/ds = \mathcal{V}^k\;\mathcal{d}_k$.

Let us assume that any mechanical system can be characterized by a stochastic (complex) action $\mathcal{S}$. The same reasoning as in classical mechanics leads us to write $d\mathcal{S}^2 = -m^2c^2\;\mathcal{d}x_k\;\mathcal{d}x^k = -m^2c^2\;\mathcal{V}_k\mathcal{V}^k\,ds^2$. The least-action principle applied on this action yields the equations of motion of a *free* particle, $\mathcal{d}\mathcal{V}_k/ds\;=0$. We can also write the variation of the action as a functional of coordinates. We obtain the usual result (but here generalized to complex quantities):

$$\delta\mathcal{S} = -mc\;\mathcal{V}_k\;\delta x^k \quad\Rightarrow\quad \mathcal{P}_k \;=\; mc\;\mathcal{V}_k \;=\; -\partial_k\,\mathcal{S} \quad , \tag{16}$$

where $\mathcal{P}_k$ is a complex 4-momentum. As in the nonrelativistic case, the wave function is introduced as being nothing but a reexpression of the action:

$$\psi \;=\; e^{i\mathcal{S}/mc\lambda} \Rightarrow \quad \mathcal{V}_k \;=\; -\,i\;\;\lambda\;\partial_k\,(ln\psi) \;\;, \tag{17}$$

so that the equations of motion ($\mathcal{d}\mathcal{V}_k/ds\;=0$) become:

$$\mathcal{d}\mathcal{V}_j/ds \;=-\,i\,\lambda\;\;(\mathcal{V}^k\;\partial_k - \;\tfrac{1}{2}\;i\,\lambda\;\;\partial^k\,\partial_k\,)\;\partial_j\,(ln\psi\,)\;\;=0\;\;. \tag{18}$$

Arrived at that stage, it is easy to show that this equation amounts to the Klein-Gordon one. Indeed, if we replace the time variable *t* by *it* in this equation, the Dalembertian is replaced by a 4-Laplacian, and we are brought back to the non-relativistic problem which has already been treated in detail in Refs. [7-9]. So Eq.(18) can finally be put under the form of a vanishing 4-gradient which is integrated in terms of the Klein-Gordon equation for a free particle:

$$\partial_j \left\{ \frac{\lambda^2 \, \partial^k \partial_k \, \psi}{\psi} \right\} = 0 \quad \Rightarrow \quad \lambda^2 \, \partial^k \partial_k \, \psi = \psi \ . \tag{19}$$

We recall that $\lambda = \hbar/mc$ is the Compton length of the particle. The integration constant is 1 in order to ensure the identification of $\rho = \psi\psi^{\dagger}$ with a probability density for the particle. In fact, there is a free normalisation factor in our whole approach, which can be taken such that $\mathcal{V}_k \mathcal{V}^k = 1$ (since it is precisely from the variation of this term that we derive the equations of motion). Such a choice simplifies the expression of our quantum-covariant partial derivative (Eq.15).

Before going on with a first trial of introduction of fields, we want to stress the physical meaning of the above result. While Eq.(19) is the equation of a free *quantum* particle, the equation of motion (18) from which we started takes exactly the form of the *classical* equation of a *free* particle. Hence quantum effects appear here through the implementation of *scale covariance*.

2.3. First Account of Electromagnetic Field.

It is easy to generalise Eq.(19) in terms of a Klein-Gordon equation including an electromagnetic potential. But we shall see in Sec.3 that we can do better, and derive the very existence of such a field from scale relativity.

Assume that the particle is subjected to an interaction with a field, with which it exchanges an energy-momentum $\Delta P^i = (e/c)\, A^i$. This quantity is real, and implies a new complex 4-momentum, $\tilde{\mathcal{P}}^i = \mathcal{P}^i + \Delta P^i = mc\, \mathcal{V}^i + (e/c)\, A^i$. Now, from Eq.(17), the potential will affect only the phase of the wave function, as expected. This leads us to introduce a “covariant” velocity:

$$\tilde{\mathcal{V}}_k = -\, i \;\lambda\, \partial_k \,(ln\psi) + \frac{e}{mc^2}\, A_k \tag{20}$$

and we recognize the well-known QED-covariant derivative, since we can write (20) as $mc\, \tilde{\mathcal{V}}_k\, \psi = [-i\hbar\, \partial_k + (e/c)\, A_k\,]\psi$. We may now introduce a scale-covariant + QED-covariant derivative:

$$\tilde{d}_k = \partial_k - \frac{1}{2}\, i\, \lambda\, \tilde{\mathcal{V}}_k\, \partial^j\, \partial_j \ . \tag{21}$$

In terms of this doubly covariant derivative, the free particle equation $\tilde{d}^2 x^k/ds^2 = 0$ now takes the form of the Klein-Gordon equation for a particle in an electromagnetic field, $[i\hbar\, \partial_k - (e/c)\, A_k\,]\, [i\hbar \partial^k - (e/c)\, A^k\,]\, \psi = m^2 c^2\, \psi$.

The case of the Dirac equation will be considered in a forthcoming work (see Refs. [10,17] for early attempts to comprehend it in terms of fractal and stochastic models).

3. Scale Relativity.

3.1. Lorentzian Laws of Dilations.

Up to now, we have still not fully used the power of the principle of scale relativity. The laws which have been demonstrated correspond only to the simplest possible laws one can construct when giving up the hypothesis of differentiability.

But if we try to forget all that we know (or believe to know) about scale laws, and decide to *deduce* them from the implementation of *scale covariance*, it is found that the currently accepted laws are only large length-scale approximations of more general scale laws [7,11].

Consider indeed the integrated length $\mathcal{L}(\varepsilon)$ along a typical fractal trajectory (this may equally be any integrated coordinate, i.e., $\int |dX^i|$), and let us search for its law of tranformation under a dilatation ρ of the resolution, $\varepsilon \to \varepsilon'$. We set $\mathbb{X} = ln(\mathcal{L}/\mathcal{L}_0)$. Assuming that this transformation depends only on the relative "state of scale" of the reference system, $\mathbb{V} = ln\rho$, and in a linear way (i.e. we consider first the case of "special scale-relativity") we can write a general expression for this transformation [7,8,11]:

$$\mathbb{X}' = A(\mathbb{V})\ \mathbb{X} + B(\mathbb{V})\ \delta \quad ; \quad \delta' = C(\mathbb{V})\ \mathbb{X} + D(\mathbb{V})\ \delta\ . \tag{22}$$

The problem now amounts to find the four functions A, B, C, D which comply with scale covariance. The general solution of this problem is well-known [11,18] and is nothing but the Lorentz transformation. The difference with motion-relativity is that, while the Einstein-Lorentz form of the law of motion transformation is universal, a breaking of the scale symmetry is known to occur beyond some scale λ_0 (there is no explicit dependence of measurement results on resolution in the classical domain). We have shown in previous works [11] that such a combination of Lorentzian scale relativity and its breaking at large length-time-scales results in the appearance of an invariant, unpassable length-scale $\mathbb{\Lambda}$ (that we have identified with the Planck length), rather than the invariant dilatation one would have get in case of an unbroken symmetry. A detailed account of the present status of the theory can be found in Refs. [7-9,11]. We shall only recall here the main new relations that are obtained in its framework.

The length of a fractal curve (more generally, any curvilinear fractal coordinate) now diverges, at resolutions $r < \lambda_o$, following the scale-covariant law, simplified thanks to the choice $\mathcal{L}_o = \mathcal{L}(\lambda_o)$

$$\mathcal{L} = \mathcal{L}_o\ (\lambda_o/r)^{\delta(r)},\ \text{with}\ \ \delta(r) = \frac{1}{\sqrt{1 - ln^2(\lambda_o/r)\ /\ ln^2(\lambda_o/\mathbb{\Lambda})}}\ . \tag{23}$$

The Compton relation is generalized as:

$$ln\ \frac{m}{m_o} = \frac{ln\ (\lambda_o/\lambda)}{\sqrt{1 - ln^2(\lambda_o/\lambda)/ln^2(\lambda_o/\mathbb{\Lambda})}}\ . \tag{24}$$

The fact that the fluctuations $d\xi^i$ have positive definite metric finally allows us to implement special scale relativity from the introduction, at small space-time scales $\varepsilon < \lambda_o$, of a 5-dimensional Minkowskian "space-time-zoom" of signature (+,-,-,-,-), whose fifth dimension is the (now variable) "scale" dimension δ:

$$d\sigma^2 = \mathbb{C}_o^{\,2}\ d\delta^2 - \Sigma_i\ (d\,ln\xi^i)^2\ , \tag{25}$$

where $\mathbb{C}_o = ln(\lambda_o/\mathbb{\Lambda})$ and $d\,ln\xi^i\ /\ d\delta = \mathbb{V}^i = ln(\lambda_o/\varepsilon^i)$.

When going back to large space-time scales: (i) the scale dimension jumps to a constant value, then $d\delta = 0$ and the 5-space degenerates into a 4-space; (ii) among the 4 variables X^i, we can singularize the one which is characterized by the smallest fractal-nonfractal transition, say X^1, and call it "time"; then one can demonstrate [7, pp.123-4] that the fractal divergences cancel in such a way that the three quantities $<dX^i/dX^1>$, i =2,3,4, remain smaller than 1 (i.e. c in units m/s); (iii) this implies Einstein's motion-relativity, i.e. equivalently, a Minkowskian metric for the mean (classical) remaining 4 variables.

3.2. Mass spectrum of charged fermions.

As is now clear from the above developments, quantum mechanics is provided to us only by the Markov-Wiener, reversible process whose fractal dimension is 2 (see [9]). On the other hand, the requirement of scale-covariance leads us to introduce a fractal dimension $D = 1+\delta(r)$ becoming larger than 2 for scales smaller than the Compton length of the electron. This seems to be in contradiction with the well-verified fact that quantum mechanics, in its gauge quantum field version, remains valid at least up to the present largest energies reached in particle accelerators ($\approx$ 100 GeV, i.e. length-scales $\approx 10^5$ times smaller than the electron Compton scale). But we have found [8,9] that the scale dependence of the self-energy of the particle (e.g., an electron) due to radiative corrections cancels the scale dependence coming from $D \neq 2$, in such a way that one can define an effective fractal dimension remaining very close to 2.

Namely, setting $\delta s/\tau \approx c\delta t/\lambda_e = cr/\lambda_e$ (valid for $r << \lambda_e$, where λ_e is the Compton length of the electron), Eq.(12) is generalized for $D = D(r) \neq 2$ as:

$$<d\xi^i_\pm \, d\xi^j_\pm> = \pm \lambda \, \delta^{ij} \, ds \, \left(\frac{\delta s}{\tau}\right)^{(2/D)-1} = \pm \delta^{ij} \, \frac{\hbar}{mc} \, ds \, \left\{ \frac{m_e}{m(r)} \left(\frac{r}{\lambda_e}\right)^{[2/D(r)]-1} \right\} \qquad (26)$$

with $D(r) = 1 + \delta(r)$ as given by Eq.(23). The variation with scale of the mass $m(r)$, due to n elementary fermion pairs of Compton length $\lambda_i = \hbar/m_i c$ and charges Q_i (in units of the electron charge) is given by the solution to its renormalization group equation [7,8,19], i.e. to lowest order:

$$\frac{m_e}{m(r)} = 1 - \frac{3\alpha_e}{2\pi} \left[\sum_{i=0}^{n} Q_i^2 \, \mathbb{V} - \sum_{i=0}^{n} (Q_i^2 \, \mathbb{V}_i) \right] , \qquad (27)$$

where α_e ($\approx$1/137.036) is the fine structure constant. In this formula we have set $\mathbb{V}_i = ln(\lambda_e/\lambda_i)$ and $\mathbb{V} = ln(\lambda_e/r)$. It is valid to lowest order and neglecting threshold effects.

Now the requirement that quantum mechanics remains true in the energy domain 0.5 MeV–100 GeV amounts to requiring the vanishing of the corrective term in (26), i.e.:

$$\frac{m_e}{m(r)} \left(\frac{r}{\lambda_e}\right)^{[2/D(r)]-1} = 1 . \qquad (28)$$

As shown in [8,9], the solution to this equation provides us with a mass and charge distribution in very good agreement with the observed spectrum of elementary charged fermions. In particular, it allows us to predict a top quark mass of 120 ± 30 GeV, agreeing with its deduction from radiative correction and precision experiments, 130 ± 40 GeV [20,21], and with our expectation of the emergence of a mass scale at 123.23 GeV (see [7] and Sec.3.5 below).

3.3. Nature of the Electromagnetic field.

The theory of scale relativity also allows us to get new insights about the nature of the fundamental fields and their associated charges. We shall only very briefly consider some possible roads in this direction and their consequences. A more detailed treatment exceeds the scope of the present contribution and will be presented elsewhere [22].

Consider an electron (or any other particle). In scale relativity, we identify the particle with its potential fractal trajectories, described as the geodesics of a nondifferentiable space-time. These trajectories are characterized by internal (fractal) structures. Now consider anyone of these structures, lying at some (relative) resolution ε (such that $\varepsilon < \lambda$) for a given position of the particle. In a displacement of the particle, the relativity of scales implies that the resolution which corresponds to the same structure will *a priori* be different from the initial one. In other words, we expect the occurrence of *dilatations of resolutions induced by translations*, which read:

$$\hbar \, \frac{\delta\varepsilon}{\varepsilon} = -\frac{e}{c} \, A_\mu \, \delta x^\mu \, . \tag{29}$$

This behaviour can be expressed in terms of a scale-covariant derivative:

$$D_\mu \, ln(\lambda/\varepsilon) \; = \; \partial_\mu \, ln(\lambda/\varepsilon) \; + \; (e/\hbar c) \;\; A_\mu \; . \tag{30}$$

However, if one wants such a “field” to be physical, it must be defined whatever the initial scale from which we started. Starting from another scale $\varepsilon' = \rho\varepsilon$ (we consider Galilean scale-relativity for the moment), we get $\hbar c\delta\varepsilon'/\varepsilon' = -e \, A'_\mu \delta x^\mu$, so that we obtain:

$$\frac{e}{c} \, A'_\mu \; = \; \frac{e}{c} \, A_\mu \; + \; \hbar \, \partial_\mu \, ln\rho \quad , \tag{31}$$

which depends on the relative “state of scale”, $\mathbb{V} = ln\rho$. However, if one now considers translation along two different coordinates (or, in an equivalent way, displacement on a closed loop), one obtains a commutator relation:

$$(D_\mu D_\nu - D_\nu D_\mu) \; ln(\lambda/\varepsilon) \; = \; (e/\hbar c) \;\; (\partial_\mu A_\nu \; - \partial_\nu A_\mu) \; . \tag{32}$$

This relation defines a tensor field $F_{\mu\nu} \; = \; \partial_\mu A_\nu \; - \partial_\nu A_\mu$ which, contrarily to A_μ, is independent of the initial scale. One recognize in $F_{\mu\nu}$ an electromagnetic field, in A_μ an electromagnetic potential and in (31) the property of gauge invariance which, in accordance with Weyl’s initial ideas [23], recovers its initial status of scale invariance. However, Eq.(31) represents a progress compared with these early attempts and with gauge invariance in today’s physics. Indeed the gauge function, which has, up to now, been considered as arbitrary and devoid of physical meaning, is now identified with the logarithm of internal resolutions. In Weyl’s theory, and in its formulation by Dirac [24], the metric element *ds* (and consequently the length of any vector) is no longer invariant and can vary in terms of some (arbitrary) scale factor. Such a theory was excluded by experiment, namely by the existence of universal and unvarying lengths such as the electron Compton length. In scale relativity, we are naturally led to introduce two “proper times”, the classical one *ds* which remains invariant, and the fractal one $d\mathcal{S}$, which is scale-divergent and can then vary from place to place. Since $d\mathcal{S} = d\sigma\,(\lambda/\varepsilon)$ from Eq.(4), we have $\delta(d\mathcal{S})/d\mathcal{S} = -\delta\varepsilon/\varepsilon = (e/\hbar c)\,A_\mu\,\delta x^\mu$, and we recover the basic relation of Weyl's theory at small scales ($\varepsilon < \lambda$).

The passage to quantum theory will now allow us to derive a general mass-charge relation and to make new theoretical predictions. We introduce a generalized action which includes motion laws and scale laws, through the motion- and scale invariants: $d\$ = a\,ds + b\,d\sigma$, with a and b constants. Now the action is directly related to the probability amplitude as recalled in Eq.(17). In the "Galilean" scale-relativistic case ($D = 2$) that we consider for the moment, we can write $d\sigma = dln(\mathcal{S}/\mathcal{S}_o) = \delta\,.\,dln(\lambda/\varepsilon)$ with $\delta = 1$, so that $d\$$ reads after identification of the two constants

$$d\$ = -i\,\hbar\,dln\psi = -mc\,\mathcal{V}_\mu\,dx^\mu - \frac{e}{c}\,A_\mu\,dx^\mu \quad . \tag{33}$$

We recover the QED+scale-covariant derivative of Eq.(21) and the interpretation of A_μ as a variation of momentum, so that we shall be led once again to the Klein-Gordon equation.

3.4. Mass-charge relation.

In a gauge transformation (i.e., in a transformation of resolution $ln\rho$ in terms of our new interpretation), the probability amplitude changes as $\psi' = exp\{(i/\hbar)\,(e/c)\,e\,ln\rho\}\,\psi$, which becomes, since $e^2 = 4\pi\,\alpha\,\hbar\,c$,

$$\psi' = e^{i\,4\pi\alpha ln\rho}\,\psi \quad . \tag{34}$$

In the Galilean case such a relation leads to no new result, since $ln\rho$ is unlimited. But if one admits that scale laws become Lorentzian below some scale λ (Sec. 3.1), then $ln\rho$ becomes limited by $\mathbb{C} = ln(\lambda/\mathbb{A})$. This implies a quantization of the charge which amounts to the relation $4\pi\,\alpha\,\mathbb{C} = 2k\pi$, i.e.:

$$\alpha\,\mathbb{C} = k/2 \quad . \tag{35}$$

Since $\mathbb{C} = ln(\lambda/\mathbb{A}) \approx ln(m/m_\mathbb{P})$, where $m_\mathbb{P}$ is the Planck mass, Eq.(35) is nothing but a new general mass-charge relation. Note that the existence of such a relation could already be expected from a renormalization group approach [22].

The first domain to which one can try to apply such a relation is QED. However we know from the electroweak theory that the electric charge is only a residual of a more general, high energy electroweak coupling. One can define an inverse 'electroweak coupling' $\bar{\alpha}_0 = \alpha_0^{-1}$ from the U(1) and SU(2) couplings:

$$\bar{\alpha}_0 = \frac{3}{8}\,\bar{\alpha}_2 + \frac{5}{8}\,\bar{\alpha}_1 \quad . \tag{36}$$

This new 'coupling' is such that $\alpha_0 = \alpha_1 = \alpha_2$ at unification scale and is related to the fine structure constant at Z scale by the relation $\alpha = 3\alpha_0/8$. We suggest that it is α_0 rather than α which must be used in Eq.(35). Indeed, even disregarding as a first step threshold effects, we get a mass charge relation for the electron (corresponding to $k = 2$ in Eq.35):

$$ln\,\frac{m_\mathbb{P}}{m_e} = \frac{3}{8}\,\alpha^{-1} \quad . \tag{37}$$

From the known experimental values, the two members of this equation agree to 2‰: $\mathbb{C}_e = ln(m_\mathbb{P}/m_e) = 51.528(1)$ while $3/8\alpha^{-1} = 51.388$. The agreement is made better if one accounts for the fact that the measured fine structure constant (at Bohr scale) differs from the limit of its asymptotic behavior. One finds that the asymptotic inverse coupling at the scale where the asymptotic mass reaches the observed mass m_e is $\alpha_0^{-1} = 51.521$, within 10^{-4} of the value of $\mathbb{C}_e$.

Now, the development of GUT's has reinforced the idea of a common origin for the various gauge interactions. The same is true from the scale relativistic approach. Indeed Eq.(29) is clearly a simplified formula, since it does not account for the fact that one can define four resolutions ε^i, one for each coordinate. We shall suggest and develop generalizations of Eq.(29) in forthcoming works [22], such as $\delta\varepsilon/\varepsilon = -(e/\hbar c)\, W_{i\mu}\, (\varepsilon^i/\varepsilon)\, \delta x^\mu$, in which four 4-potentials are introduced instead of one, that can be related to the electroweak field. Then we also expect mass-charge relations of the kind of Eq.(37) to be true for them, but at the Z scale rather than the electron one in the case of the electroweak couplings. We suggest that the following relations hold for the α_1 and α_2 couplings:

$$3\,\alpha_{1Z}\,\mathbb{C}_Z = 2 \;\; ; \;\; 3\,\alpha_{2Z}\,\mathbb{C}_Z = 4 \quad . \tag{38}$$

From the current Z mass, $m_Z = 91.187 \pm 0.007$ GeV [25 and refs. therein], we get $\mathbb{C}_Z = 39.7558(3)$, so that we predict $\alpha_{1Z}^{-1} = 59.\,6338(4)$ and $\alpha_{2Z}^{-1} = 29.8169(2)$, in good agreement with (and more precise than) the currently measured values. But, more importantly, the two relations (38) imply $\alpha_{2Z} = 2\alpha_{1Z}$, and then fix the value of the weak angle at Z scale, or, in an equivalent way, the W/Z mass ratio:

$$(sin^2\theta)_Z = \frac{3}{13} \quad ; \quad \frac{m_W}{m_Z} = \sqrt{\left(\frac{10}{13}\right)} \quad , \tag{39}$$

once again in good agreement with the measured value, $(sin^2\theta)_Z = 0.2312(4)$ [25] or 0.2306(4) [26, (for $m_t = m_H = 100$ GeV)], (while 3/13 = 0.230769...). The mass of the W is then predicted to be $m_W = 79.978 \pm 0.007$ GeV (measured, 80.22(26) GeV [25,26]).

Another possible mass-charge relation was already suggested in Ref. [7]. It reads $\alpha_{0\infty}\,\mathbb{C}_t = 1$ and, under the conjecture that the bare coupling $\alpha_{0\infty} = 1/4\pi^2$ (see Refs. [7,8]), it defines a mass scale $m_t = 123.23(1)$ GeV, that we have suggested to identify with the top mass (since the top, being the most massive charged fermion in the minimal standard model, plays the role of a new reference scale for the high energy scale-relativistic transformation). Having precision values of the W and *top* masses, we can make a first rough prediction of the value of the Higgs boson mass [21,26] thanks to the combined effect of these particles in radiative electroweak corrections. We find $m_H = 410 \pm 50$ GeV from $m_t = 123.23$ GeV and $m_W = 79.978$ GeV.

4. Discussion and Prospect.

In the present contribution, we have given only the great lines of some possible ways for future developments of the theory of scale relativity. It is clear that the present stage of the theory is very provisional, since we still mix results coming from scale relativity with results from the current quantum theory. For example, our mechanism of mass generation for elementary charged fermions is obtained by a cancellation between terms coming from the assumed new Lorentzian nature of dilation laws and radiative correction terms coming from current QED. But in Sec.3.3 we have presented first hints of a generalization of scale relativity capable of supplying the electromagnetic field from the very nature of the micro-space-time, so that a really self-consistent approach can be expected in the near future in the scale-Galilean case [22], (while the generalization of the full structure of scale + motion relativity to non-linear transformations and complete account of fields appears as a big task). The important point is that, even at this very primary stage, the theory seems to be able to provide us with new insights concerning the physical meaning of masses, charges and gauge invariance, which are achieved in terms of our theoretical predictions of Sec.3.4.

Acknowledgements. The author gratefully acknowledge Dr. J. Diaz-Alonso for very interesting discussions and for his kind invitation in this Colloquium.

References

1. A. Einstein, *Annalen der Physik* **49**, 769 (1916). English translation in "*The Principle of Relativity*", (Dover publications), p. 109-164.
2. R.P. Feynman and A.R. Hibbs. *Quantum Mechanics and Path Integrals* (MacGraw-Hill, 1965).
3. S.S. Schweber, *Rev. Mod. Phys*. **58**, 449-508 (1986).
4. B. Mandelbrot. *The Fractal Geometry of Nature*. Freeman, San Francisco (1982).
5. L. Nottale and J. Schneider, *J. Math. Phys*. **25**, 1296-1300 (1984).
6. L. Nottale, *Int. J. Mod. Phys.* **A4**, 5047-5117 (1989).
7. L. Nottale. *Fractal Space-Time and Microphysics: Towards a Theory of Scale Relativity* (World Scientific, 1993).
8. L. Nottale, in *Chaos, Solitons and Fractals*, **3**, vol. 4, Special Issue on *Quantum Mechanics and Chaotic Dynamics*, in press (1994).
9. L. Nottale, in Proceedings of "*First International Conference on Complex Systems in Computational Physics*", (Buenos-Aires), *Chaos, Solitons and Fractals*, in press (1994).
10. G.N. Ord, *J. Phys. A: Math. Gen*. **16**, 1869-1884 (1983).
11. L. Nottale, *Int. J. Mod. Phys.* **A7**, 4899-4936 (1992).
12. E. Nelson, *Phys. Rev*. **150**, 1079-1085 (1966).
13. E. Nelson, *Quantum Fluctuations*. (Princeton Univ. Press, 1985).
14. D. Dohrn and F. Guerra, *Phys. Rev.* ***D*31**, 2521-2524 (1985).
15. T. Zastawniak, *Europhys. Lett*. **13**, 13-17 (1990).
16. M. Serva, *Ann. Inst. Henri Poincaré-Physique théorique*, **49**, 415-432 (1988).
17. B. Gaveau, T. Jacobson, M. Kac & L.S. Schulman, *Phys. Rev. Lett*. **53**, 419-422 (1984).
18. J.M. Levy-Leblond, *Am. J. Phys*. **44**, 271-277 (1976).
19. A.J. Buras, J. Ellis, M.K. Gaillard, D.V. Nanopoulos, *Nucl. Phys.* ***B***135, 66-92 (1978);
20. U. Amaldi, W. de Boer, & H. Fürstenau, *Phys. Lett.* **B 260**, 447 (1991).
21. W. Marciano, *Ann. Rev. Nucl. Part. Sci*. **41**, 469-509 (1991).
22. L. Nottale, in preparation.
23. H. Weyl, *Sitz. Preus. Akad. d. Wiss.* (1918). English translation in "*The Principle of Relativity*", (Dover publications), p. 201-216.
24. P.A.M. Dirac, *Proc. Roy. Soc. Lond.* **A 333**, 403-418 (1973).
25. V.A. Novikov, L.B. Okun, & M.I. Vysotsky, *CERN Preprint-TH.6943/93* (1993).
26. P. Langacker, M. Luo, & A.K. Mann, *Rev. Mod. Phys*. 64, 87-192 (1992).