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Scale Relativity and Schrödinger's Equation

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Abstract. The theory of scale relativity generalizes the application domain of Einstein's principle of relativity to scale transformations of space-time resolutions. In this theory, we no longer assume that the space-time continuum is differentiable, this implying its fractal character. Both classical and quantum laws may emerge from a unique, more profound, scale law. The effects of nondifferentiability (complex nature of wave function, new terms in differential equations of mean motion) are accounted for by a scale-covariant derivative that transforms the equations of classical mechanics into the Schrödinger equation. Using an intermediate description in terms of a "fractal potential", we finally establish the m^{-1} dependence of the Compton-de Broglie wavelength.

1. INTRODUCTION

The theory of scale relativity [1] is a new approach to understanding quantum mechanics, and more generally physical domains involving scale laws, such as cosmology [1,2] and chaotic systems [1–4]).

It is based on a generalization of Einstein's principle of relativity to scale transformations. Namely, we redefine space-time resolutions as characterizing the state of scale of reference systems, in the same way as velocity characterizes their state of motion. Then we require that the laws of physics apply whatever the state of the reference system, of motion (principle of motion-relativity) and of scale (principle of scale-relativity). The principle of scale-relativity is mathematically achieved by the principle of scale-covariance, requiring that the equations of physics keep their simplest form under transformations of resolutions [1,5,6].

It is well-known that the geometrical tool that implements Einstein's general motion-relativity is the concept of Riemannian, curved space-time. In a similar way, the concept of *fractal space-time* [1,5,20] also independently introduced by Ord [21] and El Naschie [22], is the geometric tool adapted to construct the new theory. We use here the word "fractal" in its general meaning [7], denoting a set that shows structures at all scales and is thus explicitly resolution-dependent. More precisely, one can demonstrate [1,8] that the D_T-measure of a continuous, almost everywhere nondifferentiable set of topological dimension D_T is a function of resolution, $\mathcal{L} = \mathcal{L}(\varepsilon)$, and diverges when resolution tends to zero, $\mathcal{L}(\varepsilon) \to \infty$ when $\varepsilon \to 0$. In such a framework, resolutions are considered to be inherent to the description of the new, fractal, space-time. A new physical content may also be given to the concept of particles in this theory: the various properties of "particles" can be reduced to the geometric structures of the (fractal) geodesics of such a space-time [5,1,9].

Three levels of such a theory have been considered. (i) A "Galilean" version corresponding to standard fractals with constant fractal dimensions, and where dilations laws are the usual ones [1,8]. This theory provides us with a new foundation of quantum mechanics from first principles. (ii) A special scale-relativistic version that implements in a more general way the principle of scale relativity. It yields new dilation laws of a Lorentzian form, that imply to re-interpret the Planck length-scale as a lower, impassable scale, invariant under dilations [6,1]. The predictions of such a theory depart from that of standard quantum mechanics at large energies [1,6,4,10]. (iii) The third level, "general scale-relativistic" version of the theory deals with non-linear scale laws and accounts for the coupling between scale laws and motion laws [9,10]. It yields a new interpretation of gauge invariance and allows one to get new mass-charge relations that solve the scale-hierarchy problem [9,10].

The aim of the present letter is to describe in a precise way the new scale-relativistic foundation of quantum mechanics. We shall focus here on three particular points: (i) the emergence of both classical laws and quantum laws from single, more fundamental scale laws; (ii) the need for a complex number formalism that takes its origin in a symmetry breaking of the local time reflection invariance ($dt \to -dt$) arising from the giving up of differentiability; (iii) the demonstration and meaning of the Compton - de Broglie formula.

2. FROM SCALE EQUATIONS TO SCALE-COVARIANT DERIVATIVE

Assume that space is continuous but nondifferentiable, while time remains classical. This corresponds to the non-relativistic situation (from the viewpoint of motion laws) to which we restrict ourselves in the present paper. The case of a full fractal, nondifferentiable space-*time* has been treated elsewhere [9, 10, 11]. Consider a small increment dX^i of the nondifferentiable 3-coordinate along one of the geodesics of the fractal space. Giving up differentiability has three main consequences.

2.1. *Fractal behavior*

Strictly, the nondifferentiability of the coordinates means that the velocity $V = dX/dt$ is no longer defined. However, continuity and nondifferentiability implies scale-divergence [1,8]. Therefore the basis of our method consists in replacing the classical velocity by a function that depends explicitly on resolution, $V = V(\varepsilon)$ [1, Chap. 5.3, Fig. 5.6]. Only $V(0)$ is undefined, while $V(\varepsilon)$ is now defined for any non-zero ε. The new scale-dependence of the velocity forces us to complete the standard equations of physics by differential equations of scale. The simplest possible equation that one can write for V is a first order, renormalization-group-like differential equation, written in terms of the dilatation operator $d/dln\varepsilon$ [8], in which the infinitesimal scale-dependence of V is determined by the "field" V itself, namely:

$$\frac{dV}{dln\varepsilon} = \beta(V) \ , \tag{2.1}$$

The β–function here is *a priori* unknown, but we can use the fact that $V < 1$ (in motion-relativistic units) to expand it in terms of a Taylor expansion. One obtains:

$$\frac{dV}{dln\varepsilon} = a + b \ V + O(V^2) \ , \tag{2.2}$$

where a and b are "constants" (independent of ε, but possibly dependent on space-time coordinates). Setting $b = -\delta$ and $a = \nu\,\delta$, we obtain the solution of this equation under the form:

$$V = \mathrm{v} + k\,\varepsilon^{-\delta}\,, \tag{2.3}$$

where k is an integration "constant" (independent of ε). From dimensional analysis, we can write it under the form $k = \zeta\,\lambda^{\delta}$, with $\zeta = \zeta(t)$ dimensionless, $<\zeta^2> = 1$ and λ a constant length-scale. We get:

$$V = \mathrm{v} + \zeta\,\left(\frac{\lambda}{\varepsilon}\right)^{\delta}. \tag{2.4}$$

We recognize here a typical fractal behavior with fractal dimension $D = D_T + \delta$, where D_T is the topological dimension (= 1 here, since our description concerns displacements along geodesical curves). Namely, at large scales $\varepsilon >> \lambda$, the velocity shows a classical (i.e., scale-independent) behavior, $V \approx \mathrm{v}$, while at small scales $\varepsilon << \lambda$, it shows a power-law, scale-divergent behavior $V \approx \zeta\,(\lambda/\varepsilon)^{\delta}$. The transition scale λ (that will be interpreted in what follows as the Compton scale) thus stands out as a fractal-nonfractal transition scale (that takes place not in space, but in the new resolution dimension).

The resolution ε in the above formula is a space-resolution, $\varepsilon = \delta X$. We can relate it to time-resolution by writing (2.4) in the asymptotic domain $\varepsilon << \lambda$ under the form:

$$\frac{\delta X}{c\delta t} \approx \left(\frac{\lambda}{\delta X}\right)^{D-1}. \tag{2.5}$$

This provides us with a fundamental, well-known formula on fractals:

$$\delta X^{D} = \lambda^{D-1}\, c\delta t\,. \tag{2.6}$$

By reinserting this result in (2.4), we obtain the following expression (where we have reinserted the indices) for the elementary displacement in terms of time-resolution [1,8]:

$$dX^{i} = v^{i}\,dt + \lambda^{1-1/D}\,\zeta^{i}\,(cdt)^{1/D}\,, \tag{2.7}$$

where we have identified the time resolution with the time differential element (see below).

As we shall see in what follows, the first term yields classical physics while the second is one of the source of the quantum behavior. In our theory, they are both present whatever the scale, but the "classical" term is dominant at large scale while the "quantum" term is dominant at small scales (see Figure 1). Then in our approach the quantum and the classical laws are irreductible to each other, but both find their origin in a single, more profound, scale-dependent description whose equations take the form given by Eq. (2.1) in the simplest case. In the special case of fractal dimension $D = 2$ (see below), the time transition is easily identified with the de Broglie time-scale $\tau \approx c\lambda / v^2 = \hbar/m\, v^2$, by writing (2.7) under the form $dX = v\, dt\ [1 + \zeta\ (c\lambda / v^2 dt)^{1/2}]$. (See Ref. [8] for a more complete description of the quantum / classical transition in our framework).

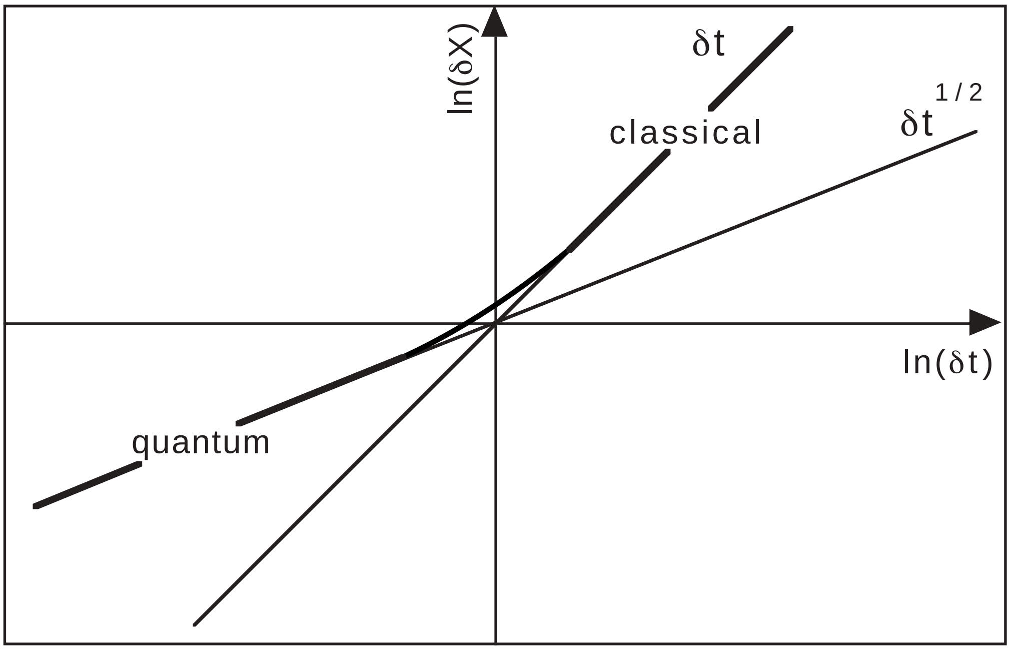

Fig. 1. Schematic description of the variation with scale of the two components of the elementary displacement in a nondifferentiable, fractal space-time (geodesics of fractal dimension 2, see text) The transition betwen the two regimes is identified with the de Broglie time scale, $\tau = \hbar/E$.

2.2. *Infinity of geodesics*

Because of nondifferentiability (and of the subsequent fractal character of space), there will be an infinity of fractal geodesics that relate any couple of points in the fractal space [5,1]. This implies jumping to a statistical description. We can decompose dX^i in terms of a mean, $<dX^i> = dx^i = v^i\, dt$, and a fluctuation respective to the mean, $d\xi^i$ (such that $<d\xi^i> = 0$ by definition):

$$dX^i = dx^i + d\xi^i \quad . \tag{2.8}$$

We recover the form of Eq.(2.7), and we can identify the fluctuation $d\xi^i$ with the "fractal" term $\lambda^{1-1/D}\, \zeta^i\, dt^{1/D}$.

2.3. *Two-valuedness of time derivative and velocity vector*:

The nondifferentiable nature of space-time implies an even more dramatic consequence, namely, a breaking of *differential* time reflection invariance. Consider indeed the usual definition of the derivative of a given function with respect to time:

$$\frac{df}{dt} = \lim_{dt\to 0} \frac{f(t+dt)-f(t)}{dt} = \lim_{dt\to 0} \frac{f(t)-f(t-dt)}{dt} \quad . \tag{2.9}$$

The two definitions are equivalent in the differentiable case. One passes from one to the other by the transformation $dt \to -dt$ (time reflection invariance at the infinitesimal level). In the nondifferentiable situation considered here, both definitions fail, since the limits are no longer defined. The scale-relativistic method solves this problem in the following way.

We attribute to the differential element dt the new meaning of a variable, identified with a time-resolution, $dt = \delta t$ ("substitution principle"). The passage to the limit $dt \to 0$ is actually devoid of physical meaning (an infinite energy would be needed to really perform a measurement at zero time resolution interval). The physics is now in the behavior of the function during the "zoom" operation on δt. The two functions f'_+ and f'_- are now defined as explicit functions of t and of dt:

$$f'_+(t, dt) = \frac{f(t+dt)-f(t)}{dt} \quad ; \quad f'_-(t, dt) = \frac{f(t)-f(t-dt)}{dt} \quad . \tag{2.10}$$

When applied to the space variable, we get for each geodesic two velocities that are fractal functions of resolution, $v_+(t, dt)$ and $v_-(t, dt)$. In order to go back to the classical macroscopic domain, we smooth out each geodesic with balls of radius larger than τ (the fractal / non fractal transition), then we take the average on the whole set of geodesics. We get two mean velocities $v_+(t) = <v_+(t, dt>>\tau)>$ and $v_-(t) = <v_-(t, dt>>\tau)>$, but after this double averaging process, there is no reason for these two velocities to be equal, contrarily to what happens in the classical, differentiable case [12,13].

In summary, while the concept of velocity was classically a single concept, we must introduce, if space-time is nondifferentiable, two velocities instead of one even when going back to the classical domain. Such a two-valuedness of the velocity vector is a new, specific consequence of nondifferentiability that has no standard counterpart (in the sense of differential physics), since it finds its origin in a breaking of the symmetry ($dt \rightarrow -dt$). Such a symmetry was considered self-evident up to now in physics (since the differential element dt disappears when passing to the limit), so that it has not been analysed on the same footing as the other well-known symmetry. Note that it is actually different from the time reflection symmetry T, even though infinitesimal irreversibility implies global irreversibility.

Now, at the level of our description, we have no way to favor v_+ rather than v_-. Both choices are equally qualified for the description of the laws of nature. The only solution to this problem is to consider both the forward ($dt > 0$) and backward ($dt < 0$) processes together. The number of degrees of freedom is doubled with respect to the classical, differentiable description (6 velocity components instead of 3).

A simple and natural way to account for this doubling of the needed information consists in complex numbers and the complex product [1,8]. As we shall recall hereafter, this is the origin of the complex nature of the wave function in quantum mechanics, since the probability amplitude is defined in terms of the complex action that is naturally introduced in such a theory. But one can demonstrate that the complex calculus is nothing but a particular choice of representation, that achieves the simplest description. Namely, using a different product would introduce additional terms in the Schrödinger equation, as we shall demonstrate in a forthcoming work. Note also that the new complex process, *as a whole*, recovers the fundamental property of microscopic reversibility.

We are then led to write:

$$dX_{\pm}^{\ i} = dx_{\pm}^{\ i} + d\xi_{\pm}^{\ i} \ , \qquad (2.11)$$

respectively for the forward process (+) and backward process (–). From our above discussion, the fluctuations $d\xi_{\pm}$ writes:

$$< \frac{d\xi^i_\pm}{dt} \frac{d\xi^j_\pm}{dt} > \; = \; \pm \, \delta^{ij} \; c^2 \left(\frac{\lambda}{cdt} \right)^{2-2/D} , \qquad (2.12)$$

This relation is invariant under translations and rotations in space between Cartesian coordinate systems. In the special, “Galilean” scale-relativistic case that we consider here, the scale-invariant is the fractal dimension itself. Indeed, as first demonstrated by Feynman [14], then confirmed using a fractal description by Abbott and Wise [15] and other authors (see [1] and Refs. therein), the fractal dimension of typical quantum mechanical paths is $D = 2$. We shall reduce our discussion in what follows to this particular case. See [4,10] for a discussion of the case $D \neq 2$ and [6,1] for D variable). When $D = 2$, Eq. (2.12) becomes:

$$<d\xi^i_\pm \; d\xi^j_\pm> \; = \pm \, \lambda \, \delta^{ij} \; cdt \quad . \qquad (2.13)$$

We can now jump to the second step of the fractal-space description, by constructing the *covariant derivative* that describes the combined effects of the new displacement laws and scale laws. We define mean forward (+) and backward (–) derivatives, which, once applied to x^i, yield the above *forward and backward mean velocities*

$$\frac{d_+ x^i(t)}{dt} \; = \; v^i_+ \;\; ; \quad \frac{d_- x^i(t)}{dt} \; = v^i_- \qquad (2.14)$$

The averaging is here taken on the family of geodesics. As a consequence the Born statistical interpretation of quantum mechanics will be ensured from the very beginning of our construction, since the "particle" can be identified with one random geodesic among their infinite set (more generally, with the subset of the geodesics that share the geometric properties that correspond to a given measurement result). The forward and backward derivatives of Eq.(2.14) can be combined in terms of a *complex* derivative operator [1],

$$\frac{\hat{d}}{dt} \; = \; \frac{(d_+ + d_-) - i \, (d_+ - d_-)}{2dt} \quad , \qquad (2.15)$$

which, when applied to the position vector, yields a complex velocity

$$\mathcal{V}^i \; = \; \frac{\hat{d}}{dt} \, x^i \; = \; V^i - i \;\, U^i \; = \; \frac{v^i_+ + v^i_-}{2} \; - \; i \; \frac{v^i_+ - v^i_-}{2} \quad . \qquad (2.16)$$

Consider a function $f(\boldsymbol{X},t)$, and expand its total differential to second order. We get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{\nabla} f \cdot \frac{d\boldsymbol{X}}{dt} + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \frac{dX_i\, dX_j}{dt} \quad . \tag{2.17}$$

We may now compute the forward and backward derivatives of f. In this averaging procedure, the mean value of dX_i/dt amounts to $d_\pm x_i/dt = v_{\pm i}$, while $<dX_i\, dX_j>$ reduces to $<d\xi_{\pm i}\; d\xi_{\pm j}>$, so that the last term of Eq. (2.17) amounts to a Laplacian thanks to Eq. (2.13). We obtain

$$d_\pm f/dt = (\partial/\partial t + v_\pm . \boldsymbol{\nabla} \pm \tfrac{1}{2} \lambda c \;\Delta\;) f \quad . \tag{2.18}$$

By combining them we get our final expression for the complex scale-covariant derivative [1]:

$$\frac{\acute{d}}{dt} = \frac{\partial}{\partial t} + \mathcal{V} \;.\; \boldsymbol{\nabla} - i \;\tfrac{1}{2} \lambda c \;\Delta \;. \tag{2.19}$$

We now apply the *principle of scale covariance*, and postulate that the passage from classical (differentiable) mechanics to the new nondifferentiable mechanics that is considered here can be implemented by a unique prescription: Replace the standard time derivative d/dt by the new complex operator $\acute{d}/dt$. As a consequence, we are now able to write the equation of the geodesics of the fractal space under its covariant form:

$$\frac{\acute{d}^2}{dt^2} x^i = 0 \;. \tag{2.20}$$

As we shall recall hereafter (and as already demonstrated in Refs. [1,8,4]), this equation amounts to the free particle Schrödinger equation.

The last step in our construction consists in writing the “field” equations, i.e., the equations that relate the geometry of space-time to its matter-energy content. At the very simplified level of description that is considered here, there is only one geometrical free parameter left in the expression of the scale-covariant derivative, namely the length-scale λ. We shall demonstrate below that λ must be the Compton length of the particle considered, i.e., $\lambda = \hbar / mc$.

3. SCALE-RELATIVITY AND SCHRODINGER EQUATION: LAGRANGIAN APPROACH

Let us finally recall the main steps by which one may pass from classical mechanics to quantum mechanics using our scale-covariance [1,10-14]. We assume that any mechanical system can be characterized by a Lagrange function $\mathbb{L}(\boldsymbol{x}, \mathcal{V}, t)$, from which an action S is defined:

$$S = \int_{t_1}^{t_2} \mathbb{L}(\boldsymbol{x}, \mathcal{V}, t)\, dt \,. \tag{3.1}$$

Our Lagrange function and action are *a priori* complex since $\mathcal{V}$ is complex, and are obtained from the classical Lagrange function $L(\boldsymbol{x}, d\boldsymbol{x}/dt, t)$ and classical action S precisely by applying the above prescription $d/dt \rightarrow \hat{d}/dt$. The principle of stationary action, $\delta S = 0$, applied to this new action with both ends of the above integral fixed, leads to generalized Euler-Lagrange equations [1]:

$$\frac{\hat{d}}{dt} \frac{\partial \mathbb{L}}{\partial \mathcal{V}_i} = \frac{\partial \mathbb{L}}{\partial x_i} \,. \tag{3.2}$$

Other fundamental results of classical mechanics are also generalized in the same way. In particular, assuming homogeneity of space in the mean leads to defining a generalized *complex* momentum given by

$$\mathcal{P} = \frac{\partial \mathbb{L}}{\partial \mathcal{V}} \,. \tag{3.3}$$

If one now considers the action as a functional of the upper limit of integration in Eq. (3.1), the variation of the action yields another expression for the complex momentum, as well as a generalized complex energy:

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\nabla} S \quad ; \quad \mathcal{E} = -\partial S / \partial t \,. \tag{3.4}$$

We now specialize and consider Newtonian mechanics. The Lagrange function of a closed system, $L = \frac{1}{2} m\, \boldsymbol{v}^2 - \Phi$, is generalized as $\mathbb{L}(\boldsymbol{x}, \mathcal{V}, t) =$

$\frac{1}{2} m\mathcal{V}^2 - \Phi$. The Euler-Lagrange equation keeps the form of Newton's fundamental equation of dynamics

$$m \frac{\hat{d}}{dt} \mathcal{V} = -\nabla \Phi \quad , \tag{3.5}$$

but is now written in terms of complex variables and operator. In the free particle case ($\Phi = 0$), we recover the geodesics equation (2.20), $\hat{d}^2 x^i / dt^2 = 0$. The complex momentum $\mathcal{P}$ now reads:

$$\mathcal{P} = m \mathcal{V} \quad , \tag{3.6}$$

so that the complex velocity $\mathcal{V}$ is the gradient of the complex action, $\mathcal{V} = \nabla S/m$.

We may now *define* the wave function ψ as *another expression for the complex action S*,

$$\psi = e^{iS/m\lambda c} . \tag{3.7}$$

It is related to the complex velocity as follows:

$$\mathcal{V} = -i \lambda c \nabla (ln\psi) . \tag{3.8}$$

From this equation and (3.4), we get a demonstration of the *correspondence principle* for momentum and energy:

$$\mathcal{P} \psi = -i m \lambda c \nabla \psi \quad ; \quad \mathcal{E} \psi = i m \lambda c \; \partial\psi/\partial t . \tag{3.9}$$

Indeed we shall demonstrate at the end of this paper that $m\lambda c$ must be a constant ($= \hbar$). Note that (3.9) are now exact equations rather than a "correspondence". We have now at our disposal all the mathematical tools needed to write the complex Newton equation in terms of the new quantity ψ. It takes the form

$$i m \lambda c \frac{\hat{d}}{dt} (\nabla ln\psi) = \nabla \Phi . \tag{3.10}$$

Replacing $\hat{d}/dt$ by its expression (Eq. 2.19) yields after some standard calculations [1]:

$$m \, \frac{\widehat{d}}{dt} \mathcal{V} = -m \, \lambda \, c \boldsymbol{\nabla} \left\{ i \, \frac{\partial}{\partial t} \ln\psi + \frac{\lambda c}{2} \frac{\Delta\psi}{\psi} \right\} = -\boldsymbol{\nabla} \Phi \, . \qquad (3.11)$$

Integrating this equation yields the Schrödinger equation (since $\lambda = \hbar/mc$, see Sec. 4):

$$\frac{1}{2} \, (\lambda c)^2 \, \Delta\psi + i \, (\lambda c) \, \frac{\partial}{\partial t} \psi - \frac{\Phi}{m} \psi = 0 \quad . \qquad (3.12)$$

It is remarkable that, in our approach, we have obtained the Schrödinger equation without explicitly introducing a probability density, nor writing Kolmogorov equations, but as a mere expression of our complex Newtonian dynamics. The Born axiom, i.e. the fact that $\rho = \psi \, \psi^\dagger$ yields the density of probability to find the particle at a given position (more generally in a given state) is a direct consequence of our basic principle that the various (wave + corpuscle) properties of what we call "particle" can be reduced to the geometric properties of the infinite set of geodesics of the fractal space-time. This have been recently corroborated by numerical simulations of our basic mechanism made by Hermann [18], who recovered solutions to the Schrödinger equation without using it. Writing the imaginary part of Eq. (3.12) in terms of the real part $\boldsymbol{V}$ of the complex velocity $\mathcal{V}$ ($\boldsymbol{V}$ is identified in the classical limit with the classical velocity), one gets the equation of continuity:

$$\partial\rho/\partial t + div(\rho \boldsymbol{V}) = 0, \qquad (3.13)$$

which confirms the identification of ρ with the probability density.

Moreover, a "measurement", in such a frame of thought, is nothing but a sorting out of the geodesics. Namely, after the measurement there remains only the sub-set of the initial set of geodesics that share the geometric property given by the measurement. Von Neuman's axiom (wave function reduction), according to which, just after the measurement, the particle is in the state given by the measurement result, is therefore also automatically verified.

In this regard, our theory differs in an essential point from Nelson’s stochastic mechanics [12,13], in which the complex Schrödinger equation is a pasting of a real Newton equation and of *Fokker-Planck equations* for a Brownian diffusion process. This is an important remark, since it has been recently demonstrated [19] that Nelson's stochastic mechanics is in contradiction with quantum mechanics concerning multitime correlations.

The source of the disagreement being precisely the Fokker-Planck equation and the wave function reduction, our theory does not come under such a problem.

4. FRACTAL POTENTIAL AND ENERGY EQUATION

In order to obtain the expression for the only remaining unknown quantity, λ, (and then demonstrate that this must be the Compton length of the particle) let us reexpress the effect of the fractal fluctuation in terms of an effective “force”. We shall separate the two effects of nondifferentiability, namely, *doubling of time derivative* expressed in terms of complex numbers, and *fractalization*, expressed by the occurence of nonclassical second order terms in the total time derivative, then treat them in a different way. Once complex numbers introduced ($V \rightarrow \mathcal{V}$), we write the time derivative as an "incomplete" covariant derivative (which is nothing but the standard total derivative, but acting on complex quantities):

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{V}.\,\boldsymbol{\nabla} \quad . \tag{4.1}$$

The equation of a free particle still takes the form of Newton’s equation of dynamics, but including now a right-hand member:

$$\frac{d}{dt}\,\mathcal{V} = i\,\frac{\lambda c}{2}\,\Delta\,\mathcal{V} \,. \tag{4.2}$$

This right-hand member can be identified with a complex "fractal force" over m , so that:

$$\mathcal{F} = i\,\frac{mc\lambda}{2}\,\Delta\,\mathcal{V} \quad . \tag{4.3}$$

In our scale-relativistic, fractal-space-time approach, this "force" is assumed to come from the very structure of space-time, so that we can require that it must be universal, independent of the mass of the particle. Then $m\lambda c$ must be a universal constant:

$$m\lambda c = \hbar \quad . \tag{4.4}$$

This result provides us with a new definition of $\hbar$, and implies that λ must be the Compton length of the particle:

$$\lambda_c = \frac{\hbar}{mc} \quad . \tag{4.5}$$

Once the Compton length obtained, it is easy to get the de Broglie length, that arises from it through a Lorentz transform (see Ref. [8] for more detail). This result solves one of the most profound questions asked in the quantum realm, i.e., why is the Compton-de Broglie wavelength *inversely* proportional to mass-energy-momentum.

The force (4.3) derives from a complex "fractal potential":

$$\varphi = -i \, \frac{\hbar}{2} \, div\, \mathcal{V} = -\frac{\hbar^2}{2m} \, \Delta \, ln\psi \tag{4.6}$$

The introduction of this potential now allows us to derive the Schrödinger equation in a very fast way, by the Hamilton-Jacobi approach. Such a derivation explains the standard quantum mechanical "derivation" via the correspondence principle, since we stress once again that we no longer use a correspondence but instead strict equalities. We simply write the expression for the total energy, including the fractal potential plus a possible external potential $\boldsymbol{\Phi}$:

$$\mathcal{E} = \frac{\mathcal{P}^2}{2m} + \varphi + \boldsymbol{\Phi} \ , \tag{4.7}$$

then we replace $\mathcal{E}$, $\mathcal{P}$ and φ by their expressions (3.9) and (4.6). This yields

$$i \ \hbar \frac{\partial}{\partial t} \ ln\psi = \frac{(-i\,\hbar\boldsymbol{\nabla}\, ln\psi)^2}{2m} - \frac{\hbar^2}{2m} \, \Delta \, ln\psi + \boldsymbol{\Phi} \, , \tag{4.8}$$

which is nothing but the standard Schrödinger equation:

$$\frac{\hbar^2}{2m} \, \Delta \, \psi + i \ \hbar \frac{\partial}{\partial t} \ \psi - \boldsymbol{\Phi} \, \psi = 0. \tag{4.9}$$

More detail on the Hamilton-Jacobi approach and new results concerning the Dirac equation in scale relativity can be found in Ref. [17].

5. CONCLUSION

The aim of the present paper was to specify the concept of a “fractal space-time” and its geodesics, then its mathematical description. The only presently existing “space-time theories” are Einstein‘s special relativity theory, implying an absolute Minkowski space-time, then Einstein's generalized relativity theory, implying a relative Riemannian space-time. The basic idea in our construction consists in keeping the concepts of general relativity (space-time, geodesics, invariants, covariant derivative, "field" equations, ...), but not its mathematical tools, since they are founded on differentiability. Clearly the development of a fractal space-time theory is far from having reached the same level of elaboration. However, at the simple level of the theory that we have considered here, that of non-relativistic quantum mechanics, we can already identify the various elements of the description that are specific of such an approach.These elements are:

(i) A description of the laws that govern the elementary space-time displacements, including in particular the quantities that remain invariant in transformations of coordinates. In general relativity, the most important is the metric $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, that contains the gravitational potentials $g_{\mu\nu}$. In the case of a fractal space-time, an enlarged group of transformations must be considered, that includes resolution transformations. In the restricted, "galilean", scale-relativity theory only considered here, the scale invariants are $d\delta$ (i.e., the fractal dimension is constant) and $\Sigma_i (d\xi^i)^2$. But we recall that a more general, special scale-relativistic invariant (that takes a Lorentzian form) may be constructed [6,1], then generalized to non-linear scale transformations (see [9,10] for first hints).

(ii) A description of the effects of elementary displacements on other physical quantities. The power of the space-time / relativity approach is that all these effects can be calculated using a unique mathematical tool, the *covariant derivative*. This covariant derivative depends on the geometry of space-time. In Einstein's general relativity, it does not affect scalars, but only vectors and then tensors. It writes $D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\rho} A^\rho$: the geometry is described by the Christoffel symbols $\Gamma^\nu{}_{\mu\rho}$, (the “gravitational field”). In scale relativity, its effects concern yet scalars. The partial scale covariant derivative writes [9,10,16], $\mathcal{d}_k = \partial_k - \frac{1}{2} i\, \lambda c\, (\mathcal{V}_k / \mathcal{V}^2)\, \Delta$. We easily recover our scale-covariant total derivative (2.19) from the relation $\mathcal{d}/dt = \partial/dt + \mathcal{V}^k \mathcal{d}_k$. Note that one must be careful in working with these operators, concerning for example the Leibniz rule, since they combine *first and second order derivatives*. See [17] for a development of this new calculus.

(iii) Equations of motion and equations of the "field", i.e., the equations that constrain the geometries that are physically acceptable, relating them to their material-energetic content (in general relativity, Einstein's equations). Covariance implies that the *equations of geodesics* take the simplest possible form, that of free motion, $D^2x^\mu/ds^2 = 0$, i.e., the covariant acceleration vanishes, or, in other words, the velocity in the "free fall" frame remains constant (inertial laws). In general relativity it is developed as $d^2x^\mu/ds^2 + \Gamma^\mu{}_{\nu\rho}$ $(dx^\nu/ds)\,(dx^\rho/ds) = 0$, which generalizes Newton's equation, $d^2x^i/dt^2 = F^i/m$. In scale relativity, we have seen that the scale-covariant free motion equation, $\not{d}^2x^i/dt^2 = 0$ (Eq. 2.20) becomes, when developed, the Schrödinger equation [1] (and, in the motion-relativistic case, the Klein-Gordon [9,10,11,17] and Dirac equations [17]). Concerning the "field" equation, it is still in a very rough and simplified form, since its role is played at this level of the theory by the Compton relation $\lambda = \hbar / mc$. Even in such a simplified case, it already owns the expected property of relating geometry (as given by λ , identified with the fractal / non fractal transition on the new resolution axis) and matter (as given by the *inertial* mass m).

To conclude, we hope to have shown that the quantum behavior can be understood in this theory as manifestations of the nondifferentiable geometry of the micro-space-time, in the same way as gravitation is understood as the manifestations of the curvature of the large scale space-time in Einstein's general relativity.

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