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On the transition from the classical to the quantum regime in fractal space–time theory

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Abstract

In the scale-relativity theory, space–time is described as a nondifferentiable continuum and the trajectories as its geodesics. In such a space–time, the coordinates are defined as the sum of a 'classical part' that remains differentiable, and a fluctuating, 'fractal part', that is divergent and nondifferentiable. The nondifferentiable geometry has three minimal consequences, namely infinite number, fractality and irreversibility of geodesics. These three effects are accounted for by the introduction of three new terms in the total derivative acting on the 'classical part' of the coordinates. When it is written using this total derivative, Newton's equation is integrated in terms of a Schrödinger equation. Such an equation is therefore both classical and quantum. In the present paper, we use this property to analyze the specific roles played by each of the individual contributions, in order to shed some light on the complicated transition from the classical to the quantum regime.

1. Introduction

It has been discovered by Feynman [1] that the typical quantum mechanical paths (i.e., those that contribute in a main way to the path integral) are nondifferentiable and fractal. Namely, Feynman has proved that, although a mean velocity can be defined, no mean mean-square velocity exists at any point, since it is given by $\langle v^2 \rangle \propto \delta t^{-1}$. Although the term 'fractal' was coined only ten years later by Mandelbrot [2], one now recognizes in this expression the behavior of a curve of fractal dimension $D_F = 2$ [3]. Based on these premises, the reverse proposal, according to which the laws of quantum physics find their very origin in the fractal geometry of space–time [4–7], has been developed along three different and complementary approaches:

Ord and co-workers [8–10], extending the Feynman chessboard model, work in terms of probabilistic models, in the framework of the statistical mechanics of binary random walks.

The scale-relativity approach [6,11–13] is founded on a nondifferentiable continuous geometry constrained by the principle of relativity (extended to scale transformations of resolutions).

El Naschie has attempted to go still one step further, and to give up also continuity. This leads him to define a 'Cantorian' space–time [7,14], and to therefore use in a preferential way the mathematical tool of transfinite Topology

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i.e. wild topology as well as number theory. Some connections between the last two approaches have been studied in Ref. [15].

In the present short contribution, we shall consider the question of the way a physical system jumps from a classical to a quantum behavior in the scale-relativity framework. One of the goals of such a preparatory work is to study whether some difference can be found (in particular around the transition scale) between the standard quantum expectation and the fractal space–time description, which would allow one to put the new theory to the test.

2. Consequences of nondifferentiable geometry

Recall that the Schrödinger equation is obtained, in the scale-relativity approach, on the basis of three fundamental conditions, namely:

(1) The trajectories are in infinite number. This condition leads one to use a statistical, fluid-like description, in which the velocity $v(t)$ is replaced by a velocity field $v[x(t), t]$. The fundamental cause for this undeterminism of trajectories is the nondifferentiability of space–time, that implies its fractality [12,13], and the subsequent identification of the trajectories with its geodesics.

(2) The trajectories are fractal curves (of fractal dimension $D_F = 2$ in the critical case that leads to standard quantum mechanics). This comes directly from the fact that the space–time itself is nondifferentiable and therefore fractal, which implies the fractality of its geodesics. This leads to generalize the concept of velocity. Indeed, the fractal geometry implies that the velocity becomes an explicit function of the scale. Let us use a time interval δt as the scale variable. In the simplest case, the velocity will be a solution of a first order scale differential equation that reads [12,13]

$$\frac{\partial V}{\partial(\ln \delta t)} = \beta(V) = a + bV + \cdots \tag{1}$$

Such an equation can be considered as a differential fractal generator, where the coefficient b is related to the fractal dimension. Its solution is the sum of a standard, classical velocity field, and of a scale-dependent, divergent fluctuation field, $V[x(t, \mathrm{d}t), t, \mathrm{d}t] = v[x(t), t] + w[x(t, \mathrm{d}t), t, \mathrm{d}t]$, where we have identified the time interval δt with the time differential element $\mathrm{d}t$. The 'fractal part' w is a fractal fluctuation of zero mean, explicitly dependent on the scale variable $\mathrm{d}t$. Since we have no explicit knowledge of this fluctuation, we replace it by a stochastic variable such that $\langle w_k \rangle = 0$ and (in the case $D_F = 2$ that is only considered here)

$$\langle w_j \times w_k \rangle = \delta_{jk}\left(\frac{\lambda}{\delta t}\right). \tag{2}$$

We have set $c = 1$ in order to simplify the writing. By setting $\mathrm{d}x_k = v_k \mathrm{d}t$, $\mathrm{d}\xi_k = w_k \mathrm{d}t$, and $\mathrm{d}X_k = \mathrm{d}x_k + \mathrm{d}\xi_k$, this relation also reads $\langle \mathrm{d}\xi_j\, \mathrm{d}\xi_k \rangle = \lambda \delta_{jk} \mathrm{d}t$, while $\langle \mathrm{d}\xi_k \rangle = 0$. The coefficient λ will be identified with the Compton length $\lambda = \hbar/mc$ of the particle.

(3) The invariance under the reflection transformation ($\mathrm{d}t \leftrightarrow -\mathrm{d}t$) is broken. Recall that this fundamental condition, that leads to a twin path process [12,13,16], is not set as an axiom, but is actually a consequence of the new geometry of space–time. Recall that it cannot be obtained from the mere fractality, nor from the nondifferentiability of the only trajectories. It is a consequence of the nondifferentiability of space–time itself. Indeed, the basic method of scale-relativity amounts to replace the standard physical quantities (which are defined at the limit $\delta t \to 0$ in the differentiable case) by explicitly scale dependent quantities (this is the mathematical expression of the fractality, in its general meaning). Therefore, while in its standard definition the velocity does not exist any longer since the coordinate $x(t)$ is nondifferentiable, we replace $x(t)$ by a fractal coordinate $x(t, \delta t)$, and the standard velocity is replaced by

$$v_+[x(t, \delta t), t, \delta t] = \frac{x(t + \delta t, \delta t) - x(t, \delta t)}{\delta t}, \tag{3}$$

$$v_-[x(t, \delta t), t, \delta t] = \frac{x(t, \delta t) - x(t - \delta t, \delta t)}{\delta t}. \tag{4}$$

Since one needs two points to define a velocity, there are two definitions instead of one, which are related by the reflexion on the scale variable ($\delta t \leftrightarrow -\delta t$). Note also that, contrarily to the interpretation given in our initial work on this subject ([12, Chapter 5]), these two velocities cannot be identified with Nelson's forward and backward velocities of stochastic mechanics. Indeed, the Nelson twin process [17] corresponds to a reversal of time, while here we deal with a reversal of the scale variable δt, which is an independent additional variable in the scale-relativity approach. Namely, we may write $v_-[x(t,\delta t),t,\delta t] = v_+[x(t,-\delta t),t,-\delta t]$, which means that we actually deal with a unique function, but which has no reason to be symmetrical with respect to the second variable δt. However, since the dilatation operator is $\partial/\partial \ln|\delta t|$, we need to define a twin process when jumping to logarithmic variables (which are the natural variables for describing scale transformations).

Such a twin process plays also a central role in Ord's statistical approach to the fractal space–time description [18]. In the scale-relativity approach it is a consequence of the nondifferentiable geometry itself, that supersedes fractality.

A new generalization of the velocity is involved by this fundamental symmetry breaking. We now deal with two fractal velocity fields,

$$V_+ = v_+[x(t),t] + w_+[x(t,\mathrm{d}t),t,\mathrm{d}t], \tag{5}$$

$$V_- = v_-[x(t),t] + w_-[x(t,\mathrm{d}t),t,\mathrm{d}t], \tag{6}$$

each of them decomposed in terms of a 'classical part' (v_+, v_-), which is differentiable and independent of resolution, and of a 'fractal part' (w_+, w_-), explicitly depending on the resolution interval $\mathrm{d}t$ and divergent at the limit $\mathrm{d}t \to 0$.

3. The triple classical/quantum transition

Before going on, recall that these three conditions are only the minimal consequences of nondifferentiability and fractality, since far more general structures originating from nondifferentiable geometry are to be considered: they include, among other structures [19], (i) a new two-valuedness of velocity which is a consequence of nondifferentiability under the space derivative $\partial/\partial x$ and which leads to the introduction of bispinor wave functions that are solution of the Dirac equation [20], and (ii) the account of space–time dependent resolutions that lead to a geometric interpretation of the gauge fields themselves [21,22].

In the present contribution, we consider only the simplest transition, i.e. that from the classical regime to the nonrelativistic quantum mechanical regime. However, since even in this simplest case the nondifferentiability and the fractality of a space–time continuum manifest themselves under three consequences, we expect this transition to be a complicated one.

Indeed, it is a combination of the passage from a deterministic velocity on a given trajectory to a velocity field (defined on the infinity of potential trajectories), then to a fractal velocity field (i.e., that is explicitly dependent on the resolution interval), and finally to a twin fractal velocity field, namely,

$$v(t) \to v(x(t),t) \to V[x(t,\mathrm{d}t),t,\mathrm{d}t] \to \{V_+[x(t,\mathrm{d}t),t,\mathrm{d}t], V_-[x(t,\mathrm{d}t),t,\mathrm{d}t]\}. \tag{7}$$

Reversely, the full transition of a system from the quantum to the classical regime becomes effective only provided all of the three new properties have disappeared (either in a change of scale or in a change of the transition scale, i.e., of velocity, temperature, etc...).

4. Total derivative in nondifferentiable geometry

The fundamental mathematical tool of scale-relativity amounts to include the new effects into the writing of a more complete expression for the total time derivative [12]. The three conditions imply the appearance of three additional terms:

{1} The condition (1), which tells us that the various physical quantities are functions of x and t, implies to replace $\mathrm{d}/\mathrm{d}t$ by the standard 'Eulerian' total derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + v \cdot \nabla. \tag{8}$$

{2} The condition (2), which tells us that the fractal fluctuations $\mathrm{d}\xi_k$ are now differential elements of order 1/2, leads one to introduce terms of second order in the total derivative. Indeed, let us consider the Taylor expansion up to order two of the derivative of a physical quantity f:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial X_k}\frac{dX_k}{dt} + \frac{1}{2}\frac{\partial^2 f}{\partial X_j \partial X_k}\frac{dX_j dX_k}{dt}. \quad (9)$$

In this expression, the differentials dX_k are the sum of a classical part dx_k and of a fractal fluctuation $d\xi_k$ of zero average. Let us consider the 'classical part' of this expression. By definition, $\langle dX_k \rangle = dx_k$, so that the second term is reduced to $v \cdot \nabla f$. Now concerning the term $dX_j dX_k/dt$, it is usually infinitesimal, but when the fractal dimension is $D_F = 2$, its 'classical part' reduces to $\langle d\xi_j d\xi_k \rangle / dt$ and it is therefore finite. Thanks to Eq. (2), the last term amounts to a Laplacian, and we obtain

$$\left\langle \frac{df}{dt} \right\rangle = \left(\frac{\partial}{\partial t} + v \cdot \nabla + \frac{1}{2}\lambda\Delta \right) f. \quad (10)$$

In other words, the effect of the second condition is to add second order derivative terms in differential equations. Due to these terms the total derivative $\langle df/dt \rangle$, despite its apparent form, should not be treated as a first order derivative, in particular as concerns the Leibniz rule for products of functions and composed functions: namely, the Leibniz rule for this derivative operator is a linear combination of the first and second order Leibniz rules (see [23] for the construction of a covariant tool allowing nevertheless to keep the form of the first order Leibniz rule, and [19] for another equivalent proposal).

{3} The condition (3), i.e. the two-valuedness of the velocity field that finds its origin in the nondifferentiability, renders the obtained description irreducible to classical, contrarily to the first two. Indeed, while the first condition is in common with the fluid description of hydrodynamics, and the second one with the description of Brownian motion, Markovian processes and random walks, there is no classical equivalent of such a twin process. The description may look similar to that of two coupled fluids, but here it is a unique 'fluid of trajectories' that is described by a twin velocity field. We have therefore suggested [12] to replace the double field (v_+, v_-) by a unique complex field

$$\widetilde{V} = V - iU = \frac{v_+ + v_-}{2} - i\frac{v_+ - v_-}{2}. \quad (11)$$

More generally, one introduces a twin classical derivative, $(d_+f/dt, d_-f/dt)$, from which we define a complex derivative operator [12]

$$\frac{d'}{dt} = \frac{1}{2}\left(\frac{d_+}{dt} + \frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt} - \frac{d_-}{dt}\right), \quad (12)$$

so that we have $d'x/dt = \widetilde{V}$. In a more recent work [19], we have shown that the choice of complex numbers can be justified as a simplifying representation, since the relation $i^2 + 1 = 0$ allows to suppress an infinite term in the equation of motion. Finally, when combining the effect of condition (2) (that leads to introduce second order terms in differential equations) and of condition (3) (that leads to jump from a real to a complex description), we have found [12] that the complex total derivative reads

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \widetilde{V} \cdot \nabla - i\frac{\lambda}{2}\Delta. \quad (13)$$

Finally, since $\widetilde{V} = V - iU$, the three minimal consequences of the nondifferentiable and fractal geometry are expressed by the appearance in the total derivative of three additional terms, namely, $V \cdot \nabla$, $-iU \cdot \nabla$and $-i(\lambda/2)\Delta$, so that it reads

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + V \cdot \nabla - iU\nabla - i\frac{\lambda}{2}\Delta. \quad (14)$$

5. Transition from Schrödinger to Newton equation

As shown in many previous works [12,13,16,19], if one now writes Newton's equation of dynamics (which is nothing but Einstein's equation of geodesics in the Newtonian limit when the potential is a gravitational one), in terms of the above total derivative, namely,

$$m\frac{d'\widetilde{V}}{dt} + \nabla\phi = 0, \quad (15)$$

one obtains after integration a Schrödinger equation

$$\frac{1}{2}\lambda^2\Delta\psi + i\lambda\frac{\partial}{\partial t}\psi - \frac{\phi}{m}\psi = 0. \tag{16}$$

In this equation $\psi = \exp(iS/S_0)$ is a mere redefinition of the action S (which is now complex since the velocity $\widetilde{V}$ is complex). The Schrödinger equation is obtained provided $S_0 = m\lambda$. The constant S_0 is introduced for simple dimensional reasons: it is nothing but $\hbar$ when the theory is applied to standard quantum mechanics, but it can also be generalized in other applications (see e.g. [12,16, Chapter 7]). Therefore ψ acquires its status of wave function only provided it is accompanied by the Compton relation $\hbar = m\lambda$ [19].

A third form of these equations can be obtained by separating the real and imaginary parts of the Schrödinger equation and by taking as new couple of variables the real part V of the complex velocity $\widetilde{V}$ and the squared modulus of the wave function $P = |\psi|^2$. We obtain a generalized Euler–Newton equation (including a 'quantum potential') and a continuity equation:

$$\left(\frac{\partial}{\partial t} + V\cdot\nabla\right)V = -\nabla\left(\frac{\phi}{m} - \frac{1}{2}\lambda^2\frac{\Delta\sqrt{P}}{\sqrt{P}}\right), \tag{17}$$

$$\frac{\partial P}{\partial t} + \mathrm{div}(PV) = 0. \tag{18}$$

The full transition from the quantum to the classical regime can now be made clear. Eq. (15) is both classical and quantum. When the three additional terms vanish, this is the standard deterministic equation of classical mechanics. When all of the three terms are present, its integral is the Schrödinger equation. Therefore a full study of the classical/quantum transition in the scale-relativity framework involves an analysis of the physical conditions under which these terms vanish. Such an analysis, concerning in particular its relation to decoherence and the role played by the de Broglie scale and by the thermal de Broglie scale, has been initiated in previous works ([12, Section 4.5;24, Section 4.5]). We also easily verify in the last form (Eqs. (17) and (18) that in the limit $\lambda \to 0$ we recover classical hydrodynamical-like equations: this corresponds to the case where only the first condition (infinity of possible trajectories described in terms of a velocity field) is fulfilled.

In what follows we shall focus on a study of the specific contribution of the two-valuedness of velocity, that leads one to jump from a real number to a complex number description.

6. A new prime integral of diffusion equations

The key condition for obtaining a genuine quantum behavior is actually the differential irreversibility condition (iii), i.e. the symmetry breaking of the reflection invariance under the transformation ($dt \leftrightarrow -dt$). Let us demonstrate this important point by studying what happens when this condition is released.

We start from only the two first conditions, namely,

(i) infinity of trajectories;
(ii) each trajectory is fractal with $D_F = 2$.

This means that we are now describing a standard diffusion process of the Brownian motion type. The elementary displacements on each trajectory are decomposed as:

$$dX = dx + d\xi \tag{19}$$

where

$$dx = V(x(t), t)dt, \tag{20}$$

and

$$\langle d\xi^2\rangle = \lambda dt \tag{21}$$

i.e., $\lambda = 2D_{\mathrm{diff}}$ is twice the 'diffusion coefficient' in a diffusion interpretation of such a process. The velocity field $V(x(t), t)$ is now real. The effect of such a behavior on the dynamics can be described in terms of a covariant derivative that writes:

$$\frac{D}{dt} = \frac{\partial}{\partial t} + V\cdot\nabla + \frac{1}{2}\lambda\Delta. \tag{22}$$

Contrarily to what happens when the differential irreversibility condition is assumed, the second order contribution $(1/2)\lambda\Delta$ is now real instead of imaginary. Namely, we jump from the quantum case to this reduced situation by making the replacement $-i\lambda \to \lambda$, and therefore $\lambda^2 \to -\lambda^2$. In terms of this total derivative operator, Newton's fundamental equation of dynamics keeps its usual form:

$$m\frac{D}{dt}V = -\nabla\phi, \tag{23}$$

where ϕ is a potential energy. Since it preserves the form of equations, one can identify the derivative operator D/dt with a 'covariant' derivative.

One can define a Lagrange function $L(x, V, t)$ and an action S such that $dS = L dt$, which are both real, since there is no longer any two-valuedness of the velocity vector. Let us now set

$$\varphi = e^{S/S_0}, \tag{24}$$

where S_0 must be introduced for dimensional reasons. The function φ is a real function, that plays a role similar to that played by the complex wave function ψ. When one considers the action as a function of coordinates, one obtains $v = \nabla S/m$, so that one can replace V in Eq. (23) by

$$V = \frac{S_0}{m}\nabla \ln \varphi. \tag{25}$$

Therefore Eq. (23) becomes

$$-S_0\left[\frac{\partial}{\partial t}(\nabla \ln \varphi) + \left(\frac{S_0}{m}(\nabla \ln \varphi \cdot \nabla)(\nabla \ln \varphi) + \frac{\lambda}{2}\Delta(\nabla \ln \varphi)\right)\right] = \nabla\phi. \tag{26}$$

Under the condition $S_0 = \lambda m$, this expression can be greatly simplified thanks to the identity [12]

$$2\nabla \ln \varphi \cdot \nabla(\nabla \ln \varphi) + \Delta(\nabla \ln \varphi) = \nabla\left(\frac{\Delta\varphi}{\varphi}\right). \tag{27}$$

We obtain:

$$\nabla\left(\lambda\frac{\partial \ln \varphi}{\partial t} + \frac{1}{2}\lambda^2\frac{\Delta\varphi}{\varphi}\right) = -\frac{\nabla\varphi}{m}. \tag{28}$$

Therefore we find a general prime integral of the motion equations, that reads:

$$\frac{1}{2}\lambda^2\Delta\varphi + \lambda\frac{\partial\varphi}{\partial t} = \left(\frac{A(t) - \phi}{m}\right)\varphi. \tag{29}$$

Though this equation may look like a Schrödinger equation, it has actually very different properties. The function φ is real, (while the wave function ψ was complex), the sign of the potential is reversed, and the integration function $A(t)$ does not vanish since φ has no phase. As a consequence, such an equation is not at all structuring, contrarily to the standard Schrödinger equation. It remains a diffusion equation, without counterterms allowing stationary solutions. This result gives the proof that the two-valuedness of the velocity issued from nondifferentiability is the fundamental key condition that allows to obtain a genuine quantum behavior.

Let us conclude this section by an application of this result to standard hydrodynamics. If one takes a negative value for the coefficient λ and sets $\nu = -\lambda/2$ and $m = 1$, Eq. (23) becomes the Navier–Stokes equation with a coefficient of viscosity ν. This form of the Navier–Stokes equation is obtained in every situations where $\nabla p/\rho$ is a gradient (achieved when there exists an univocal link between the pressure p and the density ρ, in particular in the isentropic case where $\nabla p/\rho = \nabla w$, the enthalpy by unit of mass, see [16]). Therefore Eq. (29), that becomes

$$\nu^2\Delta\varphi - \nu\frac{\partial\varphi}{\partial t} = \left(\frac{A(t) - \phi}{2}\right)\varphi, \tag{30}$$

is a prime integral of the Navier–Stokes equation in the case of a potential (irrotational) motion.

7. Conclusion

We have attempted in the present short contribution, to set the bases for an analysis of the classical to quantum transition in the scale-relativity approach. The transition is a complicated one, since it involves the combined vanishing of three conditions that can be deduced from the nondifferentiability of the space–time continuum: nondeterminism of trajectories, fractality ($D_F = 2$), and two-valuedness of velocity. These conditions manifest themselves by the appearance of three additional terms in the total derivative with respect to time. The term $V \cdot \nabla$is a fluid-like term that comes from the loss of determinism; the term $-iU \cdot \nabla$comes from the combination of nondeterminism and two-valuedness (described in terms of complex numbers), while the second order derivative term $-i(\lambda/2)\Delta$ comes from the combination of fractality and velocity two-valuedness.

We have particularly studied in this paper the influence of the velocity two-valuedness and of its representation in terms of complex numbers. We have shown that, in case this condition is not fulfilled, one still obtains a Schrödinger-like equation, but which is now real and in which the signs of the various terms are changed. As a consequence such an equation is far from structuring: on the contrary, it is describing a purely diffusing behavior. This demonstrates that the origin of the structuring behavior of quantum mechanics (described in terms of well-defined probability density distributions) is to be found in the doubling of the velocity field, whose origin is attributed in the scale relativity approach to the loss of the space–time differentiability.

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