GR IHP 06
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PART 3.

Relativistic star models, gravitational collapse and Black holes

(I) Newtonian star models

1. Static, spherically symmetric Euler-Poisson equations

Consider a perfect fluid, characterized by its mass density $\rho(t, x^\alpha)$ and velocity field $v(t, x^\alpha)$, whose motion or equilibrium configuration is determined by its pressure forces $p(t, x^\alpha)$ (with equation of state $p = p(\rho)$ for a "barotropic" fluid) and its own gravity field, represented by the gravitational potential $U(t, x^\alpha)$.

The equations of motion are, given an equation of state: the continuity, Euler et Poisson equations, which read, in an inertial frame & arbitrary coordinates:

$$(*) \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \; ; \; \frac{dv}{dt} \equiv \frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla U - \frac{1}{\rho}\nabla p \; ; \; \Delta U = 4\pi G\rho .$$

The fluid is "stationary" if no quantity depends on (universal) time t; "static" if $v = 0$.

The junction conditions at the surface Σ of the fluid can be read on $(*)$: if ρ is discontinuous, Poisson's equation imply that U and $\partial_i U$ are continuous across Σ; the continuity equation yields that the component of v orthogonal to Σ is zero (to cancel the delta-like component of $\nabla\rho$); Euler's equation finally yields that p is continuous and hence $p = 0$ on Σ.

The gravitational energy of the distribution is

$$W = \frac{1}{2}\int \rho U \, dV = -\frac{1}{8\pi G}\int (\nabla U)^2 dV \quad ; \; dV = \sqrt{\det e}\, dx^1 dx^2 dx^3$$

Use spherical coordinates $(r\theta\varphi)$ and look for spherical symmetric solutions of Poisson's equation. Outside matter, ie for $r \geq r_0$:

$$\Delta U = 0 \implies \frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} = 0 \implies U = -\frac{GM}{r} \qquad (r \geq r_0)$$

where GM depends on time only, and where we chose $U \underset{r\to\infty}{\longrightarrow} 0$.

Suppose now that the matter distribution is static; $\rho = \rho(r)$; $v = 0$. Equations (*) then reduce to:

$$(**) \quad \frac{dp}{dr} = -\rho\frac{dU}{dr} \; ; \quad \frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = 4\pi G\rho r^2 \quad ; \; r \leq r_0 .$$

which can be recast as:

$$(**) \quad \frac{dm}{dr} = 4\pi\rho r^2 \; ; \quad \frac{dU}{dr} = \frac{Gm}{r^2} \; ; \quad \frac{d\rho}{dr} = -\frac{\rho}{dp/d\rho}\frac{dU}{dr}$$

Given an equation of state, this system is integrated, with initial conditions:

$$m(0) = 0 \; ; \quad U(0) = U_0 \; ; \quad \rho(0) = \rho_0$$

The integration ends at $r = r_0$ where $p(r_0) = 0$; which defines the radius r_0 of the distribution; its mass is $M = m(r_0)$. U_0 is fixed a posteriori by imposing $U(r_0) = -GM/r_0$. Hence, given an equation of state, there $\exists$ a whole family of solutions, parametrized by the central density ρ_0.

If ρ = constant, integrating (**) is straightforward:

$$U(r) = -\frac{GM}{2r_0}\left(3 - \frac{r^2}{r_0^2}\right) \; ; \quad p(r) = \frac{GM\rho}{2r_0}\left(1 - \frac{r^2}{r_0^2}\right) \; ; \quad M = \frac{4}{3}\pi\rho r_0^3 .$$

the gravitational energy of the distribution is $W_{grav} = -\frac{3}{5}\frac{GM^2}{r_0}$

2. Polytropes & Lane-Emden equation

(for more details, see E. Gourgoulhon & P. Haensel) (Those are good models for stars)

The equation of state of a "polytrope" is $p = K\rho^\gamma$ where K & $\gamma = 1 + \frac{1}{n}$ are constants (n is the "polytropic index"; γ the "adiabatic" index).

Euler's equation ($p' = -\rho U'$) yields $\rho = \rho_0\theta^n$ with $\theta = \frac{C_1 - U}{K(n+1)}\rho_0^{-1/n}$ where $C_1 = -GM/r_0$ (junction condition) & where ρ_0 is introduced for convenience

Poisson's equation then reads: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$ and is called the "Lane-Emden" equation

where $r = \alpha\xi$ with $\alpha = \sqrt{\frac{K(n+1)}{4\pi G}}\rho_0^{\frac{1-n}{2n}}$. The initial conditions are: $\theta(0) = 1$ (so that ρ_0 is the central density), and $\frac{d\theta}{d\xi}\big|_0 = 0$ so that U is smooth at the origin.

Apart for $n = 0$, $n = 1$, $n = 5$, the Lane-Emden equation must be integrated numerically; For $n > 5$ $\theta_n(\xi)$ never vanishes so that the distribution is of infinite extent.

Once $\theta_n(\xi)$ is known, the radius is known: $r_0 = \alpha\bar{\xi}$ with $\theta_n(\bar{\xi}) = 0$.

The mass is : $M = 4\pi \int_0^{r_0} \rho r^2 dr$, which can be rewritten as, using the Lane-Emden equation :

$$M = 4\pi\rho_0 \alpha^3 \bar{\xi}\,\bar{\theta}' \qquad \text{with } \alpha = \sqrt{\frac{K(n+1)}{4\pi G}}\,\rho_0^{\frac{1-n}{2n}}$$

NB : if $n=3$, M does not depend on ρ_0 : this is the "Chandrasekhar mass" [numerical integration gives $M = 2.02 \frac{4}{\sqrt{\pi}} \left(\frac{K}{G}\right)^{3/2}$; Quantum mechanics is required to fix K (Chandra, Landau ; cf P. Haensel) and yields $M = 1.46\, M_\odot$].

Simple expressions can be obtained for the gravitational energy W_{grav} and the "internal energy" $W_{int} = \int \epsilon\, dV$ with $\epsilon = p/(\gamma-1)$

$$W_{grav} = -\frac{3}{5-n}\frac{GM^2}{r_0} \; ; \quad W = W_{int} + W_{grav} = \frac{3-n}{5-n}\frac{GM^2}{r_0}$$

(see E. Gourgoulhon for details).

3. The "isothermal sphere"

If the equation of state is $p = w\rho$ with ρ a constant, the Euler & Poisson eqns (**) yield :

$$\rho = \rho_0 e^{-U/w} \; ; \quad \Delta U = 4\pi G \rho_0 e^{-U/w}$$

which can also be cast into a Lane-Emden form : $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\Psi}{d\xi}\right) = e^{-\Psi}$ with $U \equiv w\Psi$, $r = \alpha\xi$ with $\alpha = \sqrt{\frac{w}{4\pi G\rho_0}}$ with IC : $\Psi(0)=0$; $\Psi'(0)=0$.

This equation is integrated numerically (see e.g. Binney & Tremaine) ; it turns out that the density has decreased by a factor ~ 2 at $\xi \sim 3$.

One notes that the ansatz $\Psi = 2\ln\xi - \ln 2$, called "singular isothermal sphere" solves the Lane-Emden equation ("singular" since $\rho = 2\rho_0/\xi^2$) and is a good approximation to the exact solution for large ξ.

(This is a good model for "globular clusters", see J. Silk)

Unless one invokes the fact that $p = w\rho$ is the equation of state of a perfect gaz at constant temperature, the qualification "isothermal" is not justified.

A justification of this qualification is provided by going from a fluid description to a *kinetic* description of the distribution.

Consider N identical particles of mass m, which are isolated (that is N and its total energy $\mathcal{E}$ are constant). Such a system, when in a stationary state, is described by a distribution function $f(R,v)$ such that $f\,d^3X d^3V$ represents the number of particles within the volume d^3X centered on X^α whose components V^α of their velocities $v \in [V^\alpha, V^\alpha + dV^\alpha]$ (X^α being the cartesian components of the position vector R in an inertial frame).

If the 2-point correlation function can be neglected, then f satisfies the Boltzmann equation : $v \cdot \nabla f - \frac{\partial f}{\partial v} \cdot \nabla U = 0$

(which is a direct consequence of Newton's law $ma = -\nabla U$, see eg Binney & Tremaine for its derivation).

The kinetic & gravitational energies of the distribution are :

$$\mathcal{E} = K + W \quad \text{with} \quad \begin{cases} K = \frac{m}{2}\int v^2 f(R,v)\, d^3X d^3V \\ W = \frac{1}{2}\int \rho U\, d^3X = -\frac{1}{8\pi G}\int (\nabla U)^2 d^3X \end{cases}$$

Now, the solution of Boltzmann's equation which extremizes the "entropy", defined as $S = -\int f \ln f\, d^3X d^3V$, $\mathcal{E}$ & N being constant, is easily found to be :

$$f(R,v) = \frac{\rho_0}{m}\left(\frac{\beta m}{2\pi}\right)^{3/2} e^{-\beta m\left(\frac{1}{2}v^2 + U\right)} \; , \quad \rho_0 \,\&\, \beta \text{ constants.}$$

From that we get $\rho = m\int f(R,v)\, d^3V$ and $\Delta U = 4\pi G\rho$ as :

$$\rho = \rho_0 e^{-\beta m U} \; , \quad \Delta U = 4\pi G \rho_0 e^{-\beta m U}.$$

Hence this kinetic description coincides with the description of a fluid with equation of state $p = w\rho$ with $w \equiv 1/\beta m$.

The kinetic energy is also readily obtained : $K = \frac{3N}{2\beta}$ which justifies the fact that $p = w\rho$ describes a perfect gaz with temperature $T \equiv k/\beta$ (k being Boltzmann's constant).

The gravitational energy is obtained by direct integration or, more easily, by means of the virial theorem : $2K + W = 0$.

Hence : $K = \frac{3}{2}NkT$; $W = -2K$; $\mathcal{E} = -K$

so that the "specific heat" of the distribution : $C_V = \left.\frac{d\mathcal{E}}{dT}\right)_{V,N}$, which characterizes the increase of energy when temperature rises is :

$C_V = -\frac{3}{2}Nk < 0$: Hence by loosing energy ($\mathcal{E} < 0$) the system heats up & contracts : "gravothermal catastroph

4. MacLaurin spheroids & Jacobi ellipsoids

The spherically symmetric solution of Laplace equation, $\Delta U = 0$, is $U = -\frac{GM}{r}$; the equipotentials $U = Ct$ are spheres. A way to find solutions of Laplace equation whose equipotentials are "spheroids" (that is, surfaces of equation $X^2+Y^2+\frac{Z^2}{1-e^2}=a^2$) is the following:

Go from cartesian (XYZ) coordinates to "spheroidal" ones $(\rho\,\theta\,\varphi)$ defined as:
$$X = \sqrt{\rho^2+a^2}\,\sin\theta\cos\varphi;\quad Y=\sqrt{\rho^2+a^2}\,\sin\theta\sin\varphi;\quad Z=\rho\cos\theta.$$
Since $\frac{X^2+Y^2}{\rho^2+a^2}+\frac{Z^2}{\rho^2}=1$, the surfaces $\rho = Ct$ are spheroids.

(the surface $\rho=0$ is the disk of radius a in the $Z=0$ plane;
the origin $X=Y=Z=0 \iff \rho=0$ & $\theta=0$
$\rho=0$, $\theta=\frac{\pi}{2}$ is the circle $X^2+Y^2=a^2$ in the $Z=0$ plane.)

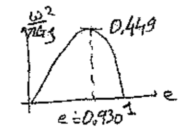
In the $(\rho\theta\varphi)$ coordinate system the euclidean metric reads:
$$dl^2 = \frac{\rho^2+a^2\cos^2\theta}{\rho^2+a^2}\,d\rho^2 + (\rho^2+a^2\cos^2\theta)\,d\theta^2 + (\rho^2+a^2)\sin^2\theta\,d\varphi^2$$
and the Laplace operator is:
$$\Delta = \frac{1}{\sqrt{\det e}}\,\partial_\alpha \sqrt{\det e}\, e^{\alpha\beta}\partial_\beta,\quad \sqrt{\det e} = (\rho^2+a^2\cos^2\theta)\sin$$

A solution of Laplace equation with spheroidal symmetry is then readily found: $U=U(\rho)$ with $[(\rho^2+a^2)U']'=0 \Rightarrow U=\frac{GM}{a}\left[-\frac{\pi}{2}+\arctan\frac{\rho}{a}\right]$ ($U \overset{r\to\infty}{\sim} -\frac{GM}{r}$). It is regular at $\rho=0$ (but U' is not). It describes the gravitational potential of a "ring singularity" at $\rho = 0$.

To find the gravitational potential created by a finite distribution of matter bounded by the spheroid $X^2+Y^2+\frac{Z^2}{1-e^2}=a^2$ is a more involved enterprise, initiated by Newton and achieved by MacLaurin in 1742 in the case the density of the fluid is constant. He found (see Chandrasekhar's "Ellipsoidal figures of equilibrium" for the demo):
$$U(XYZ) = -\pi G\rho\sqrt{1-e^2}\,[a^2 I - (X^2+Y^2)A_1 - Z^2A_3]\quad (\text{inside the fluid}).$$
with $I = 2\frac{\operatorname{Arcsin} e}{e}$; $A_1 = \frac{\operatorname{Arcsin} e - e\sqrt{1-e^2}}{e^3}$; $A_3 = 2\,\frac{e-\sqrt{1-e^2}\operatorname{Arcsin} e}{e^3\sqrt{1-e^2}}$

The surfaces $U = Ct$ are spheroids but the surface of the fluid is NOT an equipotential. (I is chosen so that the EXTERIOR potential $\to 0$ at ∞.)

Once the MacLaurin interior potential is taken for granted, the continuity & Euler equations are solved by looking for a stationary solution describing a rigidly rotating body. In the rotating frame $v' = v - \Omega\wedge R = 0$ so that $v = \Omega\wedge R = \omega(-Ye_X + Xe_Y)$ and $v.\nabla v = (V^\alpha\partial_\alpha V^\beta)e_\beta = -\omega^2(Xe_X+Ye_Y)$
$$= -\frac{\omega^2}{2}\nabla(X^2+Y^2).$$
so that the Euler equ reads: $\nabla\left(U + \frac{p}{\rho} - \frac{\omega^2}{2}(X^2+Y^2)\right) = 0$.
whose sol is: $\frac{p}{\rho} = -\pi G\rho\sqrt{1-e^2}\left[(X^2+Y^2)\left(A_1^2 - \frac{\omega^2}{2\pi G\rho\sqrt{1-e^2}}\right) + Z^2A_3 - Ct\right]$;
Now the surface of the star is given by $p=0$ and also per $X^2+Y^2+\frac{Z^2}{1-e^2}=a^2$;
The 2 surface must coincide, which gives the value of ω as:
$$\frac{\omega^2}{2\pi G\rho} = -\frac{3(1-e^2)}{e^2} + \frac{\sqrt{1-e^2}}{e^3}(3-2e^2)\operatorname{Arcsin} e.$$



The kinetic energy of the distribution is $T = \frac{1}{2}\int\rho v^2 dV = \frac{1}{2}\omega^2 I_3$; $\left(I_3 = \frac{2Ma^2}{5},\ M = \frac{4\pi}{3}a^3\sqrt{1-e^2}\right)$
its gravitational energy is $W = \frac{1}{2}\int\rho U dV$
$$= -\frac{16\pi^2}{15}G\rho^2a^3\frac{(1-e^2)}{e}\operatorname{Arcsin} e.$$
The ratio $T/|W|$ is an increasing function of e, from $T/|W|=0$ for $e=0$ (sphere) to $T/|W| = 1/2$ for $e=1$ (disk). ; $J = \omega I_3$: angular momentum

It took almost 100 years (Jacobi, 1834) to find that if $e > 0.813$ (and $\omega^2/\pi G\rho \geq 0.374$; $T/|W| \geq 0.1375$) there $\exists$ ANOTHER figure of equilibrium, an ellipsoid $X^2 + \frac{Y^2}{1-e_2^2} + \frac{Z^2}{1-e^2} = a^2$, e_2 & ω being determined in term of e (see Chandrasekhar for the explicit relations). Dedekind (1860) & Riemann (1892) generalized these results to cases when rotation is no longer imposed to be rigid.

For given mass & angular momentum a MacLaurin spheroid has a greater total energy than the corresponding Jacobi ellipsoid, it is therefore unstable (such an instability is called "secular" as it requires energy dissipation mechanism). Riemann, by studying the growing modes of the perturbations to the equilibrium state, found that a dynamical instability sets in for $e \geq 0.953$ (that is $\omega^2/\pi G\rho = 0.4402$).

Poincaré (1885) analysed the secular instability of the Jacobi ellipsoids & found a bifurcation towards "pear-shaped" configurations for $e = 0.881$, which is ALSO the dynamical instability setting point (CARTAN, 1924).

II The Schwarzschild solution

1. Spherically symmetric spacetimes

The general relativistic analog of solving Laplace's equation $\Delta U = 0$ in the case of spherical symmetry is to find spherically symmetric spacetimes whose metric g solves the Einstein vacuum equations $R_{ij} = 0$. The analog of the solution $U = -GM/r$ to Laplace's equation will be the "Schwarzschild solution".

If a ST is spherically symmetric then there $\exists$ coordinates $(t\, r\, \theta\, \varphi)$ such that, by definition, the line element can be written as:

$$ds^2 = g_{00}(dx^0)^2 + 2 g_{0\alpha}\, dx^0 dx^\alpha + g_{11}(dx^1)^2 + f^2(x^0, x^1)(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

(hence the 2-surfaces $x^0 = \text{Ct}$, $x^1 = \text{Ct}$ are 2-spheres: $\varphi \in [0, 2\pi]$; $\theta \in [0, \pi]$).

A spacetime is "stationary" if coordinate systems can be found such that the metric components do not depend on the time-coordinate $x^0 = ct$ (hence they depend g_{00}, $g_{0\alpha}$, g_{11} & f depend on x^1 only).

A spacetime is static if it is stationary & time-symmetric. Hence one can find coordinate systems where $g_{0\alpha} = 0$.

Finally one can choose the "radial" coordinate x^1 such that $f(x^1) = x^1 = r$;

Thus we restrict our attention to the class of ST whose line element can be written as:

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

2. Computing the Einstein tensor

Let us start with the pedestrian way:

- inverse metric: $g^{00} = -e^{-\nu}$; $g^{rr} = e^{-\lambda}$; $g^{\theta\theta} = r^{-2}$; $g^{\varphi\varphi} = (r^2 \sin^2\theta)^{-1}$;
- Christoffel symbols: $\Gamma^i_{jk} = \frac{1}{2} g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk})$.

  ex: $\Gamma^0_{0r} = \frac{1}{2} g^{00} \partial_r g_{00} = \nu'/2$ etc

  shorter route: $\mathcal{L} = -e^{+\nu}\dot{t}^2 + e^{\lambda}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\varphi}^2$

  $$\frac{\partial \mathcal{L}}{\partial x^i} = \left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right)^{\cdot} \Rightarrow \text{e.g.: } 0 = 2(e^{\nu}\dot{t})^{\cdot} = 2[e^{\nu}\ddot{t} + \nu' e^{\nu}\dot{r}\dot{t}]$$

  to be identified to $\frac{Du^i}{d\tau} = 0$ ie $\ddot{t} + \Gamma^0_{ij}\dot{x}^i \dot{x}^j = 0$

  yielding $\Gamma^0_{0r} = \nu'/2$ all other Γ^0_{ij} being zero.

result:
$$\begin{cases} \Gamma^0_{0r} = \frac{\nu'}{2}\ ; \quad \Gamma^r_{00} = e^{\nu-\lambda}\frac{\nu'}{2}\ ; \quad \Gamma^r_{rr} = \frac{\lambda'}{2}\ ; \quad \Gamma^r_{\theta\theta} = \Gamma^r_{\varphi\varphi} = -re^{-\lambda} \\ \Gamma^\theta_{\theta r} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}\ ; \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta\ ; \quad \Gamma^\varphi_{\varphi\theta} = \cotan\theta \end{cases}$$

- Ricci tensor: $R_{ij} = R^m{}_{imj} = \partial_m \Gamma^m_{ij} - \partial_j \Gamma^m_{im} + \Gamma^m_{mk}\Gamma^k_{ij} - \Gamma^m_{jk}\Gamma^k_{im}$

  example:
  $$R_{00} = R^\alpha{}_{0\alpha 0} = \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \cancel{\Gamma^\alpha_{0\alpha}} + \Gamma^\alpha_{\alpha i}\Gamma^i_{00} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha}$$
  $$= \partial_r \Gamma^r_{00} + \left(\Gamma^r_{rr}\Gamma^r_{00} + \Gamma^\theta_{\theta r}\Gamma^r_{00} + \Gamma^\varphi_{\varphi r}\Gamma^r_{00}\right) - \Gamma^r_{00}\Gamma^0_{0r}$$
  $$= \frac{e^{\nu-\lambda}}{2}\left(\nu'' + \frac{1}{2}\nu'^2 - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{r}\right).$$

- curvature scalar: $R = g^{ij} R_{ij}$.
- Hence Einstein's tensor:

$$\begin{cases} G_{00} = \frac{1}{r^2} e^{\nu}[r(1-e^{-\lambda})]'\ ; \quad G_{rr} = -\frac{e^{\lambda}}{r^2}(1-e^{-\lambda}) + \frac{\nu'}{r} \\ G_{\theta\theta} = \frac{1}{2} r^2 e^{-\lambda}\left[\nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} - \frac{\nu'\lambda'}{2} - \frac{\lambda'}{r}\right]\ ; \quad G_{\varphi\varphi} = \sin^2\theta\, G_{\theta\theta} \end{cases}$$

Let us perform the same calculation "à la Cartan"

- orthonormal frame: $g = \eta_{ij}\theta^i \otimes \theta^j$; $\theta^0 = e^{\nu/2} dx^0$, $\theta^1 = e^{\lambda/2} dr$; $\theta^2 = r\,d\theta$; $\theta^3 = r\sin\theta\, d\varphi$
- zero torsion, Cartan's 1st structure equations: $0 = d\theta^k + \omega^k{}_l \wedge \theta^l$

  ex: $k = 0$; $d\theta^0 = \frac{\nu'}{2} e^{\nu/2} dr \wedge dx^0$; $\Rightarrow$ one can guess $\omega^0{}_r = \frac{\nu'}{2} e^{\nu/2} e^{-\lambda/2} dx^0$

  similarly one guesses $\omega^\theta{}_r$ & $\omega^\varphi{}_r$ & checks that the $k=1$ eq is satisfied.

  result: $\omega^0{}_r = \frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}} dx^0$; $\omega^\theta{}_r = e^{-\lambda/2} d\theta$; $\omega^\varphi{}_r = \sin\theta\, e^{-\lambda/2} d\varphi$; $\omega^\varphi{}_\theta = \cos\theta\, d\varphi$

- curvature 2-forms: $\Omega^m{}_k = d\omega^m{}_k + \omega^m{}_l \wedge \omega^l{}_k$

  example:
  $$\begin{cases} \Omega^0{}_r = d\omega^0{}_r = \left(\frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}}\right)' dr \wedge dx^0 \\ \Omega^0{}_\theta = \omega^0{}_l \wedge \omega^l{}_\theta = \omega^0{}_r \wedge \omega^r{}_\theta = -\frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}} \times e^{-\frac{\lambda}{2}} dx^0 \wedge d\theta \end{cases}$$

  etc.

- Riemann tensor: $\Omega^m{}_k = R^m{}_{kpq}\, \theta^p \wedge \theta^q$,

  example $R^0{}_{rr0} = \left(\frac{\nu'}{2} e^{\frac{\nu-\lambda}{2}}\right)' e^{-\frac{(\lambda+\nu)}{2}} = \frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4}$

- The Ricci tensor & curvature scalar are then obtained in the "pedestrian" way. Powerful method if... good guess work!

3. The Schwarzschild metric

Einstein's equations:

$$\left\{\begin{array}{l} G_{00} = \frac{1}{r^2} e^{\nu}\left[r(1-e^{-\lambda})\right]' = 0 \\ G_{rr} = -\frac{e^{\lambda}}{r^2}(1-e^{-\lambda}) + \frac{\nu'}{r} = 0 \\ G_{\theta\theta} = \frac{1}{2} r^2 e^{-\lambda}\left[\nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} - \frac{\nu'\lambda'}{2} - \frac{\lambda'}{r}\right] = \frac{G_{\varphi\varphi}}{\sin^2\theta} = 0 \end{array}\right.$$

$\Big\}$: NB : 1^{st} order equations

NB : if we had allowed ν & λ to depend on time, we would have found that $\dot\nu = \dot\lambda = 0$

$G_{00} = 0 \iff e^{-\lambda} = 1 + \frac{C_1}{r}$; $G_{rr} = 0 \iff \nu = -\lambda + C_2$ ("Birkhoff theorem", (1923))

and $G_{\theta\theta} = 0$, $G_{\varphi\varphi} = 0$ are identically satisfied.

hence $ds^2 = -\left(1 + \frac{C_1}{r}\right)e^{C_2}(dt)^2 + \frac{dr^2}{1 + C_1/r} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$.

The constant C_2 can be absorbed in a redefinition of the time coordinate: $t \to t e^{C_2/2}$

At infinity the metric $\to$ minkowskian metric ; We know that in the weak field limit $g_{00} \sim -\left(1 + \frac{2U}{c^2}\right)$ where $U = -\frac{GM}{r}$ is Newton's potential, M being the mass of the central, spherically symmetric body. Hence the expression for the Schwarzschild metric in "Schwarzschild coordinates" $t r \theta \varphi$:

$$\boxed{ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{dr^2}{1 - 2GM/c^2 r} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}$$

Schwarzschild 1916 (January 1916)

This metric describes the gravitational field outside the central mass.

For the Sun: $GM_\odot/c^2 R_\odot \sim 10^{-6}$; for the Earth $GM_\oplus/c^2 R_\oplus \sim 10^{-10}$

for Neutron Stars: $GM/c^2 R \sim 0.2$

Hence for such bodies $2GM/c^2 r < 1 \quad \forall r > R$.

Beware however of the fact that r is a "radial coordinate" chosen so that the surfaces of 2-spheres are $4\pi r^2$, but must not be confused with a cartesian radial distance. Indeed the "proper" radial distance between 2 2-spheres located at $r = r_1$, $r = r_2$ is NOT $(r_2 - r_1)$ but $\int_{r_1}^{r_2} \frac{dr}{1 - 2GM/c^2 r}$; Other radial coordinates can be used; eg:

"isotropic coordinate": $r = \bar r(1 + m/2\bar r)^2$ $(m \equiv GM/c^2)$ flat spatial sections.

$$ds^2 = -\left(\frac{1 - m/2\bar r}{1 + m/2\bar r}\right)^2 c^2 dt^2 + \left(1 + \frac{m}{2\bar r}\right)^4 (d\bar r^2 + \bar r^2 d\theta^2 + \bar r^2 \sin^2\theta\, d\varphi^2)$$

"harmonic coordinate": $\tilde r = r - m$ such that $\Box x^\alpha = \frac{1}{\sqrt{g}}\partial_j(\sqrt{-g}\, g^{ij}) = 0$

$$ds^2 = -\left(\frac{\tilde r - m}{\tilde r + m}\right)c^2 dt^2 + \left[\left(1 + \frac{m}{\tilde r}\right)^2 \delta_{\alpha\beta} + \left(\frac{\tilde r + m}{\tilde r - m}\right)\frac{m}{\tilde r^4} x_\alpha x_\beta\right] dx^\alpha dx^\beta$$

III Relativistic star models

1. The Oppenheimer-Volkov equations

In Newton's theory of gravity a star model is built by solving the continuity, the Euler & the Poisson equations, given an equation of state.

In GR the matter distribution creating the gravitational field is described by its stress energy tensor $T_{ij} = (\epsilon + p)\frac{u_i u_j}{c^2} + p g_{ij}$ for a perfect fluid.

The star is static, spherically symmetric, hence the ansatz for the metric:

$$ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

The matter distribution is also static; hence $u^i = (u^0, 0, 0, 0)$;

since $u_i u^i = g_{ij} u^i u^j = -e^{\nu}(u^0)^2 = -c^2$: $u^0 = c e^{-\nu/2}$; $u_0 = -c e^{\nu/2}$

Hence $T_{00} = \epsilon e^{\nu}$; $T_{rr} = p e^{\lambda}$; $T_{\theta\theta} = p r^2$; $T_{\varphi\varphi} = p r^2 \sin^2\theta$

The functions to be determined are thus $\nu(r)$, $\lambda(r)$, $\rho(r)$, $p(r)$.

The Einstein equations read: $G_{ij} = \frac{8\pi G}{c^4} T_{ij}$, that is

$$\left\{\begin{array}{ll} \frac{1}{r^2} e^{\nu}\left[r(1-e^{-\lambda})\right]' = \frac{8\pi G}{c^4}\epsilon e^{\nu} & (00) \\ -\frac{e^{\lambda}}{r^2}(1-e^{-\lambda}) + \frac{\nu'}{r} = \frac{8\pi G}{c^4} p e^{\lambda} & (rr) \\ \frac{1}{2} r^2 e^{-\lambda}\left[\nu'' + \frac{\nu'^2}{2} + \frac{\nu'}{r} - \frac{\nu'\lambda'}{2} - \frac{\lambda'}{r}\right] = \frac{8\pi G}{c^4} p r^2 & (\theta\theta) \end{array}\right.$$

If we replace $\lambda(r)$ by the function $\frac{Gm(r)}{c^2} = \frac{r(1-e^{-\lambda})}{2}$, the (00) & (rr) eqn become:

$$dm/dr = \frac{4\pi}{c^2} r^2 \epsilon \; ; \quad d\nu/dr = 2\,\frac{Gm/c^2 + \frac{4\pi G}{c^4} p r^3}{r(r - 2Gm/c^2)}$$

Instead of the $(\theta\theta)$ eq one may use Bianchi's identity: $D_i G^{ij} = 0 \Rightarrow D_i T^{ij} = 0$

$D_i T^{ij} = \partial_i T^{ij} + \Gamma^i_{ik} T^{kj} + \Gamma^j_{ik} T^{ik} = 0$. Knowing the Christoffel symbols (see above) the $i = r$ component yields:

$$\frac{dp}{dr} = -\frac{1}{2}(\epsilon + p)\frac{d\nu}{dr}$$ which can be rewritten as:

$$\frac{dp}{dr} = -(\epsilon + p)\,\frac{Gm/c^2 + (4\pi G/c^4) p r^3}{r(r - 2Gm/c^2)}$$

That equation is known as the Oppenheimer-Volkov equation (1939).

To close the system (ie to have 4 eqns for the 4 unknowns) an equation of state must be given (cf E. Gourgoulhon & P. Haensel).

Despite the very ≠ conceptual frameworks the equation of structure for stellar equilibrium are similar in Newton & GR theories:

$$\begin{cases} \dfrac{dm}{dr} = 4\pi r^2 \dfrac{\epsilon}{c^2} & \rightsquigarrow \quad \dfrac{dm}{dr} = 4\pi \rho r^2 \\[2ex] \dfrac{dp}{dr} = -\dfrac{\epsilon + p}{2} \dfrac{d\nu}{dr} & \rightsquigarrow \quad \dfrac{dp}{dr} = -\rho \dfrac{dU}{dr} \\[2ex] \dfrac{d\nu}{dr} = \dfrac{2Gm/c^2 + (4\pi G/c^4) p r^3}{r(r - 2Gm/c^2)} \xrightarrow[r \gg \frac{Gm}{c^2}]{p \to 0} \dfrac{2Gm}{c^2 r^2} & \rightsquigarrow \quad \dfrac{dU}{dr} = \dfrac{Gm}{r^2} \end{cases}$$

(which yields $\nu \sim \frac{2U}{c^2}$ ie $g_{00} \sim -(1 + \frac{2U}{c^2})$ at the Newtonian limit as it should).

Given an equation of state the system is integrated with IC: $m(0) = 0$; $\nu(0) = \nu_0$ and $\epsilon(0) = \epsilon_0$. Integration stops at $r = r_0$ where p & ρ vanish. Outside the star the metric is Schwarzschild metric. If all quantities are smooth across the surface of the star, the continuity of λ gives $M = m(r_0)$; and $e^{\nu(r_0)} = 1 - 2GM/c^2 r_0$ which determines ν_0. Hence, as in Newton's theory, there is a whole family of solutions parametrized by the central density ρ_0.

NB: the Schwarzschild mass $M = m(r_0) = \int_0^{r_0} 4\pi \frac{\epsilon}{c^2} r^2 dr$ is NOT the "proper mass" defined as
$$M_p = \int \frac{\epsilon}{c^2} dV = \int \frac{\epsilon}{c^2} r^2 e^{\lambda/2} dr\, d\theta \sin\theta\, d\varphi$$
$$= 4\pi \int \frac{\epsilon}{c^2} r^2 \left(1 - \frac{2m(r)}{r}\right)^{-1/2} dr.$$

The difference may be interpreted as the gravitational binding energy. Indeed
$$M_p - M \sim 4\pi \int \frac{\epsilon}{c^2} \frac{m(r)}{r} r^2 dr = -\frac{1}{2} \int \rho U \, dV \quad \text{with } \rho \equiv \epsilon/c^2$$
at the Newtonian limit.

NB. The newtonian limit of the OV equation is $dp/dr = -\frac{Gm\epsilon}{c^2 r^2}$ and yields smaller p than its relativistic counterpart. It is therefore more difficult to keep a star in an equilibrium state in GR than in Newton's theory. (this property will lead to the notion of critical mass in GR.)

2. Junction conditions

Suppose the distribution of matter is discontinuous across Σ, the 3-dimensional surface which divides spacetime into an interior & an exterior region $V_\pm$. We see, intuitively, that V_+ & V_- can be smoothly pasted together if the geometry of their edges $\Sigma_\pm$ coincide with Σ, that is if the induced metrics on $\Sigma_\pm$ are the same, and if the derivatives of the metrics in the direction ⊥ to Σ are continuous.

More precisely choose "gaussian" (or "admissible") coordinate systems $(y, x^\alpha_\pm)$ in $V_\pm$ such that, at least near $\Sigma_\pm$, the line elements read:
$$ds^2_\pm = dy^2 + \gamma^\pm_{\alpha\beta}(y, x^\alpha_\pm)\, dx^\alpha_\pm dx^\beta_\pm$$
where $y = y_0$ is the equation of $\Sigma_\pm$. The two regions $V_\pm$ can be "pasted" together if $\gamma^+_{\alpha\beta}(y_0, x^\alpha_+)\, dx^\alpha_+ dx^\beta_+ = \gamma^-_{\alpha\beta}(y_0, x^\alpha_-)\, dx^\alpha_- dx^\beta_-$.

= Schwarzschild coordinates are not "admissible" since $ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 d\Omega^2$ but become easily so by going from r to y with $dy = e^{\lambda/2} dr$. Hence
$$ds^2_-|_\Sigma = ds^2_+|_\Sigma \Longrightarrow -e^{\nu_+(r_0)} dt_+^2 + \underbrace{r_+^2(y_0)}_{\equiv r_0^{+2}} d\Omega^2 = -e^{\nu_-(r_0)} dt_-^2 + \underbrace{r_-^2(y_0)}_{r_0^{-2}} d\Omega^2$$
hence $r_0^+ = r_0^-$

The pasting will be smooth if $\left.\frac{\partial \gamma^+_{\alpha\beta}}{\partial y}\right|_\Sigma = \left.\frac{\partial \gamma^-_{\alpha\beta}}{\partial y}\right|_\Sigma$, that is if:
$$\left(r_0 \left.\frac{dr}{dy}\right|_0\right)^+ = \left(r_0 \left.\frac{dr}{dy}\right|_0\right)^- \quad \text{ie if} \quad \underline{\lambda_+(r_0) = \lambda_-(r_0)}$$
and if $\left(\nu'_0\, dr/dy|_0\right)^+ = \left(\nu'_0 \left.\frac{dr}{dy}\right|_0\right)^-$ ie if $\underline{\nu'_+(r_0) = \nu'_-(r_0)}$

(in coordinate free language: continuity of "extrinsic curvature")

= These junction conditions translate as conditions on T_{ij} itself: imposing "smooth" matching, that is the continuity of the metric & its y-derivative but not (of) the second-order derivatives $\partial^2_{yy}\gamma_{\alpha\beta}$, implies, since Einstein's equations are 2nd order, that the discontinuity of T_{ij} is of the Heaviside type [if the discontinuity of T_{ij} is of the delta-type, that is if Σ is a thin shell of matter, then $\partial_y \gamma_{\alpha\beta}$ is no longer continuous and the matching is no longer smooth].

It is an easy exercise to check that if $ds^2 = dy^2 + \gamma_{\alpha\beta}(y, x^\gamma) dx^\alpha dx^\beta$ then the (yy) & $(y\alpha)$ components of Einstein's tensor do not contain second y-derivatives of $\gamma_{\alpha\beta}$. Hence, Einstein's equations impose T_{yy} and $T_{y\alpha}$ to be continuous across Σ. If $T_{ij} = (\epsilon + p)\frac{u_i u_j}{c^2} + p g_{ij}$ and if the fluid is static ($u_y = 0$), then we must have: $\underline{p = 0}$ on Σ.

3. The particular case of constant density

If the equation of state is $\rho \equiv \epsilon/c^2 = \text{Const}$, integrating the O.V. system is straightforward:

$$\begin{cases} \dfrac{dm}{dr} = 4\pi\rho r^2 \longrightarrow m(r) = \dfrac{4\pi}{3}\rho r^3 ; \\ \dfrac{dp}{dr} = -\dfrac{(\epsilon+p)}{2}\dfrac{d\nu}{dr} \longrightarrow \rho c^2 + p = B e^{-\nu/2} \quad \text{where } B \text{ is a constant} \\ \dfrac{d\nu}{dr} = 2\,\dfrac{Gm/c^2 + (4\pi G/c^4)\,p r^3}{r(r - 2Gm/c^2)} \longrightarrow z = -D\sqrt{1-\dfrac{8\pi G\rho r^2}{3c^2}}, \ D \text{ a constant} \\ \qquad\qquad \text{and } z \equiv e^{\nu/2} - 3B/2\rho c^2 \end{cases}$$

The junction conditions with the exterior Schwarzschild solution (in Schwarzschild coordinates) are:

$$\begin{cases} \lambda \text{ continuous, that is } M = \dfrac{4\pi}{3}\rho r_0^3 \\ p = 0 \text{ at } r = r_0 \text{, that is } \dfrac{Bc^2}{2D\rho} = \sqrt{1-\dfrac{2GM}{c^2}} \end{cases}$$

The constant D remains arbitrary but can be absorbed in a redefinition of t which garantees the continuity of ν' at $r = r_0$. Hence:

$$\begin{cases} ds^2 = -\left[\dfrac{3}{2}\sqrt{1-\dfrac{2GM}{c^2 r_0}} - \dfrac{1}{2}\sqrt{1-\dfrac{2GM}{c^2 r_0^3}r^2}\right]^2 c^2 dt^2 + \dfrac{dr^2}{1-\dfrac{2GM}{c^2 r_0^3}r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ p = \rho c^2\, \dfrac{\sqrt{1-\dfrac{2GM}{c^2 r_0^3}r^2} - \sqrt{1-\dfrac{2GM}{c^2 r_0}}}{3\sqrt{1-\dfrac{2GM}{c^2 r_0}} - \sqrt{1-\dfrac{2GM}{c^2 r_0^3}r^2}} \end{cases}$$

(Schwarzschild 1916)
(February)
(Schwarzschild died in June)

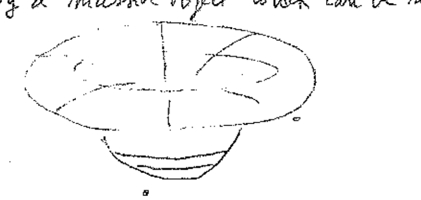
at time $t = \text{Const}$, the geometry of the $\theta = \pi/2$ 2-surfaces are:

$$d\sigma_e^2 = \frac{dr^2}{1-\dfrac{2GM}{c^2 r}} + r^2 d\varphi^2 ; \qquad d\sigma_{in}^2 = \frac{dr^2}{1-\dfrac{2GM}{c^2 r_0^3}r^2} + r^2 d\varphi^2$$

$(m \equiv GM/c^2)$

Outside the star this surface is a paraboloid with eqn $z^2 = 8m(r-2m)$ embedded in a 3D euclidean space; inside the surface is a 2-sphere. Hence the pictorial representation of the curvature of space-time induced by a massive object which can be seen in popular text-books or articles:

$\left(r_0 > \dfrac{2GM}{c^2}\right)$



The central pressure is

$$p(0) = \rho c^2\, \frac{1-\sqrt{1-2GM/c^2 r_0}}{3\sqrt{1-\dfrac{2GM}{c^2 r_0}} - 1}$$

It remains finite if $\dfrac{GM}{c^2} < 4\dfrac{r_0}{9}$, that is, if,

since $M = \frac{4}{3}\pi\rho r_0^3$, $\quad M < M_{crit} = \dfrac{4}{9\sqrt{3\pi}}\left(\dfrac{G}{c^2}\right)^{-3/2}\dfrac{1}{\sqrt{\rho}}$; for $\rho \sim 10^{15}\,g/cm^3$ $M_{crit} \simeq 4 M_\odot$

4. Rotating relativistic star models (NB: hereafter $c = G = 1$!)

We saw how, within Newton's theory, MacLaurin found the gravitational potential of a uniform density spheroid (solution of $\Delta U = 4\pi G\rho$) and found that the Euler equation imposed this spheroid to rotate with a given angular velocity. Finding the GR analog of these spheroidal figures of equilibrium is still a challenge.

The first step is to find an ansatz for the line element, ie to find the generalisation of the Schwarzschild ansatz ($ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$) to space-times which are stationary (rather than static) and axisymmetric (rather than spherically symmetric). Consider the ansatz:

(1) $ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\varphi - \omega dt)^2 + e^{2\mu}(dr^2 + r^2 d\theta^2)$ with $\varphi \in [0, 2\pi]$, $\theta \in [0, \pi]$

and with the 4-functions ν, ψ, ω & μ depending only on (r, θ).

Such spacetimes possess 2 Killing vectors:

$$\begin{cases} \xi^i_t = (1,0,0,0) & \text{which is timelike } (\text{if } e^{2\nu} > 0) \\ \xi^i_\varphi = (0,0,0,1) & \text{, spacelike with closed-orbits.} \end{cases}$$

Hence, by definition, these spacetimes are stationary & axisymmetric.

NB: Note that all metrics of ST possessing 2 such Killing vectors, do NOT reduce to (1). They do reduce to (1) if $[\xi_t, \xi_\varphi] = 0$ (when [ ] is the Lie bracket) and if the 2-surfaces orthogonal to ξ_t & ξ_φ are "integrable" (a pedestrian way to define integrability being that there ∃ coordinate systems where there are no cross-terms g_{rt}, $g_{r\varphi}$, $g_{\theta t}$, $g_{\theta\varphi}$ and where g_{ab} with $(a,b) = (r\theta)$ do not depend on t & φ). Ricci flat spacetimes (ie vacuum Einstein solution or ST with matter with no convective motion in the meridional plane do obey these extra-conditions (Carter 1969; see eg Wald's book or E. Gourgoulhon).

The metric ansatz (1) is due to A. Papapetrou (1953, 1966).

- note that: $g_{00} = -e^{2\nu} + \omega^2 e^{2\psi}$; $g_{0\varphi} = -\omega e^{2\psi}$; $g_{\varphi\varphi} = \omega^2 e^{2\psi}$.

Ansatz for the stress-energy tensor of matter: $T_{ij} = (\epsilon + p)u_i u_j + p g_{ij}$ and $u^\alpha = \dfrac{e^{-\nu}}{\sqrt{1-v^2}}(1, 0, 0, \Omega)$ where Ω depends a priori on $r\theta$ and where v is given by $u^i u_i = -1$: $v \equiv (\Omega - \omega)e^{\psi-\nu}$.

If $\Omega = \text{Cte}$ the rotation is called "rigid"; $\dfrac{d\varphi}{dt} = \dfrac{u^\varphi}{u^t} = \Omega$ is the angular velocity at ∞.

The equation of motion for matter are $D_i T^{ij} = 0$, which split into a conservation equation, which is identically satisfied for rigid motion, and the relativistic Euler equation, which can can be integrated as:

$$(*) \qquad \ln h + \nu - \ln \Gamma = \text{Const} \qquad \text{(see eg Stergioulas)}$$

where $\Gamma \equiv \frac{1}{\sqrt{1-v^2}}$; $\quad h = \frac{\epsilon + p}{n}$ (is the "specific enthalpy")
(and: $\frac{dn}{n} = \frac{d\epsilon}{\epsilon + p}$, ($n$ is the "baryon nb"))

At the Newtonian limit
$$\begin{cases} p \ll \epsilon \to n \sim \epsilon, \; h \sim 1 + p/\epsilon \Rightarrow \ln h \sim P/\rho \quad (\epsilon \equiv \rho c^2) \\ g_{00} \sim -e^{2\nu} \sim -(1+2\nu) \sim -(1+2U) \Rightarrow \nu \sim U \quad \text{(Newton's pot.)} \\ \Gamma \sim 1 + v^2/2 \sim 1 + (R \wedge \Omega)^2/2 \end{cases}$$

hence (*) reads $\quad P/\rho + U - (R \wedge \Omega)^2/2 = \text{Const} \quad$ (as seen above).

To describe the analog of constant density the following eqn of state is chosen:
$$\epsilon = \rho_0 = \text{ct.} \to n = \rho_0; \quad p = nh - \epsilon = \rho_0 (h - 1).$$

In GR Laplace's equation is replaced by Einstein's equations: $G_{ij} = 8\pi T_{ij}$.
Their explicit expressions can be found in Stergioulas' Living Reviews. There are 4 independent equations (elliptic) for the unknown ν, μ, ψ and ω.
They ~~must be~~ integrated numerically (cf. E. Gourgoulhon).

The importance of studying such models is to see if there is an analog to the Jacobi instability in GR. Indeed, if for high rotation, the configuration becomes unstable and "migrates" to an ellipsoidal figure of "equilibrium", then its quadrupole moment starts to depend on time & gravitational waves should be emitted, at a well-defined frequency (Bonazzola et al.)

Two kinds of bifurcation points can be considered, as in Newton's theory: secular or dynamical. Dynamical instabilities are found by perturbing a "Maclaurin" type star.

Secular instabilities set in when the triaxial configuration has lower total energy. They can be driven by viscosity. [NB: the Chandrasekhar-Friedman-Schutz instability is more akin to the Dedekind instability: no rotation of the surface of the ellipsoid, but internal circulation].

The question therefore becomes: how is the "total energy" of a gravitating body defined in GR?

5. Global conserved quantities

The Bianchi identity $D_i G^{ij} \equiv 0 \Rightarrow D_i T^{ij} = 0$ via Einstein's equations.
If ξ^i is a Killing vector (ie $D_i \xi_j + D_j \xi_i = 0$) then
$$0 = \xi_j D_i T^{ij} = D_i(\xi_j T^{ij}) - T^{ij} \cancel{D_i \xi_j} = \frac{1}{\sqrt{-g}} \partial_i \left(\sqrt{-g}\, \xi_j T^{ij} \right)$$

Integrating over the whole region where $T_{ij} \neq 0$ (ie over the star) we obtain:
$$\frac{d}{dt}\left(\int_V d^3x \sqrt{-g}\, \xi_j T^{0j} \right) = - \int_V d^3x\, \partial_\alpha \left(\sqrt{-g}\, \xi_j T^{\alpha j} \right) = \int_{\partial V} d^2x \sqrt{-h}\, n_\alpha \xi_j T^{\alpha j}$$

where, if h_{ab} is the 2-metric induced on the surface of the star ∂V, and n^α is the unit ($h_{ab} n^a n^b = 1$) vector pointed out of the surface. Now $T_{ij} = (p+\epsilon) u_i u_j + p g_{ij}$.
In the $(t r \theta \varphi)$ coordinates $n_\alpha \propto (0,1,0,0)$; since T_{r0} & $T_{r\varphi}$ are zero, the surface integral is zero for both Killing vectors $\xi^i_t = (1000)$ & $\xi^i_\varphi = (0001)$.

Hence $\int_V d^3x \sqrt{g}\, \xi_j T^{0j} = \text{cte}$

Now, in flat SP, minkowskian coordinates these constants identify to the total mass (for $\xi^i = \xi^i_t$) & the total angular momentum (for $\xi^i = \xi^i_\varphi$).
Hence the following definitions for the mass & angular momentum of the star:
$$E = -\int_V d^3x \sqrt{g}\, \xi^j_t T^0_j \; ; \quad J = \int_V d^3x \sqrt{g}\, \xi^j_\varphi T^0_\varphi \qquad \text{(up to a sign)}$$

Other global quantities can be obtained. (Komar 1959)
We have for $\forall$ vector ξ^i: $D_i D_j (D^j \xi^i - D^i \xi^j) = 0$.
Hence, for a Killing vector: $D_i \Box \xi^i = D_i (\underbrace{D_j D^i \xi^j}_{\equiv V^i}) = 0$.

$D_i V^i = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g}\, V^i) = 0$: then we have the conservation law:
$$\frac{d}{dt}\left(\int_V d^3x \sqrt{-g}\, V^0 \right) = + \int_{\partial V} d^2x \sqrt{h}\, n_\alpha V^\alpha$$
where the integral is taken over all 3-space and its boundary

Now: $-V^i = -D_j D^i \xi^j = +D_j D^j \xi^i = \partial_j (D^j \xi^i) + \underbrace{\Gamma^j_{jk}}_{= \frac{1}{\sqrt{-g}} \partial_k \sqrt{-g}} D^k \xi^i + \Gamma^i_{jk} \underbrace{D^j \xi^k}_{=0}$
$$= \frac{1}{\sqrt{-g}} \partial_k \left(\sqrt{-g}\, D^k \xi^i \right)$$

hence: $\quad \int d^3x \sqrt{-g}\, V^0 = + \int d^2x \sqrt{-g}\, D^\alpha \xi^0 n_\alpha$

which transforms a volume integral into a integral on the 2-sphere at infinity.

This constant can be related to T_{ij} by means of Einstein's equations:
Indeed :

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi T_{ij} \qquad (\Rightarrow R = -8\pi T)$$

$$\Rightarrow \xi_j R^{ij} - \frac{1}{2} R \xi^i = 8\pi \xi_j T^{ij}$$; now for a Killing vector : $D_i D_j \xi^i = R^k{}_j \xi_k$

hence : $D_k D^i \xi^k = 8\pi \xi_j (T^{ij} - \frac{1}{2} g^{ij} T)$

so that $\Gamma^i = 8\pi \xi_j (T^{ij} - \frac{1}{2} g^{ij} T)$

From this one first sees that $\Gamma^\alpha = 0$ at ∞ so that :

$$-\frac{1}{8\pi} \int_{\partial V_\infty} d^2x \sqrt{-g}\, D^\alpha \xi^0 n_\alpha = -\int_{V_{star}} d^3x\, \xi_j (T^{0j} - \frac{1}{2} g^{0j} T) = \text{Constant}.$$

These Komar integrals identify (up to factors 2!) to the SR definition of mass & angular momentum & can also be used as definitions of M & J.

Following Friedmann et al (1986) numerical relativists define thus a number of global quantities which 'reduce' to their Newtonian counterparts in the weak field limit :

- the gravitational mass : $M = \int (T_{ij} - \frac{1}{2} g_{ij} T) \xi^i_t \hat{n}^j \, dV$ — where $\hat{n}^j$ is the unit $\perp$ to the 3-surfaces $t = $ const
- baryon mass : $M_0 = \int \rho u_i \hat{n}^i \, dV$
- internal energy : $U = \int (\epsilon - \rho c^2) u_i \hat{n}^i \, dV$
- proper mass : $M_p = M_0 + U$
- angular momentum : $J = \int T_{ij} \xi^i_\varphi \hat{n}^j \, dV$
- kinetic energy : $T = \frac{1}{2} J \Omega$
- gravitational binding energy : $W = M - M_p - T$.
  OR : $W = M - M_0 - T$

It is clear that such definitions require sounder foundations. ~~Since~~ Energy linear & angular momenta are "best defined within a Lagrangian framework by means of Noether-type theorems" : cf Deser.

IV Gravitational collapse

We arrived at the concept of critical mass within the over-simplified model of a spherically symmetric star of constant density. It turns out however that the result is quite general : if one imposes the equation of state to be such that the velocity of perturbations (that is, of "sound") is less than c that a compact object cannot have a mass $\geq$ a few solar masses, if gravity is to be described by GR (cf Barsota & e.g. Ruffini).

1. "Newtonian collapse"

In Newton theory a static, spherically symmetric gravitationally bound object is stable. Suppose, nonetheless, that the pressure suddenly becomes negligible. Then the surface of the star, because of Gauss theorem, would undergo free fall, as if attracted by a point-like mass M at the centre.

the radial eom are $\begin{cases} F = -m \nabla U \; ; & U = -GM/r. \\ F = m a \; ; & a = \ddot{r} \end{cases}$

$\Rightarrow \frac{1}{2} \dot{r}^2 = \frac{GM}{r} + E$ where E is a constant. Setting $E = -\frac{GM}{r_0}$ (r_0 initial radius)

integration yields $t = \frac{r_0}{2} \sqrt{\frac{r_0}{2GM}} (\eta + \sin \eta)$; $r = \frac{r_0}{2}(1 + \cos \eta)$

Hence $t_{collapse} = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{2GM}}$; $\begin{cases} \text{for the Sun} & t_{coll} \sim 1/2\,h \quad (r_0 \sim 7 \times 10^5\,km) \\ \text{for a NS} & t_{coll} \sim 10^{-3}\,sec \quad (r_0 \sim 10\,km) \end{cases}$

t, being universal time all observers agree on the duration of the collapse.

In a corpuscular theory of light, supposing that the "photons" are emitted with velocity c from the surface of the collapsing star, then the eom of these particles is :

$$\frac{1}{2} \dot{r}^2 = \frac{GM}{r} + E \quad \text{with} \quad \frac{1}{2} c^2 = \frac{GM}{r(t_{em})} + E \Rightarrow \dot{r}^2 = c^2 - \frac{2GM}{r(t_{em})} + \frac{2GM}{r}$$

Hence, in accordance with the results of Blair Michell Laplace no light will reach a distant observer once the star has entered its "gravitational radius" $r_S = 2GM/c^2$. If these particles of light are sent from the surface at regular time intervals Δt_e then they are received by a distant observer at larger and larger time intervals Δt_r (as a short calculation shows); to be checked
and : $\Delta t_r \longrightarrow \infty$ when the star crosses its gravitational radius : the star becomes "black".

On the other hand, in a wave-theory of light, the star becomes "black" only when it reaches $r = 0$. (cf J. Eisenstaedt).

2. Radial infall in General Relativity ($G=c=1$)

When pressure becomes negligible the stress-energy tensor of matter becomes: $T_{ij} = \epsilon u_i u_j$ & the eom $D_i T^{ij} = 0 \Leftrightarrow Du^i/d\tau = 0$. Hence the surface of the star follows a radial geodesic of the Schwarzschild ST.
In Schwarzschild coordinates the metric is:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

The metric coefficients do not depend on time. Hence:

$$u_0 = g_{00}u^0 = -\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = -E \quad \text{where } E \text{ is a constant.}$$

For radial infall, $dr/d\tau$ is most simply given by writing that $ds^2 = -d\tau^2$ on the trajectory (that is $g(u,u) = -1$):

$$-1 = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \frac{\dot{r}^2}{1-2M/r}\quad \text{, which yields, choosing}$$

E so that $\dot{r} = 0$ at $r = r_0$:

$$\left(1-\frac{2M}{r}\right)\dot{t} \overset{(1)}{=} \sqrt{1-\frac{2M}{r_0}} \;;\quad \dot{r}^2 \overset{(2)}{=} \frac{2M}{r} - \frac{2M}{r_0}$$

The equation for $\dot{r}$ is the same as in Newton's theory. Hence:

$$r = \frac{r_0}{2}(1+\cos\eta) \;;\quad \tau = \frac{r_0}{2}\sqrt{\frac{r_0}{2GM}}\,(\eta + \sin\eta).$$

Hence the collapse, AS MEASURED by an observer falling with the star, lasts:

$$\tau_{coll} = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{2GM}}. \quad \text{(Oppenheimer-Snyder 1939)}$$

redshift calculation:

1) $\Delta t_{em} = \Delta\tau_{em}\sqrt{1-\frac{2M}{r_0}}\,\frac{1}{1-2M/r_{em}}$ from (1)

2) photons follow null radial geodesics $\frac{dr}{dt} = \left(1-\frac{2M}{r}\right) \Rightarrow t = t_{em} + \int_{r_{em}}^{r}\frac{dr}{1-2M/r}$

$$\Rightarrow \Delta t_{rec} = \Delta t_{em} - \frac{\Delta r_{em}}{1-2M/r_{em}}$$

3) from (2): $\Delta r_{em} = -\Delta\tau_{em}\sqrt{\frac{2M}{r_{em}} - \frac{2M}{r_0}}$

Hence

$$\Delta\tau_{rec} \underset{\text{at } \infty}{=} \Delta t_{rec} = \Delta t_{em} - \frac{\Delta r_{em}}{1-2M/r_{em}} = \frac{\Delta\tau_{em}}{1-2M/r_{em}}\sqrt{1-\frac{2M}{r_0}} + \frac{\Delta\tau_{em}}{1-2M/r_{em}}\sqrt{\frac{2M}{r_{em}}-\frac{2M}{r_0}}$$

$$= \frac{\Delta\tau_{em}}{1-2M/r_{em}}\left[\sqrt{1-\frac{2M}{r_0}} + \sqrt{\frac{2M}{r_{em}}-\frac{2M}{r_0}}\right] \xrightarrow[r_{em}\to 2M]{} \infty$$

Hence, as seen from infinity, the star turns black when reaching its gravitational radius $r_g = 2M$. It becomes a "black hole" (J. Wheeler

3. The interior solution

If pressure is negligible $T_{ij} = \epsilon u_i u_j$ with $u^\alpha = 0$, $T^0_0 = -\epsilon$ (in comoving coordinates)
Outside the star the metric is Schwarzschild's. Inside the solution for general $\epsilon(t,r)$ was obtained by Tolman (1934). (cf J.P. Uzan)

If we suppose that $\epsilon = \epsilon(t)$ only that the interior of the star is not only isotropic but homogeneous: Hence the metric is a Robertson-Walker metric:

$$ds^2 = -d\tau^2 + a^2(\tau)\left[\frac{d\rho^2}{1-k\rho^2} + \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

The solution of Einstein's equation is then (cf JP Uzan), for $k=+1$:

$$a = \frac{a_m}{2}(1+\cos\eta) \quad \tau = \frac{a_m}{2}(\eta + \sin\eta)$$

To match the ext & int solution, compute the circumference of the star:
in Schw coordinates $C = 2\pi r$ where $r = \frac{r_0}{2}(1+\cos\eta)$, $\tau = \frac{r_0}{2}\sqrt{\frac{r_0}{2GM}}(\eta+\sin\eta)$
in RW coordinates $C = 2\pi a(\tau)\rho_0$

hence: $r_0 = \rho_0 a_m$ and $r_0\sqrt{\frac{r_0}{2M}} = a_m$.

The more general Tolman (1934) - Bondi (1936) is obtained as follows:
Take as metric ansatz:

$$ds^2 = -d\tau^2 + e^{\lambda(\rho,\tau)}d\rho^2 + r^2(\rho,\tau)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Compute Einstein's tensor (see e.g. Stephani).
Choose $T^0_0 = -\epsilon(\rho,\tau)$.
Integrate Einstein's equation. Find in particular the solution:

$$e^\lambda = \left(\frac{\partial r}{\partial\rho}\right)^2 \;;\quad r = \left(\frac{9M(\rho)}{2}\right)^{1/3}(\tau_0(\rho) - \tau)^{2/3}$$

where $M(\rho)$ and $\tau_0(\rho)$ are 2 arbitrary functions of ρ.

and: $4\pi\epsilon = \frac{1}{r^2}\frac{\partial M}{\partial r}$.

4. Tidal forces.

The equation for geodesic deviation is $a^i = R^i{}_{mjk}u^m u^j n^k$
Cf the calculation of the curvature 2-form "à la Cartan".
In the orthonormal basis θ^i the non-zero components of the Riemann-tensor are all $\propto \frac{M}{r^3}$; Hence tidal forces remain finite at $r = 2M$.

(VI) The Schwarzschild black hole $(G=c=1)$

1. The "Flamm diagram"

In Schwarzschild coordinates the Schwarzschild metric reads:

$$g = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$

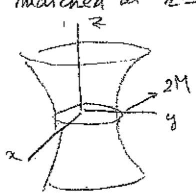
and exhibits singularities: $\begin{cases} r=2M & \text{in } g_{00}, g_{rr} \text{ components} \\ \theta = 0 & \text{in } g_{\varphi\varphi} \\ r=0 & \text{in all components}\end{cases}$

The singularity in the 2-sphere metric $(d\theta^2+\sin^2\theta\, d\varphi^2)$ at $\theta=0$ is well-known & due to the fact that one needs 2 maps (ie 2 distinct sets of coordinates) to cover the full 2-sphere.

Let us consider then the sections $\theta=\frac{\pi}{2}$ of the $t=$ cte spatial surfaces: the induced metric is $h = \frac{dr^2}{1-2M/r} + r^2 d\varphi^2$. For $r>2M$ and

As already mentioned this is the metric on a paraboloid $z^2 = 8M(r-2M)$

At the end of gravitational collapse there is no more interior solution to be matched at $r=r_0>2M$. Hence one must envision the full diagram:



The very curious feature of this "Flamm" diagram (1920) is that is exhibits TWO asymptotically flat regions.

It took 50 years to understand that.
(cf J. Eisenstaedt)

2. The Schwarzschild singularities

The quantity which characterizes the geometry of spacetime is is curvature tensor, but only scalar quantities do not depend on the chosen coordinate system. In Schwarzschild space-time $R=0$; $R_{ij}=0$, $R_{ijkl}\neq 0$ (and what computed above). One finds:

$R_{ijkl}R^{ijkl} = 48M^2/r^6$. Hence $r=0$ is a curvature singularity

$r=2M$ is not and hence may just be a coordinate singularity only.

NB: defining properly what is a "singularity" is an ongoing topic in GR. see Hawking & Ellis, Wald

Lemaître (1933) (?) was the first to find a coordinate change which eliminated the singularity at $r=2M$.

Look for a coordinate change $(t,r)\to(\tau,\rho)$ such that:

$$g = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$
$$= -d\tau^2 + g_{\rho\rho}(\tau,\rho)\,d\rho^2 + r^2(\tau,\rho)(d\theta^2+\sin^2\theta\, d\varphi^2).$$

In the new (τ,ρ) system the coordinate lines $\rho=$ const are geodesics (as can be easily checked: $du^i/d\tau + \Gamma^i_{00}=0$) and τ their proper time.

Hence: $\begin{cases} d\rho = \frac{\partial\rho}{\partial t}dt + \frac{\partial\rho}{\partial r}dr = 0 & \text{is a radial geodesic} \\ d\tau = \frac{\partial\tau}{\partial t}dt + \frac{\partial\tau}{\partial r}dr & \text{is their proper time}\end{cases}$

We have already studied the radial geodesics; they are such that:

$\begin{cases} \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \\ \frac{dt}{d\tau} = \frac{1}{1-2M/r}\end{cases}$ (radial infall with zero velocity at ∞)

Hence the system of equations characterizing the coordinate change:

$$\begin{cases} \frac{\partial\rho}{\partial t}\frac{1}{1-2M/r} - \frac{\partial\rho}{\partial r}\sqrt{\frac{2M}{r}} = 0 \\ 1 = \frac{\partial\tau}{\partial t}\frac{1}{1-2M/r} - \frac{\partial\tau}{\partial r}\sqrt{\frac{2M}{r}}\end{cases}$$

If we choose $\partial\rho/\partial t = 1$ and $\partial\tau/\partial t = 1$ the solution is:

$$\rho = t + \int\frac{dr}{(1-2M/r)\sqrt{2M/r}} \; ; \quad \tau = t + \int\frac{dr\sqrt{2M/r}}{1-2M/r}$$

Note that the transformation is singular at $r=2M$.

We now need the explicit expression for $g_{\rho\rho}(\tau,\rho)$.

g being a $\binom{0}{2}$ type tensor: $g'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}g_{kl}$, we have:

$$g_{rr} = \frac{1}{1-2M/r} = \underbrace{\left(\frac{\partial\tau}{\partial r}\right)^2}_{\frac{1}{(1-2M/r)^2}\frac{2M}{r}} \underbrace{g_{\tau\tau}}_{-1} + \underbrace{\left(\frac{\partial\rho}{\partial r}\right)^2}_{\frac{1}{(1-2M/r)^2}\frac{1}{2M/r}} g_{\rho\rho}$$

which yields: $g_{\rho\rho} = \frac{2M}{r}$

The final step is to obtain the explicit expression for $r(\rho,\tau)$.

$$\rho - \tau = \int\frac{dr}{(1-2M/r)}\left(\frac{1}{\sqrt{2M/r}} - \sqrt{2M/r}\right) = \frac{2}{3}\frac{1}{\sqrt{2M}}r^{3/2}$$

so that the Schwarzschild metric reads in Lemaître coordinates:

$$g = -d\tau^2 + \frac{2M\, d\rho^2}{\left[\frac{3}{2}\sqrt{2M}\,(\rho-\tau)\right]^{2/3}} + \left[\frac{3}{2}\sqrt{2M}\,(\rho-\tau)\right]^{4/3}(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

Remarks:
- the metric coefficients now depend on time τ: the staticity of Schwarzschild spacetime is no longer manifest
- $r = 2M \iff \rho - \tau = \frac{4M}{3}$: the metric coefficients are no longer singular at $r = 2M$.
- $\rho = \tau \iff r = 0$: the curvature singularity still shows up
- once Schwarzschild's metric is given in Lemaître's form one may "forget" that the transformation $(t, r) \to (\tau, \rho)$ is well defined for $r > 2M$ only.

as an aside: consider the sections $\tau = $ constant, eg $\tau = 0$. The induced metric is: $h = \frac{2M d\rho^2}{\left[\frac{3}{2}\sqrt{2M}\,\rho\right]^{2/3}} + \left[\frac{3}{2}\sqrt{2M}\,\rho\right]^{4/3}(d\theta^2 + \sin^2\theta\, d\varphi^2)$

Perform the change of coordinate $\rho \to R = \left(\frac{3}{2}\right)^{2/3}(2M)^{1/3}\rho^{2/3}$ and realise that $h = dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$. In Lemaître coordinates, the spatial sections $\tau = $ constant are Euclidean.

other aside: The Lemaître form of Schwarzschild's metric is a particular case of the Tolman-Bondi metric, see above where $M(\rho) = M$ is a constant (so that $\epsilon = 0$; empty space) and $\tau_0 = \rho$.

3. An example of "maximal extension": from Rindler ST to M_4

Consider the 2-D metric: $ds^2 = -x^2 dt^2 + dx^2$, $t \in [-\infty, +\infty]$; $x \in [0, +\infty[$ (This is the 'Rindler metric'). The game here is to get rid of the coordinate singularity at $x = 0$ using the powerful method devised by Finkelstein (1958) [and anticipated by Eddington (1922)]

Consider the null geodesics: $ds^2 = 0$ $\quad t = \pm \ln x + $ Const and introduce new, "null" coordinates: $u = t - \ln x$, $v = t + \ln x$ $(u, v \in [-\infty, +\infty]$; in these coordinates $ds^2 = -e^{v-u}\, du\, dv$.

u & v are "Eddington-Finkelstein" coordinates.

then introduce $U = -e^{-u}$, $V = e^{v}$, $U \in [-\infty, 0]$; $V \in [0, +\infty]$

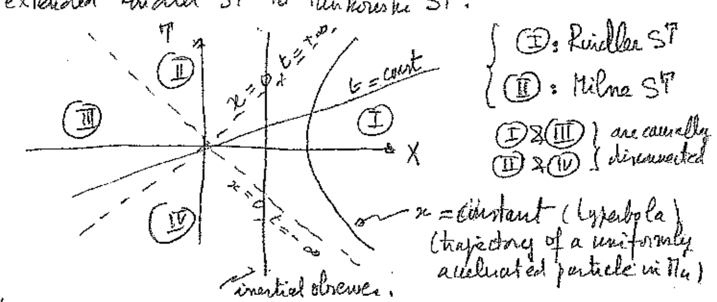
the metric becomes: $ds^2 = -dU\, dV$.

This metric is regular everywhere, hence one can extend the intervals of variation of the coordinates U & V to the whole real line.

Finally: $T = \frac{U+V}{2}$; $X = \frac{V-U}{2}$; $(T, X) \in [-\infty, +\infty]$ yields

$ds^2 = -dT^2 + dX^2$. $\quad (T = x \cosh t;\ X = x \sinh t)$

Hence: in going from the Rindler coordinates (t, x) to the Minkowski-coordinates (T, X) we have eliminated the coordinate singularity at $x = 0$ AND extended Rindler ST to Minkowski ST.



(I): Rindler ST
(II): Milne ST
(I) & (III), (II) & (IV) are causally disconnected

4. The Kruskal-Szekeres extension of Schwarzschild space-time (1960)

Consider the sections $\theta = \pi/2$, $\varphi = $ const of Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1 - 2M/r} \qquad t \in [-\infty, +\infty],\ r \in]0, +\infty]$$

the null geodesics are $t = \pm r_* + $ Const where r_* is the "tortoise" coordinate (Wheeler) $r_* = r + 2M \ln(|r/2M - 1|)$

- Introduce the Eddington-Finkelstein null coordinates:

  $u = t - r_*$, $\quad v = t + r_*$, $\quad u, v \in [-\infty, +\infty]$

  so that $ds^2 = -\frac{2M}{r} e^{-r/2M} e^{\frac{(v-u)}{4M}}\, du\, dv$.

- Introduce $U = -e^{-u/4M}$, $V = e^{v/4M}$ $\quad U \in [-\infty, 0]$ $\quad V \in [0, +\infty]$

  so that $ds^2 = -\frac{32M^3}{r^3}\, dU\, dV$ and EXTEND (U, V) to the whole plane

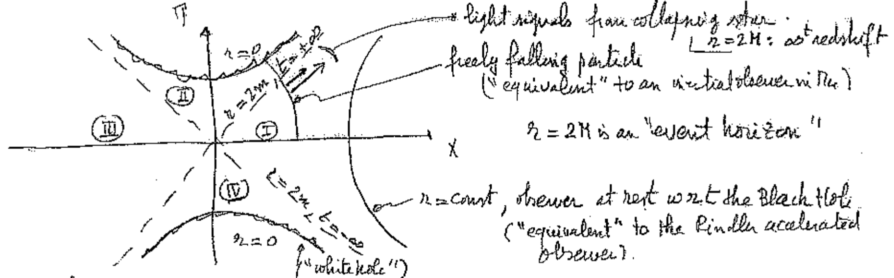
- finally set $T = \frac{U+V}{2}$ $\quad X = \frac{V-U}{2}$ $\quad (T, X) \in [-\infty, +\infty]$

  to obtain the 'Kruskal metric' (1960).

$$ds^2 = \frac{32M^3}{r}\exp\left(-\frac{r}{2M}\right)\left(-dT^2+dX^2\right)+r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

where $T = \pm\sqrt{\left|\frac{r}{2M}-1\right|}\,\exp\frac{r}{4M}\,\sinh\frac{t}{4M}$; $X = \pm\sqrt{\left|\frac{r}{2M}-1\right|}\,\exp\frac{r}{4M}\,\cosh\frac{t}{4M}$; $r \gtrless 1$

(+ signs in regions I & II ; − sign in regions III & IV).



Schwarzschild coordinates (t,r) cover regions I & II only ; the extension of Schwarzschild ST to regions III & IV gives a meaning to the 2 asymptotic regions of the Flamm diagram.

5. The Penrose – Carter conformal diagram

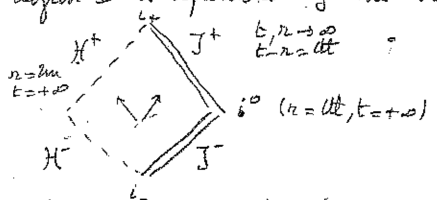
Consider region I of Kruskal's diagram.
Go to (T,X) to (Ψ,ξ) such that $T+X=\tan\frac{1}{2}(\Psi+\xi)$; $T-X=\tan\frac{1}{2}(\Psi-\xi)$ ($\Psi+\xi\in[0,\pi]$; $(\Psi-\xi)\in[-\pi,0]$; $\xi>0$). The Schwarzschild metric reads, in that region:

$$ds^2=\Omega^2 d\bar{s}^2 \quad \text{with} \quad d\bar{s}^2=-d\Psi^2+d\xi^2+r^2\Omega^{-2}\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)$$

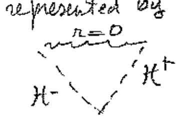
and

$$\Omega^2=\frac{32M^3}{r}\,\frac{e^{-r/2M}}{4\cos^2\frac{1}{2}(\Psi+\xi)\cos^2\frac{1}{2}(\Psi-\xi)}$$

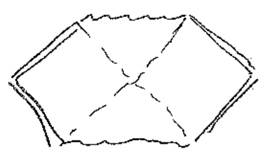
In those (Ψ,ξ) coordinates infinity is brought to finite distances, and region I is represented by the block:



Similarly region II is represented by



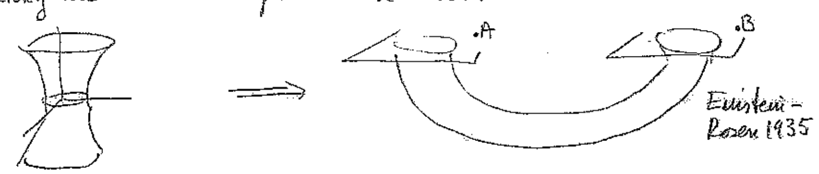
So that the conformal diagram of the full Kruskal ST is



see Carter, les Houches, 1972 for details.

6. Einstein – Rosen bridge (Wheeler's "wormhole")

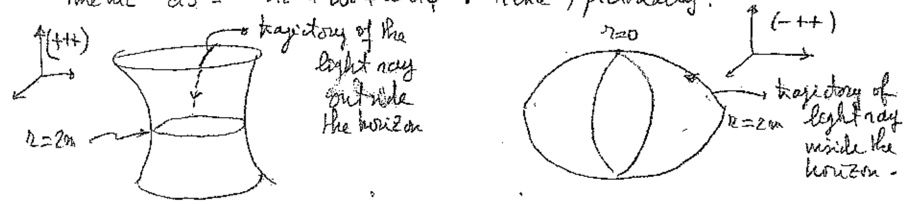
distorting the Flamm diagram to a "wormhole":



Question: if we identify the 2 asymptotically flat region can A send signals to B through the wormhole faster than by using the asymptotically flat route?

The answer can be read off Kruskal's diagram and is NO since regions I and III are causally disconnected.

One can also answer by studying how the "wormhole" which is a snapshot of Schwarzschild space time at time t evolves as the light ray approaches the horizon. As long as $r>2M$ the trajectory of the light ray is represented by a curve on the Flamm paraboloid. But for $r<2M$ the geometry of the surfaces $t=$ constant, $\theta=\pi/2$ is no longer a paraboloid. Indeed the surface with metric $d\sigma^2=-\frac{dr^2}{\frac{2M}{r}-1}+r^2d\varphi^2$ is the closed surface with equation $z^2=8M(2M-r)$ embedded in the pseudo-euclidean space with metric $dS^2=-dz^2+dr^2+r^2d\varphi^2$. Hence, pictorially:



VI Charged black-holes

1. The Reissner-Nordström solution (1916, 1918)

Consider a static, spherically symmetric ST in Schwarzschild coordinates:

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

and solve Einstein's equs $G_{ij} = 8\pi T_{ij}$ when matter is electrostatically charged, that is when $T_{ij} = F_{ik}F^{k}{}_{l} - \frac{1}{4} F_{kl}F^{kl}$ with $F_{ij} = \partial_i A_j - \partial_j A_i$ and $A_i = (\Phi(r), 0)$.

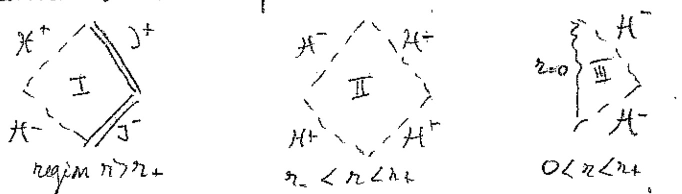
Find the Reissner-Nordström solution:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + Q^2/r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

with $\Phi = +Q/r$.

If $Q^2 > m^2$ (which is the case for all electrically charged particles) the curvature singularity at $r = 0$ is "naked": light signal can propagate from $r=0$ to infinity. Penrose's "cosmic censorship hypothesis" is the (still unproven) conjecture that "realistic" gravitational collapse will never end up in the occurrence of such naked singularities.

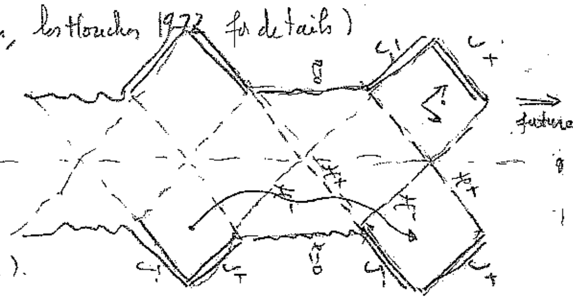
If $Q^2 < m^2$ there are 2 horizons and the Penrose-Carter diagram of the maximally extended Reissner-Nordström spacetime is the card-board castle built out of 3 blocks:



Graves & Brill 1960
Carter 1966

and is (see Carter, les Houches 1972 for details)

Hence the $\exists$ of an infinite nb of asymptotic regions which are now causally connected (Note that $\nexists$ Cauchy surface).



2. On the stability of the solution...

An important question is of course that of the stability of the Schwarzschild & Reissner Nordström solutions.

A first approach to the problem is to consider the propagation of test-fields in such geometries & see if they exhibit growing modes. The "paradigmatic" equation to be solved being the Klein-Gordon equation:

$$\Box\phi = 0 \quad \text{with} \quad \Box = \frac{1}{\sqrt{-g}}\partial_i \sqrt{-g}\, g^{ij}\partial_j$$

spherical symmetry & staticity (in Schw. coordinates) allow the mode decomposition: $\Phi = e^{-iEt} Y_\ell^m(\theta,\varphi) R(r)$ and one is left with solving the radial equation for $R(r)$.

Similar equations are found for various test fields

A similar equation is also found for the perturbations of the metric (solution to the vacuum Einstein equations). (Teukolski, Regge Wheeler, Zerilli) (see Chandrasekhar's book & M. Sasaki for details).

In the case of Schwarzschild's BH $\Box\phi = 0$ does not exhibit any growing mode if one solves it in the region I only. BUT if one solves it in regions I + II of the Kruskal diagram one discovers the Hawking instability (1974): black holes radiate (at a very slow rate though, which is negligible for BH with masses $M > 10^5 g$).

The inner horizons of Reissner-Nordström's BHoles are "classically" unstable as there are growing modes in region II, even with no extension of the modes to the other regions. (Israel et al, cf workshop on num relativity

Hence the aside: what is the generic geometry of spacetime near a curvature singularity?

This investigation started with ~~Lifshitz~~, Khalatnikov & Belinski (cf Damour, Henneaux, Spindel & cosmology workshop for recent developments.) Two generic features: curvature singularities are spacelike (ie Schwarzschild rather than RN-like) & oscillatory.

3. Reissner-Nordström "irreducible mass"

Consider a charged test particle, of mass m & charge e in the gravitational field of a charged black hole. Its eom is Lorentz equation:

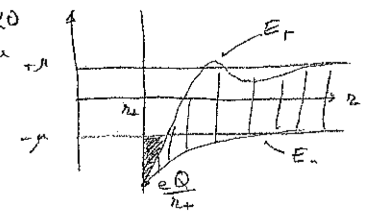
$$m\frac{Du^i}{d\sigma} = e\,F_j{}^i u^j \qquad F_{ij} = \partial_i A_j - \partial_j A_i$$

First integrals are easy to obtain: $\theta = \pi/2$; $\dot\varphi = L/r^2$; $\dot t = \dfrac{E - eQ/r}{1 - 2M/r + Q^2/r^2}$
and:

$$\dot r^2 = (E-E_+)(E-E_-) \quad \text{with} \quad E_\pm = \frac{eQ}{r} \pm \sqrt{\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(\mu^2+\frac{L^2}{r^2}\right)}$$

where L & E are constant of integration which can be interpreted as the angular momentum of the charge wrt the BH and its energy (indeed the $\exists$ of L is linked to the existence of the Killing vector $\xi^i_\varphi \doteq (0001)$ expressing axial symmetry & E is related to $\xi^i_t = (1000)$ the KV expressing stationarity).

case $eQ<0$
$eQ<-\mu$



$r_+ = M + \sqrt{M^2 - Q^2}$

From this plot one can describe trajectories qualitatively.

Send a charge A with charge e_A, angular momenta L_A, energy $E_A > \mu$ from infinity down the hole; Suppose that close to the hole this charge splits in 2 charges $B(e_B, L_B, E_B)$ & $C(e_C, L_C, E_C)$; In the local, freely falling, inertial frame, Special relativity applies, hence:

$$e_A = e_B + e_C\,; \quad E_A = E_B + E_C\,, \quad L_A = L_B + L_C.$$

Suppose now that $\frac{eQ}{r_+} < E_B < -\mu_B$ $\therefore$ the B particle will fall into the BH, hence decreasing its charge by e_B and its mass by $|E_B|$; as for the other particle it will fly to ∞ carrying that energy. This "Penrose process" (1969) is a "gedanken experiment" to show how energy can be extracted from a BH. Hence also the name "ergoregion" given to the shaded part of the diagram (Christodoulou-Ruffini 1970-1971).

The maximum energy which can be extracted corresponds to the case $E = eQ/r_+$; therefore setting $\delta Q = |e$; $\delta M = E$ we have

$$\delta M \geqslant \frac{Q\,\delta Q}{r_+}$$

Can one put this inequality under the form $\delta F \geqslant 0$?

The answer is yes. Choose F to be a function of the geometry of the BH, that is its area: $F = F(A)$ with $A = 4\pi r_+^2$, $r_+ = M + \sqrt{M^2-Q^2}$

then $\delta F = \dfrac{dF}{dA}\,\dfrac{r_+}{\sqrt{M^2-Q^2}}\left(\delta M - \dfrac{Q\delta Q}{r_+}\right)$;

hence $\delta M \geqslant Q\delta Q/r_+ \iff \delta F \geqslant 0$ whith $F = F(A)$ (& $dF/dA > 0$).

Are there "natural choices" for the function $F(A)$?

The Princeton school (Wheeler, Ruffini, Christodoulou ...) was interested in relativistic astrophysics, aiming at explaining supernovae, quasars etc by means of energy extraction from BH. They chose $S(A)$ to have the dimension of a mass: $F \equiv M_{irr}$, $M_{irr} = \frac{1}{2} r_+$ $\boxed{}$

Hence the inequality $\delta M_{irr} \geqslant 0$.

Recall its meaning: if we start with a black hole of mass M, charge Q and devise processes near the horizon which produce particles in the ergosphere, energy can be thus extracted from the BH with the restriction that $\delta M_{irr} \geqslant 0$. A "reversible" transformation corresponds to $\delta M_{irr} = 0$. Hence the maximum energy which can be extracted from a charged BH is

$$M - M_{irr} = \frac{M - \sqrt{M^2-Q^2}}{2}$$

For an "extreme" BH, that is for $M = Q$, that is ½ its rest mass!
(cf Ruffini for developments).

Simultaneously Bekenstein, Bardeen, Carter, Hawking, Israel chose for F the area itself & interpreted $dA \geqslant 0$ in a thermodynamical terms & conjectured that A should measure somehow the "entropy" of the BH.

Now, if we look more carefully at dimensions we see that if $F(A)$ must be interpreted as an entropy A must be rescaled to became dimensionless; Now A has dimension L^2; with only G & c at our disposal one cannot build a quantity having dimension of a length. $\hbar$ has to be introduced to build the Planck length $l_P = (\hbar G/c^3)^{1/2}$
Hawking (1974) understood that quantum processes were at stake ....

VII The Kerr solution

1. Kerr-Schild metrics et Kerr's solution

Einstein's equations are non linear. There $\exists$ however classes of metrics which make them linear. Consider metrics whose coefficients in appropriate coordinate systems read:

$$\begin{cases} g_{\mu\nu} = \bar g_{\mu\nu} + l_{\mu\nu} & \text{where } \bar g_{\mu\nu} \text{ is a given "background" metric} \\ \text{and where } \; l_{\mu\nu} \mathrel{\dot=} f(x^\alpha)\, l_\mu l_\nu & \text{with } l^\mu \equiv \bar g^{\mu\nu} l_\nu \text{ null \& geodesic,} \\ \text{that is: } \; \bar g_{\mu\nu} l^\mu l^\nu = 0 & \text{and } \; l^\mu \bar D_\mu l^\nu = 0. \end{cases}$$

It is a (good!) exercise to compute the Ricci tensor for such metrics. The result is:

$$\left.\begin{aligned} R^\mu{}_\nu &= \bar R^\mu{}_\nu - l^{\mu\sigma}\bar R_{\sigma\nu} + \bar D_\rho\left(\bar g^{\mu\sigma}\Delta^\rho_{\nu\sigma}\right) \\ \text{with} \quad \Delta^\rho_{\nu\sigma} &= \tfrac{1}{2}\left[\bar D_\nu l^\rho{}_\sigma + \bar D_\sigma l^\rho{}_\nu - \bar D^\rho l_{\nu\sigma}\right] \end{aligned}\right\}$$

This (exact) expression is linear in the "perturbation" $l_{\mu\nu}$.

As for the scalar curvature $R = R^\mu{}_\mu$, it reduces to:

$$R = \bar R - l^{\mu\nu}\bar R_{\mu\nu} + \left|\begin{aligned} &\bar D_\rho\left(l^\rho \bar D_\sigma (f l^\sigma)\right) \\ &\frac{1}{\sqrt{-\bar g}}\,\partial_\rho\left[l^\rho \partial_\sigma\left(\sqrt{-\bar g}\, f l^\sigma\right)\right] \end{aligned}\right.$$

example: $\begin{cases} d\bar s^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) & : \bar R_{\mu\nu} = 0. \\ l^\mu = (1, 1, 0, 0) & (\bar g_{\mu\nu} l^\mu l^\nu = 0;\; l^\mu \bar D_\mu l^\nu = 0) \end{cases}$

trace of Einstein's vacuum equations: $\quad (\sqrt{-\bar g} = r^2 \sin\theta)$

$$R = 0 \iff \frac{1}{r^2}\frac{d^2}{dr^2}(r^2 f) = 0 \iff f = \frac{a + br}{r^2}$$

- One must then check if this solution is indeed Ricci-flat:

$$R^0{}_0 = R^1{}_1 = \frac{1}{2r^2}(r^2 f')' \; ; \; R^2{}_2 = R^3{}_3 = \frac{f^2}{r^2}(rf)'$$

Hence $R^2{}_2 = R^3{}_3 = 0$ if $a = 0$ and then $R^0{}_0 = R^1{}_1 = 0$.

We thus recover Schwarzschild's metric, in Kerr-Schild coordinates: (setting $a \equiv 2M$)

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}dt\,dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

(the coordinate change $t \to t - t_0(r)$ with $t_0' = -\frac{2M}{r - 2M}$ yields Schw's form)

Suppose now we still take a flat background but write its metric in spheroidal coordinates $(\rho\theta\varphi)$ (see p 5):

$$d\bar s^2 = -dt^2 + \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2}d\rho^2 + (\rho^2 + a^2\cos^2\theta)d\theta^2 + (\rho^2 + a^2)\sin^2\theta\, d\varphi^2;$$

and choose for the null vector: $l_\mu = \left(1, \frac{\rho^2 + a^2\cos^2\theta}{\rho^2 + a^2}, 0, a\sin^2\theta\right)$

The trace of the Einstein vacuum equation $R = 0$ is easily solved (Mathematica helps!) and yields: $f = \frac{a(\theta) + \rho\, b(\theta)}{\rho^2 + a^2\cos^2\theta}$

One must then ensures that the metric is indeed Ricci-flat, this imposes $b(\theta) = 0$; $a(\theta) = \text{const} \equiv 2M$; hence: $f = \frac{2M\rho}{\rho^2 + a^2\cos^2\theta}$.

This solution is Kerr's solution (1963) in Kerr-Schild coordinates.

Comments:
- the solution can be generalized to the case when the background is (A)dS & Einstein's equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ (Carter, 1968)
- It can be extended to the charged case: Kerr-Newmann
- The uncharged (A)dS solution can be extended to all dimensions (Myers-Perry, Gibbons et al)
- There is no Kerr-Schild type solution to the Einstein-Maxwell equations in $D > 4$ (and $a \neq 0$).

In 1967 Boyer & Lindquist rewrote the Kerr metric as:

$$ds^2 = -\frac{\Delta}{\Sigma^2}(dt - a\sin^2\theta\, d\varphi)^2 + \frac{\sin^2\theta}{\Sigma^2}\left[(r^2 + a^2)d\varphi - a\,dt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma^2 d\theta^2$$

with $\Delta \equiv r^2 - 2Mr + a^2$; $\Sigma^2 \equiv r^2 + a^2\cos^2\theta$.

NB: for the explicit transformation from the Kerr-Schild coordinates $(t, \rho, \theta, \varphi)$ to the Boyer-Lindquist ones $(t_{BL}, r, \theta, \varphi)$, see, e.g. Gibbons et al.

2. The Kerr black hole

In Boyer-Lindquist coordinates the stationarity & axisymmetry of Kerr's geometry is manifest. It possesses 2 Killing vectors $\xi^i_{(t)} = (1,0,0,0)$ and $\xi^i_{(\varphi)} = (0,0,0,1)$. It is also asymptotically flat.

The asymptotic form of the metric is :

$$ds^2 = -\left[1-\frac{2M}{r}+\mathcal{O}\left(\frac{1}{r^2}\right)\right]dt^2 - \left[\frac{4aM}{r}\sin^2\theta + \mathcal{O}\left(\frac{1}{r^2}\right)\right]dt\,d\varphi +$$
$$+\left[1+\mathcal{O}\left(\frac{1}{r}\right)\right]\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

As we saw "global charges" can be associated to space-times by integration of various conservation laws $\partial_i \Theta^{ij} = 0$ where Θ_{ij} is some pseudo-tensor which can be rewritten as integrals over the 2-sphere at ∞ (by means of "superpotentials" or "à la Komar"). All definitions & construct agree to interpret M as the gravitational mass of Kerr's spacetime and $J \equiv Ma$ as its angular momentum (see Deser for a review): Kerr's solution describes a stationary, rotating object.

The Kerr metric in Boyer-Lindquist coordinates is singular at :

$$\begin{cases} \Delta = 0 : & r = r_\pm \qquad r_\pm = M \pm \sqrt{M^2-a^2} \\ \Sigma^2 = 0 & \text{ie } r=0 \text{ AND } \theta = \pi/2 \end{cases}$$

$\Sigma^2 = 0$ is a curvature singularity (see Carter 1972 for the explicit expression of the Kerr Riemann tensor) of a "ring type".

$r = r_\pm$ are coordinate singularities ; surfaces of ∞ redshift, horizons

The Kruskal-like maximal extension of the Kerr manifold is similar to that of the Reissner-Nordström ST ; see Carter 1972, Hawking & Ellis.

Important remarks :

(1) There is no known "realistic" (ie other than ∞ thin disks in the equatorial plane) interior solution which would match Kerr outside the star

(2) Unicity : the Kerr solution is the only solution of Einstein's vacuum equations which is stationary, axisymmetric and possesses an horizon (See eg Carter gr-qc/0604064 for a (vivid) review.

3. The Kerr black hole ergoregion

The detailed analysis of geodesic motion in the Kerr metric was performed by Carter (1966) and will be reviewed by JP Lasota.

We shall here only note the following :

In Boyer-Lindquist coordinates the metric coefficients do not depend on t & φ.

Hence $\begin{cases} u_t = g_{tt}\dot{t} + g_{t\varphi}\dot{\varphi} = -E \\ u_\varphi = g_{\varphi\varphi}\dot{\varphi} + g_{t\varphi}\dot{t} = L \end{cases}$ are constant (isometries)

the angular velocity of the test particle, as defined asymptotically, that is wrt time t is : $\dfrac{d\varphi}{dt} = \dfrac{\dot{\varphi}}{\dot{t}} = -\dfrac{E g_{t\varphi} + L g_{tt}}{E g_{\varphi\varphi} + L g_{t\varphi}}$.

Thus a zero-angular momentum test particle ($L=0$) is such that $\dfrac{d\varphi}{dt} = -\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}$

Now : $g_{t\varphi} = -\dfrac{2Mra\sin^2\theta}{\Sigma^2}$; $g_{\varphi\varphi} = (r^2+a^2)^2 - a^2\Delta\sin^2\theta = (r^2+a^2)\Sigma^2 + 2Mra^2\sin^2\theta$

hence $\dfrac{d\varphi}{dt} > 0$: A zero-angular momentum test particle is "dragged" by the BH.

Consider now a light ray with $\dot\theta = 0$ & $\dot r = 0$; its trajectory is given by

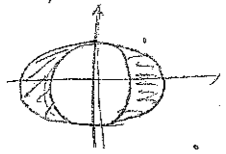
$$0 = g_{tt}\dot{t}^2 + 2g_{t\varphi}\dot{t}\dot{\varphi} + g_{\varphi\varphi}\dot{\varphi}^2 \quad \text{(zero-length world-line).}$$

with solutions : $\dfrac{d\varphi}{dt} = \dfrac{-g_{t\varphi} \pm \sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}{g_{\varphi\varphi}}$ (prograde & retrograde trajectories)

if r lies on the surface $g_{tt} = 0$ then $\dfrac{d\varphi}{dt} = 0$, $\dfrac{d\varphi}{dt} = -\dfrac{2g_{t\varphi}}{g_{\varphi\varphi}} > 0$

Hence : In the region $\theta = $ cst, r such that $g_{tt} > 0$, <u>all</u> particles <u>must</u> corotate with the BH.

Now : $g_{tt} = -\dfrac{r^2 - 2Mr + a^2\cos^2\theta}{\Sigma^2}$, so that $g_{tt} > 0 \Leftrightarrow r < M + \sqrt{M^2 - a^2\cos^2\theta}$ (outside horizon).



($g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi} = 0$ is the horizon r_+)

In this region Penrose processes can take place.

The maximum energy which can be extracted is when $E = -\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_+ L$ $\left(\dfrac{d\varphi}{dt} = 0\right)$

Now $-\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_+$ is the angular velocity (as seen from ∞) of photons orbiting on the horizon ; it is identified with the angular velocity of the BH : $\Omega = -\dfrac{g_{t\varphi}}{g_{\varphi\varphi}}\Big|_+$

Hence $\delta M \geq \Omega_{hole}\,\delta J$. (2nd law of BH dynamics).