Cosmological helium production simplified

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The authors present a simplified model of helium synthesis in the early universe. The purpose of the model is to explain clearly the physical ideas relevant to the cosmological helium synthesis in a manner that does not overlay these ideas with complex computer calculations. The model closely follows the standard calculation, except that it neglects the small effect of Fermi-Dirac statistics for the leptons. The temperature difference between photons and neutrinos during the period in which neutrons and protons interconvert is also neglected. These approximations permit the expression of neutron-proton conversion rates in a closed form, which agrees to 10% accuracy or better with the exact rates. Using these analytic expressions for the rates, the authors reduce the calculation of the neutron-proton ratio as a function of temperature to a simple numerical integral. They also estimate the effect of neutron decay on the helium abundance. Their result for this quantity agrees well with precise computer calculations. Their semianalytic formulas are used to determine how the predicted helium abundance varies with such parameters as the neutron lifetime, the baryon-to-photon ratio, the number of neutrino species, and a possible electron-neutrino chemical potential.

CONTENTS

I. INTRODUCTION

In the early 1950s George Gamow and his collaborators, Ralph Alpher and Robert Herman, inaugurated the study of physical processes in the early universe, including the origin of light elements such as helium and deuterium. The essential idea was that, given an initial equilibrium concentration of neutrons and protons, they would fuse into deuterium as soon as the ambient temperature of the universe dropped substantially below the binding energy of the deuteron. Once deuterium was formed it would rapidly enter into a chain of nuclear reactions that would ultimately produce the other elements. The original notion was to build up all the elements, light and heavy, by a series of captures and decays. It was soon realized, however, that this scheme would not work to produce the heavy elements, since there is no stable mass-five nucleus and the Coulomb barriers block element formation as the temperature falls. Hence the emphasis became focused on increasingly detailed calculations of light-element formation and, in particular, the formation of helium. The amount of deuterium, and hence helium, formed was governed by the neutron-to-proton ratio at the time corresponding to the capture temperature at which the deuterium could be formed. Any neutrons available at that time would be conscripted into this activity. The first attempt to calculate the n/p ratio at the capture temperature was made by Alpher and Herman (1950). In making this estimate they assumed that the only effect causing the depletion of neutrons was their instability against beta decay. This, as we shall see, and as was first pointed out by Hayashi (1950), is a relatively small effect. The major effect in the interconversion of neutrons and protons in the inelastic scattering of neutrinos from nucleons, reactions such as $\nu_e+n\rightarrow e^-+p$. Hayashi found that the n/p ratio at the capture temperature was about 0.25. This is to be compared to the modern estimate, which we shall derive, of about 0.12. In 1953 Alpher, Follin, and Herman redid Hayashi's calculation in greater detail. They analyzed how the result depended on the neutron lifetime and found answers that varied between 0.22 and 0.17, depending on what they took for the lifetime. As we shall explain in detail, the longer the lifetime, the larger the ratio. This calculation was done prior to the development of the modern theory of weak interactions, and Alpher *et al.* took "for lack of a better estimate" unity for the matrix element for the inelastic scattering. In the present weak-interaction theories this constant, which is of order unity, can be computed from the theory.[1] That was in

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[1]One feature of these early calculations was that they involved Dirac neutrinos. This doubled the number of neutrino states. This does not, however, affect the final answer, since the factor of 2 is compensated in the relation between the effective weak coupling constants and the neutron mean life.

fact done by Hoyle and Tayler (1964), Peebles (1966), and Wagoner, Fowler, and Hoyle (1967). We shall not discuss these results in detail here, except to note that since, for helium production, these independent calculations give nearly the same answers, we have, for convenience, made our numerical comparisons to the results given by Peebles (1966), after having corrected his results for modern improvements in such experimental data as the neutron lifetime and the number of neutrino types.

It is probably fair to say that this work, when it was first done, was primarily of interest to a relatively small community of astrophysicists and cosmologists—then also a relatively small community. What broadened the interest was the gradual realization that the early universe in general, and the helium production in particular, was such a good probe of some aspects of particle physics. Of special interest to us is the use of the observed helium abundance in the universe by weight, some 25%, to set limits on the number of neutrino types. The fact that these limits seem to be confirmed in terrestrial laboratories—notably by the study of the width of the Z^0 boson—has made the theory of helium production in the early universe of wide general concern to a large community of physicists.

The first suggestion that the presence of various exotic particles in the early universe could change the amount of primordial helium produced seems to be due to Shvartsman (1969). He did not comment, however, on the use of the observed production ratio to limit the numbers of such particles and to limit the number of neutrino types in particular. That step was taken, at least qualitatively, by Peebles (1971). In the chapter of his book *Physical Cosmology* devoted to early universe helium production, Peebles observed that adding additional types of neutrinos beyond the electron and muon neutrinos, the only ones known at the time he wrote the book, would increase the expansion rate of the universe and hence—as we shall later explain in detail—the n/p ratio. He remarked that "extra classes of neutrinos could only cause trouble for the model." This qualitative observation was made quantitative in the paper of Steigman *et al.* (1977). Given the uncertainties that then existed in the observed helium abundance, these authors were only able to set an upper limit of seven types. Since that time, a vast amount of work has gone into refining both the experimental and theoretical results. [See Cline *et al.* (1987) for a recent review and for the connection between these limits and the ones being derived from the Z^0 experiments.] From these results one can conclude that at most four flavors of neutrinos exist. This result is consistent with the Z^0 experiments.[2]

[2]The conclusion about the number of neutrino types assumes that there is no significant lepton chemical potential. This amounts to assuming that the difference between the number of neutrinos and antineutrinos of any flavor is small compared to their sum. The consequences of relaxing this assumption have been discussed by various authors. See, e.g., Dimopoulos and Feinberg (1979) and our later discussion.

The work that we have described contains some of the most quantitatively accurate results in all of cosmology. As impressive as this work is, there is something about it that a physicist finds somewhat unsatisfying. While it is possible to give a simple equilibrium theory of helium production [see Peebles (1971) or Weinberg (1972) for nice presentations], this theory has only a very limited accuracy, since the particles do not remain in thermal equilibrium during the regime of interest. To go beyond the equilibrium theory has required the use of elaborate computer codes, giving the subject a kind of black box character. This point was made by Weinberg (1972) when he wrote, "Unfortunately, if we want to describe the behavior of $T(t)$ and $R(t)$ throughout the whole history of the universe, we have to do a numerical calculation." It seems to us therefore useful and instructive to present a version of the theory of helium production that would be both substantially more accurate than the equilibrium theory and would have an intuitive appeal to a physicist. As the reader will learn, our results agree to within a few percent with those generated by computer code, indicating that we have managed to isolate the essential physics; the details can be worked out using any pocket calculator that can perform one-dimensional numerical integrals.

In rough outline, the formation of the primordial helium occurs through the following series of events. At early times when the temperature of the universe, T, was on the order[3] of 100 MeV, the energy and number density were dominated by relativistic and therefore effectively massless particles: leptons (electrons, positrons, and neutrinos) and photons. At this early time, the smattering of neutrons and protons (with number fraction $\sim 10^{-9}$) contributed very little to the total energy density. All of the particles were kept in thermal equilibrium by their rapid collisions. The interactions of the neutrons and protons with the leptons,

$$\nu_e + n \leftrightarrow p + e^- , \tag{1.1a}$$

$$e^+ + n \leftrightarrow p + \bar{\nu}_e , \tag{1.1b}$$

and

$$n \leftrightarrow p + e^- + \bar{\nu}_e , \tag{1.1c}$$

also kept these baryons in chemical equilibrium. It is usually assumed that all of the leptons have a vanishing chemical potential, which means that the total numbers of particles and antiparticles were equal (except that the electrons have a very small chemical potential, so that they, together with the protons, maintain electrical charge neutrality). Under this assumption, the ratio of

[3]At temperatures somewhat in excess of 100 MeV, there was, presumably, a phase transition to a plasma of massless gluons and essentially massless quarks. Recently there has been some discussion of the effects on the primordial nucleosynthesis brought about by fluctuations in the quark-gluon phase transition. See, for example, Alcock *et al.* (1987) and Applegate *et al.* (1987). We do not consider such effects here.

the neutron to proton number densities at such early times is given by the simple Boltzmann factor

$$\frac{n_n(T)}{n_p(T)}=\exp\left[\frac{-\Delta m}{T}\right], \tag{1.2}$$

where Δm is the neutron-proton mass difference,

$$\Delta m = m_n - m_p \simeq 1.29 \text{ MeV} . \tag{1.3}$$

At the initial temperature of 100 MeV, the neutron-proton ratio is very close to unity. It proves convenient to express the ratio as the number of neutrons to the total number of baryons,

$$X(T)=\frac{n_n(T)}{n_n(T)+n_p(T)} . \tag{1.4}$$

So long as chemical equilibrium is maintained,

$$X(T)=X_{\text{eq}}(T)=\frac{1}{1+e^{\Delta m/T}} . \tag{1.5}$$

As we shall see in Sec. III, the helium production occurs when the age of the universe, t, is about 180 sec. Since this is a time that is short in comparison with the mean life of the neutron $\tau=896\pm16$ sec, it is a good first approximation to neglect the neutron decay process (and its reverse) displayed in Eq. (1.1c). This omission will be corrected in Sec. III. Now, as the universe expands and cools, the chemical equilibrium for the baryons is broken because the neutrino interactions are too weak to enforce it and, with the neglect of the neutron decay, the fraction of neutrons to baryons approaches a constant and non-vanishing value $X(T\simeq0)$.

To see how this happens we first calculate the expansion rate of the universe. During the epoch that concerns us the curvature term in the expansion of the universe is negligible. [See, for example, Weinberg (1972).] Taking the cosmological constant to vanish, we find from the Einstein equation that the scale factor of the universe $R(t)$ obeys

$$\left[\frac{dR/dt}{R}\right]^2=\frac{8\pi\rho}{3M_{\text{Pl}}^2} . \tag{1.6}$$

Here ρ is the energy density and M_{Pl} is the Planck mass, which, in the units which we shall use where $\hbar=c=1$, is simply related to Newton's gravitational constant G by

$$M_{\text{Pl}}=G^{-1/2}\simeq 1.22\times10^{19} \text{ GeV} . \tag{1.7}$$

During the times that concern us, assuming that there were no unknown massive species present in large numbers, the energy density is dominated by the massless particles, and we have

$$\rho=N\frac{\pi^2}{30}T^4 , \tag{1.8}$$

where N is the effective number of degrees of freedom. The number N is the sum of 2 for the photon, $\frac{7}{2}$ for the electron-positron, and $\frac{7}{4}$ for each of the three types of neutrino-antineutrinos, giving $N=\frac{43}{4}$. (This assumes that the muon and tau neutrinos have a mass small compared to the effective temperature, and that no other massless species are present in equilibrium. If not, N will be different.)

Now there comes a time t_F, or equivalently a temperature T_F, when the universal expansion rate $(dR/dt)/R$ exceeds the rate Λ at which the processes (1.1a) and (1.1b) maintain the baryon chemical equilibrium. At about this time the baryons become uncoupled from the leptons and, as we later show, the neutron-to-baryon ratio is frozen at the value

$$X(T\simeq0)=X_{\text{eq}}(T_F) . \tag{1.9}$$

The rate $\Lambda(T)$ is roughly given by

$$\Lambda(T)\simeq n_\nu(T)\langle\sigma v\rangle_T , \tag{1.10}$$

where $n_\nu(T)$ is the electron-neutrino number density and $\langle\sigma v\rangle_T$ is an average of the cross section σ for the reactions (1.1a) and (1.1b) times the relative velocity v. Since $n_\nu\sim T^3$ and $\sigma\sim G_F^2T^2$, where G_F is the Fermi constant of the weak interactions, and $v\sim1$, we have

$$\Lambda\sim G_F^2T^5 . \tag{1.11}$$

Setting

$$\Lambda\sim(dR/dt)/R , \tag{1.12}$$

and recalling Eqs. (1.6) and (1.8), one sees that

$$T_F^3\sim\frac{N^{1/2}}{G_F^2M_{\text{Pl}}} . \tag{1.13}$$

Since $G_F\simeq1\times10^{-5}$ GeV^{-2}, we have

$$T_F\sim N^{1/6}\times1 \text{ MeV} . \tag{1.14}$$

Upon substituting this T_F into Eqs. (1.5) and (1.9) with $\Delta m\sim1$ MeV, we find that a significant fraction of neutrons are left at the freezing temperature T_F. We have exhibited the dependence of the freeze-out temperature T_F on the effective number of massless species N to indicate how the number of neutrino types plays a role in helium production, a point to which we return later.

At temperatures of order T_F, the light nuclei D, T, ^{3}He, and ^{4}He are kept in thermal and chemical equilibrium by reactions such as

$$n+p\leftrightarrow \text{D}+\gamma , \tag{1.15a}$$

$$\text{D}+\text{D}\leftrightarrow\text{T}+p , \tag{1.15b}$$

and

$$\text{T}+\text{D}\leftrightarrow{}^4\text{He}+n . \tag{1.15c}$$

These equilibrium populations of D, T, and ^{4}He amount to a very small fraction. However, as we shall discuss in Sec. III, once the temperature T has fallen below about $\frac{1}{25}$ of the deuteron binding energy $\varepsilon_{\text{D}}=2.23$ MeV, a temperature that is much smaller than the freeze-out temperature T_F, the reactions shown in Eqs. (1.15) proceed almost entirely to the right. Because of the large binding energy of ^{4}He, $\varepsilon_{\mathrm{He}}=28.3$ MeV, nearly all of the original neutrons present at the freeze-out temperature are, after a short period of free neutron decay, captured in ^{4}He. Thus, at the conclusion of this big-bang nucleosynthesis, the ratio of the number of ^{4}He nuclei to the total number of baryons is given by

$$X_4=\tfrac{1}{2}X(T\simeq 0)\ , \tag{1.16a}$$

or, equivalently, the ^{4}He mass fraction of the total baryonic mass is given by

$$Y_4=2X(T\simeq 0)\ . \tag{1.16b}$$

Thus the freeze-out ratio $X(T\simeq 0)$ of the neutron-to-total-baryon number determines the amount of primordial ^{4}He production. In Sec. II we shall give a simple but accurate model that yields a detailed calculation of $X(T\simeq 0)$. Then, in Sec. III, we shall discuss the period following the freeze-out, including the small correction brought about by neutron decay and the reason why the temperature must be below about $\frac{1}{25}$ of the deuteron binding energy (the "deuteron bottleneck") before the nucleosynthesis can proceed. In the Appendix we present an accurate analytic formula that relates temperature and time during this epoch. Finally, in Sec. IV we discuss the sensitivity of the helium abundance to variations of the parameters on which it depends.

Before turning to the details, we pause to describe briefly some of the scales of the universe during the epoch that concerns us. Here we shall just quote some numerical values whose justification can be found in various formulas appearing throughout the text. First we note that a temperature $T=1$ MeV corresponds to the age of the universe given by $t\simeq 1$ sec. Time and temperature are related by $t\sim 1/T^2$. Thus our initial temperature of $T=100$ MeV corresponds to a time $t\simeq 10^{-4}$ sec. At the temperature $T=1$ MeV, there is a photon density given by $n_\gamma\simeq 10^{31}/\mathrm{cm}^3$ and a baryon density given by $n_B\simeq 10^{22}/\mathrm{cm}^3$. These densities vary as T^3. Thus at our initial temperature $T=100$ MeV there is a baryon density $n_B\sim 10^{28}/\mathrm{cm}^3$. Although this is a large density by ordinary standards (10^4 times that of water), it is yet very dilute in comparison to the nuclear matter density of $10^{39}/\mathrm{cm}^3$, and so the nucleons can be treated as an ideal gas.

II. NEUTRON-PROTON RATIO IN THE EXPANDING UNIVERSE

We now describe our simple model that provides an accurate account of the neutron abundance as the universe evolves. We denote by $\lambda_{pn}(t)$ the rate for the weak processes to convert protons into neutrons and by $\lambda_{np}(t)$ the rate for the reverse processes that convert neutrons into protons. These rates are time dependent because they depend on temperature, which in turn is a function of time. Thus the basic rate equation for the ratio $X(t)$ of the number of neutrons to the total number of baryons reads

$$\frac{dX(t)}{dt}=\lambda_{pn}(t)[1-X(t)]-\lambda_{np}(t)X(t)\ . \tag{2.1}$$

This rate equation has the solution

$$X(t)=\int_{t_0}^{t}dt'I(t,t')\lambda_{pn}(t')+I(t,t_0)X(t_0)\ , \tag{2.2}$$

where the integrating factor is given by

$$I(t,t')=\exp\left[-\int_{t'}^{t}dt''\Lambda(t'')\right]\ , \tag{2.3}$$

with

$$\Lambda(t)=\lambda_{pn}(t)+\lambda_{np}(t)\ . \tag{2.4}$$

As we shall soon see, the rates $\lambda_{pn}(t)$ and $\lambda_{np}(t)$ are very large at early times t when the temperature T is on the order of 100 MeV. Hence, if the initial time t_0 in Eq. (2.2) is such an early time, the integration factor $I(t,t_0)$ will be very small in a short time $t\sim 1/\Lambda(t_0)$ later. Therefore the initial value $X(t_0)$ [which must lie in the interval (0,1)] is unimportant in Eq. (2.2), and it can be omitted. We see that for times t somewhere later than t_0, the fast reaction processes wash out the initial value. Moreover, because the initial interaction rates are large, the integral in Eq. (2.2) is not sensitive to the value of t_0 and we may simplify the expression by setting $t_0=0$; the integral from $t'=0$ to $t'=t_0$ is negligible. We now have

$$X(t)=\int_{0}^{t}dt'I(t,t')\lambda_{pn}(t')\ . \tag{2.5}$$

To show that the neutron population is in equilibrium until fairly late times and also to exhibit the onset of the breaking of this equilibrium, we note that

$$I(t,t')=\frac{1}{\Lambda(t')}\frac{d}{dt'}I(t,t')\ , \tag{2.6}$$

so that one may integrate by parts to obtain

$$X(t)=\frac{\lambda_{pn}(t)}{\Lambda(t)}-\int_{0}^{t}dt'I(t,t')\frac{d}{dt'}\left[\frac{\lambda_{pn}(t')}{\Lambda(t')}\right]\ . \tag{2.7}$$

In the regime where the total reaction rate $\Lambda(t)$ is large in comparison with the rate of time variation of the rates, the last term in Eq. (2.7) gives a small correction. This correction is shown by again performing an integration by parts to obtain

$$X(t)\simeq\frac{\lambda_{pn}(t)}{\Lambda(t)}-\frac{1}{\Lambda(t)}\frac{d}{dt}\left[\frac{\lambda_{pn}(t)}{\Lambda(t)}\right]\ , \tag{2.8}$$

where terms involving $(d\lambda/dt)^2$ and $d^2\lambda/dt^2$ have been dropped. Now with the leptons kept in tight thermal equilibrium by scattering processes among themselves and photons, the principle of detailed balance requires that

$$\lambda_{pn}(t)=\exp\left[-\frac{\Delta m}{T(t)}\right]\lambda_{np}(t)\ , \tag{2.9}$$

a result we derive below. Thus

$$\frac{\lambda_{pn}(t)}{\Lambda(t)}=\frac{1}{1+e^{\Delta m/T}}=X_{\rm eq}(T)\ , \tag{2.10}$$

and so, to our present level of accuracy, we can take Eq. (2.8) to be the Taylor expansion of

$$X(t)\simeq X_{\rm eq}\left[T\left[t-\frac{1}{\Lambda(t)}\right]\right]. \tag{2.11}$$

We see that, to this first approximation, the population has an equilibrium value, but at an earlier time, a time retarded by the reaction time $1/\Lambda(t)$. To put this result in another form, we note that the conservation of entropy for the (relativistic) leptons is equivalent to the statement that their quantum wavelengths expand with the general expansion of the universe or that

$$R(t)T(t)=\text{const}\ . \tag{2.12}$$

Thus $\dot{T}/T=-\dot{R}/R$ and we may write

$$X(T)\simeq X_{\rm eq}(T_{\rm eff}(t))\ , \tag{2.13}$$

in which

$$T_{\rm eff}(t)=\left[1+\frac{1}{\Lambda}\frac{dR/dt}{R}\right]T(t)\ . \tag{2.14}$$

We see that the neutron-to-baryon ratio $X(t)$ follows its equilibrium ratio at the temperature T of the universe until a time at which $\dot{R}/R\sim\Lambda$. This is just the condition described in the Introduction [see Eq. (1.12)]. When $\dot{R}/\Lambda R$ becomes of order unity, the ratio $X(t)$ becomes an equilibrium ratio $X_{\rm eq}(T_{\rm eff})$ with an effective temperature $T_{\rm eff}(t)$ that is higher than the temperature of the universe giving a population $X_{\rm eq}(T_{\rm eff})$ that is larger than $X_{\rm eq}(T)$. [Since $X_{\rm eq}(T)$ is a decreasing function of T, it follows that at these times $X(T)$ will exceed $X_{\rm eq}(T)$.] This effect is the onset of the freezing in of the neutron and proton numbers. However, the result of Eq. (2.14) is not quantitatively correct when $\dot{R}/\Lambda R\sim 1$, and so we must extend the analysis in this case.

To do this we need an explicit form for the rate $\lambda_{np}(t)$. This rate is the sum of the rates for the individual processes proceeding to the right in Eqs. (1.1),

$$\lambda_{np}=\lambda(\nu+n\to p+e^-)+\lambda(e^++n\to p+\bar{\nu})$$
$$+\lambda(n\to p+\bar{\nu}+e^-)\ . \tag{2.15}$$

These individual weak-interaction rates are given by [see, for example, Weinberg (1972)]:

$$\lambda(\nu+n\to p+e^-)=A\int_0^\infty dp_\nu p_\nu^2 p_e E_e(1-f_e)f_\nu\ , \tag{2.16a}$$

$$\lambda(e^++n\to p+\bar{\nu})=A\int_0^\infty dp_e p_e^2 p_\nu E_\nu(1-f_\nu)f_e\ , \tag{2.16b}$$

and

$$\lambda(n\to p+\bar{\nu}+e^-)=A\int_0^{p_0} dp_e p_e^2 p_\nu E_\nu(1-f_\nu)(1-f_e)\ . \tag{2.16c}$$

Here A is an overall effective coupling constant that we shall later eliminate in favor of the free-neutron decay rate, so that its precise value need not concern us. The magnitudes of the neutrino and electron momenta in the various processes are denoted by p_ν and p_e, with corresponding energies $E_\nu=p_\nu$ and $E_e=(p_e^2+m_e^2)^{1/2}$. In the relatively low temperature or equivalently low-energy domain that concerns us, the recoil of the nucleons can be neglected. Thus the electron momentum p_e in Eq. (2.16a) is determined by the energy conservation condition $E_e=E_\nu+\Delta m$, and the neutrino energy in Eq. (2.16b) is determined by $E_\nu=E_e+\Delta m$. The neutrino energy in Eq. (2.16c) is determined by $E_\nu=\Delta m-E_e>0$, which gives the upper limit on the integration range of $p_0=(\Delta m^2-m_e^2)^{1/2}$. The p^2dp factors are, of course, just the usual phase space, while the remaining factors of pE arise from the square of the transition matrix element. The lepton phase-space density functions are given by their equilibrium values

$$f_\nu=\frac{1}{e^{E_\nu/T_\nu}+1} \tag{2.17a}$$

and

$$f_e=\frac{1}{e^{E_e/T_e}+1}\ . \tag{2.17b}$$

The factors f_ν and f_e in Eqs. (2.16a) and (2.16b) correspond to the density of the incident flux of particles. The remaining factors of $(1-f_e)$ and $(1-f_\nu)$ in Eqs. (2.16) are "blocking factors" that represent the Pauli exclusion principle.

Here we have noted that, in general, the electron and neutrino temperatures, T_e and T_ν, may differ. The reason for this is that towards the end of the freezing-out period of the neutrons, the temperature drops somewhat below m_e and the electrons and positrons annihilate, heating the photons but not the neutrinos, which are now decoupled. The reaction $e^-+\gamma\to e^-+\gamma$ occurs rapidly in comparison with $\dot{R}/R$, so that electrons maintain thermal equilibrium with the photons, $T_e=T_\gamma$. In addition, the reaction $e^++e^-\leftrightarrow 2\gamma$ occurs rapidly in comparison with $\dot{R}/R$ during almost all of the freeze-out period, so that the overall entropy of the system is conserved. Using the conservation of entropy, one finds, however, that T_ν and T_γ differ by at most 10% during the freeze-out period that we are about to consider. (We shall discuss this point in some detail in Sec. III.) Our first approximation is then to set $T_\nu=T_e=T_\gamma=T$. Within this approximation, the rates for reverse reactions such as $e^-+p\to n+\nu$, which have forms similar to those of Eqs. (2.16), obey the principle of detailed balance, and we have, for example,

$$\lambda(e^-+p\to n+\nu)=\exp\left[-\frac{\Delta m}{T}\right]\lambda(\nu+n\to p+e^-)\ . \tag{2.18}$$

To see this, we first note that the rate for the reaction which is the reverse of that presented in Eq. (2.16a) is given by

$$\lambda(e^- + p \to n + \nu) = A\int_{p_0}^{\infty} dp_e p_e^2 p_\nu E_\nu (1-f_\nu) f_e \ . \quad (2.19a)$$

Using $p_e dp_e = E_e dE_e$ and energy conservation (with the neglect of nucleon recoil energy) so that $dE_e = dE_\nu = dp_\nu$, we have

$$\lambda(e^- + p \to n + \nu) = A\int_0^{\infty} dp_\nu p_\nu^2 p_e E_e (1-f_\nu) f_e \ . \quad (2.19b)$$

It follows from Eqs. (2.17) that $1-f_\nu = e^{E_\nu/T} f_\nu$ and $f_e = (1-f_e)e^{-E_e/T}$. Using these substitutions in Eq. (2.19b), the energy relation $E_e - E_\nu = \Delta m$, and comparing with Eq. (2.16a), we verify the detailed balance statement (2.18). Adding up all the processes yields the detailed balance statement (2.9) that we have already used.

The next approximation that we shall make follows from the observation that during the freeze-out period the temperature T is low in comparison with the typical energies E that contribute in the integrals for the rates. Hence we may replace the Fermi-Dirac distributions by the Boltzmann weights

$$f_{e,\nu} \simeq \exp\left[-\frac{E_{e,\nu}}{T}\right] , \quad (2.20a)$$

and correspondingly neglect the effects of the Pauli blocking,

$$1 - f_{e,\nu} \simeq 1 \ , \quad (2.20b)$$

since the Boltzmann weights are small in this dilute gas limit. Accordingly, we now have

$$\lambda(\nu + n \to p + e^-) = A\int_0^{\infty} dp_\nu p_\nu^2 p_e E_e e^{-E_\nu/T} , \quad (2.21a)$$

$$\lambda(e^+ + n \to p + \bar{\nu}) = A\int_0^{\infty} dp_e p_e^2 p_\nu E_\nu e^{-E_e/T} , \quad (2.21b)$$

and

$$\lambda(n \to p + \bar{\nu} + e^-) = A\int_0^{p_0} dp_e p_e^2 p_\nu E_\nu \ . \quad (2.21c)$$

Our final approximation is to neglect the electron mass in Eqs. (2.21a) and (2.21b) in comparison to the energies E_ν, E_e, which give the main contributions to these integrals. In this approximation these two rates become identical. Placing $p_e = E_e = \Delta m + E_\nu$ (and $p_\nu = E_\nu$) in Eq. (2.21a), we obtain

$$\lambda(\nu + n \to p + e^-) = AT^3(4!T^2 + 2\times 3!T\Delta m + 2!\Delta m^2)$$
$$= \lambda(e^+ + n \to p + \bar{\nu}) \ . \quad (2.22)$$

The effect of setting $m_e = 0$ in Eqs. (2.21) is to neglect terms of order m_e^2/T^2 in Eq. (2.22). Even when $T \simeq m_e$, these terms produce only 15% corrections to the rates. Although this approximation breaks down when the temperature T falls below the electron mass m_e, in this regime the rates themselves are very small and their effects are not significant. Equation (2.21c) is just the decay rate $1/\tau$ for a free neutron. In that equation we cannot neglect m_e. Setting $E_\nu = \Delta m - E_e$ and recalling that $p_0 = (\Delta m^2 - m_e^2)^{1/2}$ we find that several elementary integrations yield

$$\frac{1}{\tau} = \lambda(n \to p + \bar{\nu} + e^-)$$
$$= \frac{A}{5}(\Delta m^2 - m_e^2)^{1/2}(\tfrac{1}{6}\Delta m^4 - \tfrac{3}{4}\Delta m^2 m_e^2 - \tfrac{2}{3}m_e^4)$$
$$+ \frac{A}{4} m_e^4 \Delta m \cosh^{-1}\left[\frac{\Delta m}{m_e}\right] . \quad (2.23)$$

Using the numerical values $\Delta m = 1.29$ MeV and $m_e = 0.511$ Mev, we obtain

$$\frac{1}{\tau} = 0.0157 A \Delta m^5 \ . \quad (2.24)$$

We shall use this last result to provide the scale for the rates in terms of the directly observable mean life of the neutron. Thus we write

$$4A = \frac{a}{\tau}\Delta m^{-5} , \quad (2.25a)$$

in which a is the pure number

$$a = 255 \ . \quad (2.25b)$$

As was discussed in the Introduction, we are neglecting the free-neutron decay process in this section, and so the value for the total rate $\lambda_{np}(t)$ that we use is just twice that given in Eq. (2.22). It is convenient to introduce the dimensionless temperature variable,

$$y = \Delta m / T , \quad (2.26)$$

and express this rate as

$$\lambda_{np}(t) = \left[\frac{a}{\tau y^5}\right](12 + 6y + y^2) \ . \quad (2.27)$$

The inclusion of neutron decay and the elimination of other simplifying assumptions would add extra terms to Eq. (2.27). These extra terms do not change λ_{np} significantly for $y < 5$. We should note that for $T > 1$ MeV or $y < 1$, this rate is 3 orders of magnitude larger than the free-neutron decay rate. The two become comparable at $y \simeq 10$, $T \simeq 0.13$ MeV. The rate λ_{np} given by Eq. (2.27) agrees with the precise calculation of Peebles to within 15% (see Table I), when we correct for the difference in the weak coupling constant that he used.

We can now evaluate Eq. (2.7) for the neutron abundance. Changing variables from the time t to the scaled inverse temperature y, we have

$$X(y) = \frac{\lambda_{pn}(y)}{\Lambda(y)} - \int_0^y dy' I(y,y') \frac{d}{dy'}\left[\frac{\lambda_{pn}(y')}{\Lambda(y')}\right] . \quad (2.28)$$

The detailed balance relation (2.9) gives

$$\lambda_{pn}(y) = e^{-y}\lambda_{np}(y) \quad (2.29)$$

and

TABLE I. Comparison of interaction rates and neutron-proton ratio calculated with our Eqs. (2.27) and (2.38) with the computer calculation of Peebles (1966). To facilitate the comparison, we have used the same weak coupling constants as Peebles, which correspond to a neutron mean life of 1013 sec and, with the two neutrino types used by Peebles, to a value of our constant given by $b=0.243$. The numbers for the rates λ_{np} are in $\sec^{-1}$.

y	λ_{np} [Eq. (2.27)]	λ_{np} (Peebles)	X [Eq. (2.38)]	X (Peebles)
0.25	3.5×10^{3}	3.2×10^{3}	0.438	0.438
0.50	1.2×10^{2}	1.1×10^{2}	0.380	0.380
0.75	1.8×10^{1}	1.6×10^{1}	0.330	0.331
1.5	7.7×10^{-1}	6.8×10^{-1}	0.239	0.241
2.5	8.6×10^{-2}	7.1×10^{-2}	0.192	0.197
5.0	5.4×10^{-3}	3.4×10^{-3}	0.165	0.172
∞	0	0	0.155	0.164

$$\Lambda(y)=(1+e^{-y})\lambda_{np}(y)\ , \tag{2.30}$$

while the integrating factor (2.3) now becomes

$$I(y,y')=\exp\left[-\int_{y'}^{y}dy''\left[\frac{dt''}{dy''}\right]\Lambda(y'')\right] . \tag{2.31}$$

To evaluate the Jacobian dt''/dy'' we recall that the conservation of entropy (2.12) gives $\dot{T}/T=-\dot{R}/R$. Therefore the universal expansion equation (1.6), together with the energy density formula (1.8), may be expressed as

$$\frac{dT}{dt}=-\left[\frac{4\pi^3}{45M_{\rm Pl}^2}N\right]^{1/2}T^3\ , \tag{2.32}$$

with, we recall, $N=\frac{43}{4}$. Using this result and Eqs. (2.27) and (2.30), we see that the integrating factor (2.31) has the form

$$I(y,y')=\exp[K(y)-K(y')]\ , \tag{2.33}$$

with

$$K(y)=-b\int dy'\left[\frac{12}{y'^4}+\frac{6}{y'^2}+\frac{1}{y'^2}\right](1+e^{-y'})\ , \tag{2.34}$$

where b is the pure number

$$b=a\left[\frac{45}{4\pi^3N}\right]^{1/2}\frac{M_{\rm Pl}}{\tau\Delta m^2} . \tag{2.35}$$

It is quite remarkable that the numerical coefficients inside the integration are such as to give a simple closed form

$$K(y)=b\left[\left[\frac{4}{y^3}+\frac{3}{y^2}+\frac{1}{y}\right]+\left[\frac{4}{y^3}+\frac{1}{y^2}\right]e^{-y}\right]\ , \tag{2.36}$$

as one can check by differentiation.

Introducing

$$X_{\rm eq}(y)=\frac{\lambda_{pn}(y)}{\Lambda(y)}=\frac{1}{1+e^{y}}\ , \tag{2.37}$$

the neutron abundance ratio now reads

$$X(y)=X_{\rm eq}(y)+\int_0^y dy' e^{y'}X_{\rm eq}(y')^2\exp[K(y)-K(y')]\ . \tag{2.38}$$

It is a simple matter to numerically compute the integral that appears here for a range of y values. Using the numerical values that we have given before for the various parameters that enter into the dimensionless constant

$$b=\frac{0.823}{\sqrt{N}}=0.251\ , \tag{2.39}$$

we obtain the curve for $X(y)$ shown in Fig. 1. The curve asymptotes to

$$X(y=\infty)=0.151\ . \tag{2.40}$$

At the end of our paper (Sec. IV), we assess the sensitivity of the helium abundance to the parameters upon which it depends. We pause here to derive some results that will be needed for this later discussion.

First we consider the dependence on the number of neutrino types. To do this, we note that a variation of N gives

$$\frac{\delta b}{b}=-\frac{1}{2}\frac{\delta N}{N}\ . \tag{2.41}$$

Such a variation alters the scale of $K(y)$ and, according

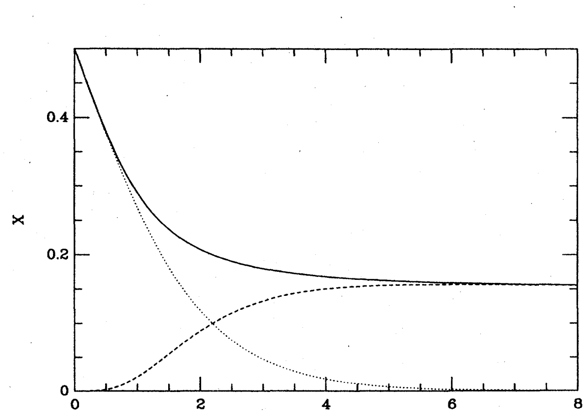

FIG. 1. Neutron-to-total-baryon ratio $X(y)$ as a function of $y=\Delta m/T$ (solid line). The dotted line gives the thermal equilibrium abundance $X_{\rm eq}(y)$, while the dashed line gives the correction due to the integral in Eq. (2.38), the "freeze-out" correction.

to Eq. (2.38), produces

$$\frac{\delta X(y=\infty)}{X(y=\infty)}=-C\frac{\delta b}{b}\ , \tag{2.42}$$

where

$$C=X(y=\infty)^{-1}\int_0^\infty dy' e^{y'} X_{\rm eq}(y')^2 K(y')\exp[-K(y')]\ . \tag{2.43}$$

It is an easy task to evaluate this integral numerically, which gives

$$C=0.52\ . \tag{2.44}$$

Since $N=\frac{43}{4}$ for three neutrino types and the addition of one more type entails the change $\delta N=\frac{7}{4}$, we see that such a change produces a fractional increase in the neutron population given by

$$\frac{\delta X(y=\infty)}{X(y=\infty)}=0.042\ . \tag{2.45}$$

The sign of this result is easy to understand. Increasing the number of neutrinos increases the energy density of the universe and speeds up its expansion. Therefore the neutron comes out of equilibrium sooner at a higher temperature and with a larger population.

Finally, we consider the effect of a possible chemical potential for the electron neutrino. In this case the density functions for the neutrinos and antineutrinos differ, with

$$f_\nu=\frac{1}{\exp(-\alpha+E_\nu/T)+1} \tag{2.46a}$$

and

$$f_{\bar\nu}=\frac{1}{\exp(\alpha+E_{\bar\nu}/T)+1}\ . \tag{2.46b}$$

Here we use the parameter $\alpha=\mu/T$ rather than the usual chemical potential μ, because α remains constant for the freely streaming neutrino gas in the expanding universe. These distributions are solutions of the collisionless Boltzmann equation in the expanding universe,

$$\left[\frac{\partial}{\partial t}-\frac{\dot R}{R}E\frac{\partial}{\partial E}\right]f(t,E)=0\ , \tag{2.47}$$

with constant α and with $T(t)$ obeying Eq. (2.12). With such a nonvanishing chemical potential, the detailed balance statement (2.9) is altered to read

$$\lambda_{pn}(t)=\exp\left[-\frac{\Delta m}{T(t)}-\alpha\right]\lambda_{np}(t)\ , \tag{2.48}$$

as one can readily check by writing down the rate formulas analogous to Eqs. (2.16) and the corresponding formulas for the reverse processes. Hence the equilibrium neutron-to-baryon ratio is now given by

$$X_{\rm eq}^{(\alpha)}(y)=\frac{1}{1+e^{\alpha+y}}\ , \tag{2.49}$$

which is to say that for large y the population is multiplied by the factor $e^{-\alpha}$. (With $\alpha>0$ the neutron population is reduced, since there are more neutrinos than antineutrons and the $n\to p$ reactions are favored.) The asymptotic neutron abundance is given by a simple modification of Eq. (2.38):

$$X^{(\alpha)}(y=\infty)=\int_0^\infty dy\frac{e^{-\alpha-y}}{(1+e^{-\alpha-y})^2}\exp[-K^{(\alpha)}(y)]\ . \tag{2.50}$$

To agree with the presently accepted values of the helium abundance, these α corrections must be small. Therefore we need compute only to first order in α. Replacing the previous neutrino distribution by the Boltzmann limits of Eqs. (2.46), we see that to this order

$$K^{(\alpha)}(y)=b\left[\left[1+\frac{\alpha}{2}\right]\left[\frac{4}{y^3}+\frac{3}{y^2}+\frac{1}{y}\right]+\left[1-\frac{\alpha}{2}\right]\left[\frac{4}{y^3}+\frac{1}{y^2}\right]e^{-y}\right]. \tag{2.51}$$

[The neutrino energy density has no first-order correction in α; hence the Jacobian factor dt''/dy'' in Eq. (2.31) is not altered to this order.] Pulling out the factor $e^{-\alpha}$ that appears in the numerator in Eq. (2.50) and then expanding the remainder to first order in α gives

$$X^{(\alpha)}(y=\infty)=e^{-\alpha}X(y=\infty)-\alpha X_1\ , \tag{2.52}$$

where

$$X_1=\int_0^\infty dy\, e^y X_{\rm eq}(y)^2 G(y)\exp[-K(y)]\ , \tag{2.53}$$

in which

$$G(y)=\frac{b}{2}\left[\frac{4}{y^3}+\frac{3}{y^2}+\frac{1}{y}\right]-\frac{b}{2}\left[\frac{4}{y^3}+\frac{1}{y^2}\right]e^{-y}-\frac{2}{1+e^y}\ . \tag{2.54}$$

The simple one-dimensional integral (2.53) is readily performed numerically. Using the value $b=0.25$, one finds a very small correction,

$$X_1=9.0\times10^{-4}=6.0\times10^{-3}X(y=\infty)\ . \tag{2.55}$$

We thus find rather remarkably that, to within an accuracy of better than 1%, the presence of a small chemical potential for the electron neutrino alters the asymptotic neutron abundance by the same factor $e^{-\alpha}$ that changes the equilibrium abundance.

III. NEUTRON DECAY CORRECTION

Thus far we have solved for the evolution of the neutron abundance with the neglect of neutron decay. Let us change notation by using an overbar to denote the result already obtained, $X(y)\to\bar X(y)$. Including the effect of the neutron decay (with mean life τ) in the rate equation (2.1), we now have

$$X(t)=e^{-t/\tau}\bar{X}(y(t)) , \tag{3.1}$$

because $\bar{X}(y)$ does not vary much during the period in which neutrons decay.

As was discussed in the Introduction, when the temperature drops somewhat below the deuteron binding energy ε_D, the neutrons are captured in deuterons, and the deuterons collide to place essentially all of the neutrons present at this time t_c into ^{4}He. For the sake of completeness, we present here an approximate evaluation of the capture time t_c. Inserting this time into Eq. (3.1) and using the asymptotic value of $\bar{X}(y)$ determined in the previous section yields the relative neutron abundance at the time of the helium formation and thus the mass fraction Y_4 of helium that is produced in the early universe. Since the effect of the neutron decay provides only a small correction, an approximate computation of t_c will provide a reasonably accurate number for Y_4.

As we shall soon see, at the capture time t_c the temperature of the universe is well below the electron mass m_e. At such low temperatures the electrons and positrons have annihilated into photons, heating the photon gas but not altering the neutrino distribution. Thus at times in the vicinity of t_c, the photon temperature T_γ differs from the neutrino temperature T_ν. Since we shall need the connection between T_γ and T_ν, we pause to review briefly this well-known relationship.

At early times, the collision rates are rapid in comparison with the expansion rate $\dot{R}/R$ of the universe, and the temperatures of all of the effectively massless gas components are equal, $T_e=T_\gamma=T_\nu=T$. As we have discussed before [Eq. (2.12)], the conservation of entropy requires that $R(t)T(t)$ be constant. At later times and lower temperatures, the collision rates for the neutrinos become less than the expansion rate $\dot{R}/R$. (This happens when the temperature is roughly on the order of 1 MeV.) The neutrinos are now in a decoupled, freely expanding massless gas with a number density proportional to $T_\nu(t)^3$. Since neutrinos can no longer be created or destroyed, their number in a comoving volume $R(t)^3$ is fixed and so for all times we have

$$R(t)T_\nu(t)=\text{const} . \tag{3.2}$$

We should note that this temperature constraint is such as to ensure that the thermal, Fermi-Dirac neutrino distribution at the time-dependent temperature $T_\nu(t)$ obeys the collisionless Boltzmann equation in the expanding universe, so that a thermal distribution of neutrinos is always maintained.

The rapid Compton scattering of the photons on the electrons, together with electron-positron annihilation that changes the photon number, always keeps the e-γ system in thermal equilibrium with $T_e(t)=T_\gamma(t)$. Hence the entropy of this subsystem in a comoving volume, $s_{e\gamma}(t)R(t)^3$, remains constant. In view of Eq. (3.2), we can write this constraint as

$$\frac{s_{e\gamma}(t)}{T_\nu(t)^3}=\text{const} . \tag{3.3}$$

When the temperature is much higher than the electron mass, the electron-photon subsystem is described as a massless gas with an entropy density given by $s_{e\gamma}=\frac{4}{3}\rho_{e\gamma}/T_\gamma=\frac{4}{3}N_{e\gamma}(\pi^2/30)T_\gamma^3$, where $N_{e\gamma}=\frac{7}{2}+2$. At these high temperatures $T_\gamma=T_\nu$. When the temperature is much lower than the electron mass, the electrons have disappeared and the electron-photon subsystem is a simple photon gas, so that now $s_{e\gamma}=\frac{4}{3}\rho_\gamma/T_\gamma$ $=\frac{4}{3}N_\gamma(\pi^2/30)T_\gamma^3$, where $N_\gamma=2$. Therefore, with $T_\gamma(t)$ and $T_\nu(t)$ denoting the late-time temperatures, the entropy constraint (3.3) requires that

$$N_{e\gamma}=N_\gamma\frac{T_\gamma(t)^3}{T_\nu(t)^3} , \tag{3.4}$$

giving the well-known result

$$T_\gamma(t)=(\tfrac{11}{4})^{1/3}T_\nu(t) . \tag{3.5}$$

The conservation of entropy (3.3) can be used to relate $T_\gamma(t)$ and $T_\nu(t)$ at intermediate times, but this requires the numerical evaluation of the entropy density for the Fermi-Dirac distribution of semirelativistic, massive electrons. Here we only need to note that T_γ departs substantially from T_ν (halfway towards its final value) at the rather low temperature $T_\nu\approx m_e/4$ [see Peebles (1966)]. This departure corresponds to a value of the variable used in the previous section, $y=\Delta m/T$, given by $y\simeq 10$. This is much larger than the value $y\simeq 5$ at which the neutron fraction $\bar{X}(y)$ has reached its asymptotic limit, justifying our previous approximation of setting $T_\gamma=T_\nu$ in the rate equations.

To compute the capture time t_c, we shall need to translate temperature into time. To do this, we use Eqs. (1.6) and (3.2) to obtain

$$\frac{1}{T_\nu(t)}\frac{dT_\nu(t)}{dt}=-\left[\frac{8\pi\rho}{3M_{\rm Pl}^2}\right]^{1/2} , \tag{3.6}$$

which gives

$$t=\int_{T_\nu}^{\infty}\frac{dT_\nu'}{T_\nu'}\left[\frac{3M_{\rm Pl}^2}{8\pi\rho}\right]^{1/2} . \tag{3.7}$$

As we have seen, the functional form of the energy density changes from the high-temperature form $\rho=N(\pi^2/30)T_\nu^4$, where $N=\frac{43}{4}$, to the low-temperature form

$$\rho_0=N_\nu\frac{\pi^2}{30}T_\nu^4+N_\gamma\frac{\pi^2}{30}T_\gamma^4=N_{\rm eff}\frac{\pi^2}{30}T_\nu^4 , \tag{3.8}$$

where

$$N_{\rm eff}=N_\nu+N_\gamma(\tfrac{11}{4})^{4/3}=\tfrac{21}{4}+2(\tfrac{11}{4})^{4/3}\simeq 13.0 . \tag{3.9}$$

It is the later, low-temperature form that is relevant at the temperature of the neutron capture and also for somewhat higher temperatures. On the other hand, since $N_{\rm eff}/N \simeq 1.2$, and it is the square root of the energy that enters into the universe age formula (3.7), one should be able to get a reasonable accurate if approximate evaluation of the time t in the vicinity of the time of neutron capture. To do this, we note that we can use the low-temperature form (3.8) of the energy density to solve Eq. (3.6) in terms of an additional integration constant:

$$t=\left[\frac{45}{16\pi^3 N_{\rm eff}}\right]^{1/2}\frac{M_{\rm Pl}}{T_\nu^2}+t_0$$
$$=\left[\frac{45}{16\pi^3 N_{\rm eff}}\right]^{1/2}\left[\frac{11}{4}\right]^{2/3}\frac{M_{\rm pl}}{T_\gamma^2}+t_0 \; . \tag{3.10}$$

The integration constant t_0 can be estimated by performing a perturbation analysis of the exact result (3.7) about ρ_0. This is done to first order in the small quantity $\rho-\rho_0$ in the Appendix. We find that to first order

$$t_0 \simeq 2 \text{ sec} \; . \tag{3.11}$$

We may now turn to determine the time t_c at which the neutrons are captured. First we recall that in equilibrium the neutrons, protons, and deuterons behave as free nonrelativistic gases with number densities

$$n_a=g_a e^{-(\mu_a+m_a)/T_\gamma}\int\frac{d^3p}{(2\pi)^3}e^{-p^2/2m_aT_\gamma}$$
$$=g_a e^{-(\mu_a+m_a)/T_\gamma}\left[\frac{m_aT_\gamma}{2\pi}\right]^{3/2} \; . \tag{3.12}$$

Here g_a is the statistical spin weight $g_n=g_p=2$, $g_D=3$, and μ_a, m_a are the chemical potentials and masses of the particles. At early times, these gases are in chemical equilibrium, so that

$$\mu_{\rm D}=\mu_n+\mu_p \; . \tag{3.13}$$

Therefore

$$\frac{n_n n_p}{n_{\rm D}}=\frac{g_p g_n}{g_{\rm D}}\left[\frac{m_p m_n}{m_{\rm D}}\right]^{3/2}\left[\frac{T_\gamma}{2\pi}\right]^{3/2} e^{-\varepsilon_{\rm D}/T_\gamma} \; , \tag{3.14}$$

where

$$\varepsilon_{\rm D}=m_p+m_n-m_{\rm D} \tag{3.15}$$

is the deuteron binding energy. This is the Saha equation.

As we shall soon see, the Saha formula (3.14) gives a very small fraction of deuterons until the temperature T_γ is well below the deuteron binding energy $\varepsilon_{\rm D}$. This is the "deuteron bottleneck" that inhibits the formation of ^{4}He through reaction sequences such as

$$n+p\to {\rm D}+\gamma \; , \tag{3.16a}$$
$${\rm D}+{\rm D}\to {\rm T}+p \; , \tag{3.16b}$$
$${\rm D}+{\rm T}\to {}^4{\rm He}+n \; . \tag{3.16c}$$

(These are all of the most important reactions except for the two involving ^{3}He that we omit to simplify the discussion.) Since we are now discussing the epoch when other elements are formed, it is convenient to use number abundance fractions X_a normalized to the total baryon number density n_B, which also counts the number of nucleons in D, T, ^{4}He, and so forth. Thus $X_a=n_a/n_B$ and we have $X_p+X_n+2X_{\rm D}+3X_{\rm T}+4X_{^4{\rm He}}+\cdots=1$. It is also convenient to introduce the photon number density

$$n_\gamma=\frac{2\zeta(3)}{\pi^2}T_\gamma^3 \; , \tag{3.17}$$

where $\zeta(3)\simeq 1.202$ is the Riemann zeta function, and the baryon-photon number ratio

$$\eta=n_B/n_\gamma \tag{3.18}$$

to rewrite the Saha formula in the form

$$\frac{X_nX_p}{X_{\rm D}}=G_{np} \; , \tag{3.19}$$

where

$$G_{np}=\frac{\pi^{1/2}}{12\zeta(3)}\frac{1}{\eta}\left[\frac{m_p}{T_\gamma}\right]^{3/2}e^{-\varepsilon_{\rm D}/T_\gamma} \; . \tag{3.20}$$

Since[4] $\eta\approx 5\times 10^{-10}$, one finds, for $T_\gamma \gtrsim 0.1$ MeV, that G_{np} is a very large number. For example, at $T_\gamma\simeq 0.1$ MeV one has $G_{np}\simeq 10^5$. Therefore for temperatures above 0.1 MeV the deuteron fraction is less than 10^{-4} (since $X_n\simeq 0.1$), and there is an insufficient population of deuterium to produce much ^{4}He by the reactions (3.16b) and (3.16c).

To quantitatively determine the neutron capture time t_c, we need to examine the rate equations that reflect the reaction sequence of Eqs. (3.16). It is convenient to write the rate equations in terms of the scaled temperature variable

$$z=\frac{\varepsilon_{\rm D}}{T_\gamma} \; , \tag{3.21}$$

so that they involve scaled rate parameters of the form

$$R=\frac{dt}{dz}\langle\sigma v\rangle_T n_B \; , \tag{3.22}$$

where $\langle\sigma v\rangle_T$ denotes the thermal average of the relevant

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[4]The value of the η parameter is rather uncertain. It is obtained in part by studying the mass-to-light ratios for a large number of galaxies. (See, for example, Faber and Gallagher, 1979.) This gives the mass-to-light ratio for visible baryons. The (visible) baryonic mass density is then obtained from the average luminosity density of the universe. (See, for example, Efstathiou, Ellis, and Paterson, 1988.) The result scales as the square of the Hubble constant H, which is the present value of $\dot R/R$. Since this number has an error on the order of 50%, there is a factor 2 uncertainty in η. Further uncertainty arises from the possibility of the presence of baryons in nonluminous material.

cross section times relative velocity. Using Eq. (3.10) to compute dt/dz and using Eqs. (3.17) and (3.18) to write the baryon number density n_B in terms of the baryon-photon number ratio η, we have

$$R=\frac{\eta}{z^2}\left[\frac{45}{\pi^7 N_{\mathrm{eff}}}\right]^{1/2}\left[\frac{11}{4}\right]^{2/3}\zeta(3)\varepsilon_{\mathrm{D}}M_{\mathrm{Pl}}\langle\sigma v\rangle_T \ . \tag{3.23}$$

First we note that the neutron and proton populations are governed by

$$\frac{dX_n}{dz}=-R_{np}(X_pX_n-G_{np}X_D)+\cdots \tag{3.24a}$$

and

$$\frac{dX_p}{dz}=-R_{np}(X_pX_n-G_{np}X_D)+\cdots \ . \tag{3.24b}$$

The Saha factor G_{np} must appear in the reverse reaction so as to give the proper equilibrium expressions for X_n and X_p. The ellipsis represents processes [such as the reverse of (3.16c)] which feed (or deplete) the neutron or proton populations but which are not important for our discussion. Since the neutron-proton capture process is exothermic, the product $\sigma_{np}v$ is constant at the low energies that concern us. With the value $\sigma_{np}v=4.55\times10^{-20}$ cm^3/sec [see, e.g., Peebles (1966)], we compute from Eq. (3.23) that

$$R_{np}\simeq5\left[\frac{29}{z}\right]^2\left[\frac{\eta}{\eta_0}\right], \tag{3.25}$$

where

$$\eta_0=5\times10^{-10} \tag{3.26}$$

is the nominal value for the photon-baryon ratio that we are using. As we shall see, for times prior to the neutron capture, $z<29$, and the rate constant is reasonably large, $R_{np}\gtrsim5$. Therefore, if the deuteron population is not depleted by other reactions such as (3.16b) and (3.16c), the protons, neutrons, and deuterons are kept in equilibrium with $X_p+X_n+2X_{\mathrm{D}}=1$ and with $X_{\mathrm{D}}=G_{np}^{-1}X_pX_n$. Since G_{np}^{-1} is very small, the deuteron population will be very small, and we can write the first approximation as

$$X_{\mathrm{D}}^{(1)}=G_{np}^{-1}X_p^{(0)}X_n^{(0)}, \tag{3.27}$$

where $X_p^{(0)}$ and $X_n^{(0)}$ are the unperturbed populations that obey $X_p^{(0)}+X_n^{(0)}=1$. Using this first approximation we now have

$$X_p+X_n\simeq1-2G_{np}^{-1}X_p^{(0)}X_n^{(0)} \ . \tag{3.28}$$

Recalling Eq. (3.20), one sees that for the z values that will concern us ($z\sim30$), the major z dependence of G_{np}^{-1} is controlled by the factor e^z. Hence we find that to first order

$$\frac{d}{dz}(X_p+X_n)\simeq-2G_{np}^{-1}X_p^{(0)}X_n^{(0)} \ . \tag{3.29}$$

Adding Eqs. (3.24) and remembering Eq. (3,27) we find that to a first approximation

$$R_{np}(X_pX_n-G_{np}X_{\mathrm{D}})\simeq X_{\mathrm{D}}^{(1)} \ . \tag{3.30}$$

We have just described the situation when the deuteron population is not depleted by other reactions. The first in the chain of these is that of Eq. (3.16b), giving

$$\frac{dX_{\mathrm{D}}}{dz}=+R_{np}(X_pX_n-G_{np}X_{\mathrm{D}})-R_{\mathrm{DD}}[2X_{\mathrm{D}}^2-G_{\mathrm{DD}}X_{\mathrm{T}}X_p]+\cdots \ . \tag{3.31}$$

Here R_{DD} is the scaled rate for the reaction (3.16b) and G_{DD} is the Saha factor which gives the equilibrium value for the ratio $X_{\mathrm{D}}^2/X_{\mathrm{T}}X_p$. The same argument that led to Eq. (3.20) gives

$$G_{\mathrm{DD}}=\frac{9}{4}\left[\frac{m_{\mathrm{D}}^2}{m_{\mathrm{T}}m_p}\right]^{3/2}e^{-B/T_\gamma}, \tag{3.32}$$

where B is the energy release of the reaction, $B=2m_{\mathrm{D}}-m_p-m_{\mathrm{T}}\simeq4.02$ MeV. In contrast to G_{np}, G_{DD} is always a small number

The rate R_{DD} involves the cross section σ_{DD} for the process $\mathrm{D}+\mathrm{D}\rightarrow\mathrm{T}+p$. This process is inhibited by the Coulomb barrier between the two incident charged deuterons, which is accounted for at the relevant low energies by the Coulomb penetration factor $(2\pi\alpha/v)\exp[-(2\pi\alpha/v)]$, where v is the relative velocity of the two deuterons and $\alpha\simeq\frac{1}{137}$ is the fine-structure constant. The basic size scale of the cross section is roughly the deuteron radius $\sim(1/m_p\varepsilon_{\mathrm{D}})^{1/2}$. The reaction is exothermic and so the cross section must contain a factor of $1/v$, which we may write in terms of the dimensionless parameter $(\varepsilon_{\mathrm{D}}/m_pv^2)^{1/2}$. These remarks lead to a phenomenological fit to the cross section given by

$$\sigma_{\mathrm{DD}}=\frac{0.87}{m_p\varepsilon_{\mathrm{D}}}\left[\frac{\varepsilon_{\mathrm{D}}}{m_pv^2}\right]^{1/2}\frac{2\pi\alpha}{v}\exp\left[-\frac{2\pi\alpha}{v}\right], \tag{3.33}$$

where we have obtained the numerical constant 0.87 by referring to the fit given by Peebles (1966). We need the thermal average of $\sigma_{\mathrm{DD}}v$. To obtain this we write the Boltzmann factors for the deuterons $f(v_1)f(v_2)\sim\exp[-\frac{1}{2}m_{\mathrm{D}}(v_1^2+v_2^2)]$ in terms of the relative velocity $\mathbf{v}=\mathbf{v}_1-\mathbf{v}_2$ and center-of-mass velocity $V=\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)$ to obtain

$$\langle\sigma_{\mathrm{DD}}v\rangle_T=\left[\frac{m_p}{2\pi T}\right]^{3/2}\int(d^3\mathbf{v})\sigma_{\mathrm{DD}}v\exp\left[-\frac{m_pv^2}{2T}\right], \tag{3.34}$$

where we have approximated the deuteron mass m_{D} by twice the proton mass m_p. Using Eq. (3.33) for the cross section in this average, one finds that the integrand is a sharply peaked function. Hence it may be evaluated by the method of steepest descent to obtain the good approximation

$$\langle\sigma_{\mathrm{DD}}v\rangle_T=0.87\frac{4\pi\alpha v_S}{(3\varepsilon_{\mathrm{D}}m_pT^2)^{1/2}}\exp\left[-\frac{3\pi\alpha}{v_S}\right], \tag{3.35}$$

in which

$$v_S = \left[\frac{2\pi\alpha T}{m_p} \right]^{1/3} . \tag{3.36}$$

Placing this result in the rate formula (3.23) and evaluating the numbers gives

$$R_{\rm DD} = 2.4\times 10^7 \left[\frac{\eta}{\eta_0} \right] z^{-4/3} e^{1.44 z^{1/3}} . \tag{3.37}$$

When the supply of deuterons begins to decrease, the number of neutrons is no longer maintained by the photodisintegration of the deuterons, and the chain of reactions is initiated that rapidly converts almost all of the neutrons into helium. We may therefore identify the temperature $T_{\gamma,c}$ at which the neutrons are captured, or equivalently $z_c = \varepsilon_{\rm D}/T_{\gamma,c}$, by the condition that

$$\left. \frac{dX_D}{dz} \right|_{z=z_c} \simeq 0 . \tag{3.38}$$

As we shall soon see, $z_c \approx 30$, and so the factor $G_{\rm DD}$ is of order e^{-60}, a completely negligible number. Hence the condition (3.38) in conjunction with Eqs. (3.27) and (3.30) and the approximation $X_{\rm D} \simeq X_{\rm D}^{(1)}$ gives

$$2X_{\rm D}^{(1)} R_{\rm DD} \simeq 1 . \tag{3.39}$$

The result of the previous section gave $X_n^{(0)} \simeq 0.15$, and so $X_n^{(0)} X_p^{(0)} \simeq 0.13$. Thus, using Eqs. (3.27) for $X_{\rm D}^{(1)}$ together with Eqs. (3.20) and (3.37), and putting in the numbers, we find that the capture condition reduces to

$$2.9\times 10^{-6} \left[\frac{\eta}{\eta_0} \right]^2 z_c^{-17/6} e^{-1.44 z_c^{1/3}} e^{z_c} \simeq 1 . \tag{3.40}$$

Taking $\eta = \eta_0$, we have $z_c = 26$ and

$$T_{\gamma,c} = \varepsilon_{\rm D}/26 = 0.086 \text{ MeV} . \tag{3.41}$$

Clearly our argument has yielded only an approximate condition, which we have indicated by the symbol $\simeq$. Nonetheless, it does provide a good determination of $T_{\gamma,c}$, since the condition entails the large and rapidly varying factor $\exp(z_c)$.

Placing now $T_\gamma = T_{\gamma,c} = 0.086$ MeV in the formula (3.10), we find that the neutron capture time is given by

$$t_c = 180 \text{ sec} . \tag{3.42}$$

At the temperature $T_{\gamma,c}$ the neutron population $\overline{X}(y)$ in the absence of decay has reached its asymptotic value. Therefore the neutron fraction available for capture into deuterium and ultimately into ^{4}He is given by

$$X(180 \text{ sec}) = \exp(-\tfrac{180}{896}) \overline{X}(y=\infty)$$

$$= 0.818 \times 0.151 = 0.123 . \tag{3.43}$$

We conclude that the helium abundance by weight is given by

$$Y_4 = 2X(180 \text{ sec}) = 0.247 . \tag{3.44}$$

IV. DISCUSSION

We have just presented an approximate and semianalytical calculation of the helium abundance produced in the early universe. Our aim has been to provide a treatment that illustrates clearly the physical principles at work in this helium production. We have been careful to get the essential physics right so that our result is accurate to within a few percent. Clearly the extensive computer computations that exist in the literature are needed for a more accurate evaluation and to compute the formation of other light elements such as deuterium, helium three, and lithium seven. However, a great advantage of having a semianalytical model for the helium production is that its variation with respect to the parameters on which it depends is easily computed. Moreover, the physical origins of these variations can be explicitly traced. We conclude our work with a discussion of this topic.

The cosmological helium production depends upon four essential parameters. They are $\mathcal{N}_\nu$, the number of neutrino types; η, the baryon-to-photon ratio; τ, the neutron's mean life; and α, a possible chemical potential for the electron neutrino. A chemical potential for the other neutrinos would be relevant if it were large, since then it would affect the energy density ρ. We will limit our discussion to small chemical potentials[5] that give no first-order change in ρ. We shall first present our final result for the change in the helium abundance ΔY_4 brought about by changes in the parameters and then discuss its origin in some detail. It is given by

$$\Delta Y_4 = -0.25\alpha + 0.014 \Delta\mathcal{N}_\nu + 0.18 \frac{\Delta\tau}{\tau} + 0.009 \ln \left[\frac{\eta}{\eta_0} \right] , \tag{4.1}$$

where $\Delta\mathcal{N}_\nu$ and $\Delta\tau$ are the variations of the number of neutrinos and neutron's mean lifetime about the values that we have employed, $\Delta\mathcal{N}_\nu = \mathcal{N}_\nu - 3$, $\Delta\tau = (\tau - 896)$ sec. We again denote the nominal value of the baryon-photon ratio by η_0, with $\eta_0 = 5\times 10^{-10}$.

The variation with an electron-neutrino chemical potential α follows immediately from the result [Eq. (2.52)] given in Sec. II together with the unaltered helium abun-

[5]It is interesting to note that a small value of α can lead to a large neutrino-antineutrino number density difference n_L relative to the baryon number density n_B. Using

$$n_L = \int \frac{d^3p}{(2\pi)^3} (f_\nu - f_{\bar\nu}) ,$$

a short calculation gives

$$n_L \simeq \frac{\pi^2}{12\zeta(3)} \left[\frac{\alpha}{\eta} \right] n_B .$$

Since η is so very small, n_L/n_B can be large even for small α.

dance $Y_4=0.25$ given at the end of the previous section. The variation of Y_4 with respect to the number of neutrino types $\mathcal{N}_\nu$ has two sources. The first we have already discussed at the end of Sec. II [Eq. (2.45)], where we showed that adding an additional neutrino type produces a fractional increase in the neutron population given by

$$\frac{\delta\bar{X}(y=\infty)}{\bar{X}(y=\infty)}=0.042 . \tag{4.2}$$

The second source of the variation of Y_4 with $\mathcal{N}_\nu$ comes about because the capture time t_c depends on the neutrino density according to Eq. (3.10), with

$$\frac{\Delta t_c}{t_c}=-\frac{1}{2}\frac{\Delta N_{\text{eff}}}{N_{\text{eff}}} . \tag{4.3}$$

Recalling that [Eq. (3.9)] $N_{\text{eff}}=7\mathcal{N}_\nu/4+2(\frac{11}{4})^{4/3}\simeq 13$ for three neutrino types, we see that

$$\frac{\Delta t_c}{t_c}\simeq -\tfrac{7}{104}\Delta\mathcal{N}_\nu . \tag{4.4}$$

This alteration of the decay factor $\exp(-t_c/\tau)$ from its value at $t_c/\tau=0.20$, together with the first source of variation (4.2), gives

$$\Delta Y_4=Y_4(0.042+0.014)\Delta\mathcal{N}_\nu . \tag{4.5}$$

[In obtaining this result we have omitted the variation of the capture temperature $T_{\gamma,c}$ with neutrino number which is negligible since $T_{\gamma,c}$ is governed by the large and rapidly varying factor $\exp(z_c)$ in Eq. (3.40).] We have already commented in Sec. II that increasing the number of neutrinos speeds up the evolution of the universe, which accounts for the positive sign of the first term in parentheses in Eq. (4.5). It is for this same reason that the second term in the parentheses also has a positive sign. Using $Y_4=0.25$ in Eq. (4.5) yields the coefficient of $\Delta\mathcal{N}_\nu$ shown in Eq. (4.1).

The variation of Y_4 with respect to the neutron mean life τ also has two sources. According to Eq. (2.35), the parameter b that controls the lepton-nucleon interaction rates has the dependence $b\sim 1/\tau$, and so the change $\Delta\tau$ gives

$$\frac{\Delta b}{b}=-\frac{\Delta\tau}{\tau} . \tag{4.6}$$

In view of Eqs. (2.42) and (2.44), this produces a neutron population change

$$\frac{\Delta\bar{X}(y=\infty)}{\bar{X}(y=\infty)}=0.52\frac{\Delta\tau}{\tau} . \tag{4.7}$$

The second source involves the obvious change in the decay factor $\exp(-t_c/\tau)$ about the value $t_c/\tau=0.20$. Therefore a change in the neutron lifetime gives

$$\Delta Y_4=Y_4(0.52+0.20)\frac{\Delta\tau}{\tau} . \tag{4.8}$$

and yields the coefficient of $\Delta\tau/\tau$ shown in Eq. (4.1). Note that the signs of the two contributions to ΔY_4 given in Eq. (4.8) are rather obvious. Increasing the neutron lifetime decreases the leptonic collision rate and causes the neutrons to freeze out sooner at a larger population and there is less decay before the neutrons are captured into helium.

Finally we note that the dependence of the helium abundance Y_4 on the baryon-photon ratio η comes about because of the dependence of the capture time t_c in the decay factor on η. This, in turn, arises from the condition (3.40), which determines the capture temperature $T_{\gamma,c}$ in terms of η and the connection between time and temperature given in Eq. (3.10). The variation of the decay factor $\exp(-t_c/\tau)$ entails

$$\Delta Y_4=-Y_4\frac{t_c}{\tau}\frac{\Delta t_c}{t_c} . \tag{4.9}$$

To compute $\Delta t_c/t_c$, we recall that $t_c\sim T_{\gamma,c}^{-2}$ while the variation in $T_{\gamma,c}$ is determined by the constraint (3.40). Taking the logarithm of Eq. (3.40) and then examining its variation about $\eta=\eta_0$ and $z_c=26$, one finds that

$$2\ln\left[\frac{\eta}{\eta_0}\right]+0.84\Delta z_c=0 . \tag{4.10}$$

Therefore changing the baryon-photon ratio from η_0 to η changes the capture temperature from $T_{\gamma,c}$ to $T'_{\gamma,c}$, with

$$\frac{1}{T'_{\gamma,c}}=\frac{1}{T_{\gamma,c}}\left[1-0.092\ln\left[\frac{\eta}{\eta_0}\right]\right] . \tag{4.11}$$

This gives

$$\frac{\Delta t_c}{t_c}=-0.18\ln\left[\frac{\eta}{\eta_0}\right] , \tag{4.12}$$

and, with $t_c/\tau=0.20$,

$$\Delta Y_4=0.036Y_4\ln\left[\frac{\eta}{\eta_0}\right] , \tag{4.13}$$

which yields the coefficient of $\ln(\eta/\eta_0)$ shown in Eq. (4.1). The sign of this correction is understood easily. Increasing η is equivalent to decreasing the photon number relative to the number of baryons. Thus the photodisintegration of the deuteron becomes less effective, and helium is formed sooner when fewer neutrons have decayed.

To compare our results for the dependence of Y_4 on these parameters with the results of computer calculations, we consider an interpolation formula given in Boesgaard and Steigman (1985). This formula represents the computer calculations of Y_4 in the region $1.5<\eta\times 10^{10}<10$ to ± 0.001:

$$Y_4^B=0.230+0.013(\mathcal{N}_\nu-3)+0.014(\tau_n-10.6)$$
$$+0.011\ln(\eta\times 10^{10}) , \tag{4.14}$$

where τ_n is the neutron *half-life* in minutes.

We may compare three of these terms directly to Eq. (4.1). There is no term in (4.14) for the dependence of Y_4

on an electron chemical potential α, and to our knowledge no such expression that goes beyond one obtained from the equilibrium neutron population appears in the literature. The coefficients of the first and second terms in (4.1) are about the same as those terms in (4.14) when one converts from mean life to half-life. Our coefficient for the last term in (4.1) is 20% smaller than that inferred from (4.14).

We can compare our answer for Y_4 with that of Boesgaard and Steigman by inserting their central values, i.e., those that yield 0.230 in Eq. (4.14), into our Eq. (4.1). When this is done, we obtain 0.23, which means that the helium abundances that we obtain with our semianalytic methods are within a few percent of what the computer codes give.

Finally, our calculation of Y_4 can be compared with observation. The most precise empirical value for Y_4 seems to be that of Kunth and Sargent (1983), who measure the helium abundance in metal-poor galaxies. They give

$$Y_4 = 0.245 \pm 0.003 \ . \tag{4.15}$$

To the extent that this represents the true cosmological number, an empirical limit on $\mathcal{N}_\nu$ can be obtained by comparing Eq. (4.1) to this number. It may be seen that any value of $\mathcal{N}_\nu > 4$ does not fit the data, unless one of the other parameters, such as η or α, is changed. Furthermore, if $\mathcal{N}_\nu$ is kept at the presently observed value of 3, then $|\alpha|$ is constrained to be less than about $\frac{1}{10}$.

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APPENDIX

Here we shall derive the approximate result given in Eq. (3.11) for the integration constant t_0 that appears in Eq. (3.10) of the text. This we shall do by expanding the exact energy density ρ in the time formula (3.7) about the low-temperature form ρ_0 given by Eq. (3.8). With $\rho=\rho_0$, Eq. (3.7) yields the first term in Eq. (3.10). Expanding Eq. (3.7) to first order in the small quantity $\rho-\rho_0$ thus provides the first order value for the integration constant t_0 appearing in Eq. (3.10), and we have

$$t_0 = -\frac{1}{2}\left[\frac{3M_{\rm Pl}^2}{8\pi}\right]^{1/2}\int_{T_\nu}^{\infty}\frac{dT'_\nu}{T'_\nu}\frac{\rho-\rho_0}{\rho_0^{3/2}} \ . \tag{A1}$$

Here

$$\rho-\rho_0=\rho_{e\gamma}-2(\tfrac{11}{4})^{4/3}\frac{\pi^2}{30}T_\nu^4 \ , \tag{A2}$$

where $\rho_{e\gamma}$ is the energy density of the electron-photon subsystem. To reduce the integral in Eq. (A1), we note that since $\rho_0 \sim T_\nu^4$, one has

$$\frac{1}{T'_\gamma}\frac{1}{\rho_0^{3/2}} = -\frac{1}{6}\frac{d}{dT'_\nu}\rho_0^{-3/2} \ . \tag{A3}$$

Hence we may integrate by parts to obtain

$$t_0 = -\frac{1}{12}\left[\frac{3M_{\rm Pl}^2}{8\pi}\right]^{1/2}\int_{T_\nu}^{\infty}dT'_\nu\rho_0^{-3/2}\frac{d}{dT'_\nu}(\rho-\rho_0) \ . \tag{A4}$$

There are no end-point contributions in this partial integration, since the upper limit involves the vanishing quantity $T_\nu'^{-2}$ while at the lower limit $\rho=\rho_0$.

We may now make use of the thermodynamic relation

$$\frac{d}{dT'_\nu}\rho_{e\gamma}=T'_\gamma(T'_\nu)\frac{ds_{e\gamma}}{dT'_\nu} \ , \tag{A5}$$

where $s_{e\gamma}$ is the entropy density of the $e\gamma$ subsystem and $T'_\gamma(T'_\nu)$ is the photon temperature when the neutrinos are at the temperature T'_ν. As was discussed in the text [cf. Eq. (3.3)], $s_{e\gamma}/T_\nu'^3$ is a constant. Therefore

$$\frac{d}{dT'_\nu}\rho_{e\gamma}=3T'_\gamma T_\nu'^2\left[\frac{s_{e\gamma}}{T_\nu^3}\right] \ . \tag{A6}$$

The constant $(s_{e\gamma}/T_\nu^3)$ may be evaluated in the low-temperature limit where $s_{e\gamma}=s_\gamma=\frac{4}{3}\rho_\gamma/T_\gamma=(8\pi^2/90)T_\gamma^3$ and $T_\gamma^3=\frac{11}{4}T_\nu^3$. Thus

$$\frac{d}{dT'_\nu}\rho_{e\gamma}=\frac{11\pi^2}{15}T'_\gamma T_\nu'^2 \ , \tag{A7}$$

and using Eq. (A2) we have

$$\frac{d}{dT'_\nu}(\rho-\rho_0)=\frac{11\pi^2}{15}T_\nu'^3\left[\left[\frac{T'_\gamma}{T'_\nu}\right]-\left[\frac{11}{4}\right]^{1/3}\right] \ . \tag{A8}$$

Finally, using Eq. (3.8) we write

$$T_\nu'^3\rho_0^{-3/2}=-\frac{d}{dT'_\nu}\left[\frac{1}{2}\left[N_{\rm eff}\frac{\pi^2}{30}\right]^{-3/2}T_\nu'^{-2}\right] \ , \tag{A9}$$

insert Eqs. (A8) and (A9) into Eq. (A4) and integrate by parts yet once again, to obtain

$$t_0=-\frac{11}{6}\frac{1}{N_{\rm eff}}\left[\frac{45M_{\rm Pl}^2}{16\pi^3}\right]^{1/2}\int_{T_\nu}^{\infty}\frac{dT'_\nu}{T_\nu'^2}\frac{d}{dT'_\nu}\left[\frac{T_\gamma(T'_\nu)}{T'_\nu}\right] \ . \tag{A10}$$

We define the neutrino temperature $T_{\nu,0}$ at which the neutrino temperature departs from the photon temperature by the condition

$$\int_{T_\nu}^{\infty}\frac{dT'_\nu}{T_\nu'^2}\frac{d}{dT'_\nu}\left[\frac{T_\gamma(T'_\nu)}{T'_\nu}\right] = \frac{1}{T_{\nu,0}^2}\int_{T_\nu}^{\infty}dT'_\nu\frac{d}{dT'_\nu}\left[\frac{T_\gamma(T'_\nu)}{T'_\nu}\right] \ . \tag{A11}$$

Clearly, $T_{\nu,0}$ is approximately the value of the neutrino temperature at the point where T_γ/T_ν is most rapidly varying. The integral remaining is simply

$$\int_{T_\nu}^{\infty} dT'_\nu \frac{d}{dT'_\nu}\left[\frac{T_\gamma(T'_\nu)}{T'_\nu}\right] = 1-(\tfrac{11}{4})^{1/3} . \tag{A12}$$

We may now write the additional integration constant as

$$t_0 = \frac{11}{6N_{\text{eff}}}[(\tfrac{11}{4})^{1/3}-1]t_1 , \tag{A13}$$

where

$$t_1 = \left[\frac{45}{16\pi^3 N_{\text{eff}}}\right]^{1/2} \frac{M_{\text{Pl}}}{T^2_{\nu,0}} . \tag{A14}$$

This decomposition is illuminating because t_1 is (to first approximation) the time at which the neutrino and photon temperatures depart, while Eq. (A13) shows that the integration constant t_0 is only a small fraction of this time. Although the precise evaluation of $T_{\nu,0}$ would entail a numerical computation, it suffices to use the estimate $T_{\nu,0} \sim m_e/4$ discussed in the text. This gives $t_1 \simeq 42$ sec, and in turn

$$t_0 = 0.056 t_1 \simeq 2 \text{ sec} . \tag{A15}$$

This is the result (3.11) quoted in the text. Since this is such a short time in comparison with the capture time $t_c \simeq 180$ sec, we see that our crude estimate for $T_{\nu,0}$ does indeed suffice even though t_1 is a somewhat rapidly varying function of $T_{\nu,0}$. The point to be made is that the additional integration constant t_0 differs from the time t_1 at which the neutrino and photon temperatures depart by the very small factor displayed in Eqs. (A13) and (A15), a factor which we have derived in this appendix.

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