VIII - BEYOND FLRW

- 3+1 covariant description:

In FLRW, we had a family of fundamental observers with 4-velocity u^μ.

The cosm. Principle implied $\nabla_\mu u_\nu = \frac{1}{3}\theta h_{\mu\nu}$. We can drop this hypothesis and just consider the dynamics of these fundamental observers.

- $\left.\begin{matrix} u^\mu \\ g^{\mu\nu} \end{matrix}\right\} \longrightarrow h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projection operator

- time derivative: derivative along the world line

$$\dot{T}^\mu{}_\nu = u^\alpha \nabla_\alpha T^\mu{}_\nu$$

- spatial gradient: fully orthogonally projected derivative.

$$D_\mu T^\alpha{}_\beta = h^\alpha_\delta \, h^\gamma_\beta \, h^\nu_\mu \, D_\nu T_\gamma^\delta$$

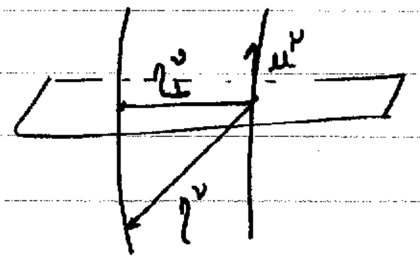
$$\left[\begin{array}{l} v^{\langle\alpha\rangle} = h^\alpha_\beta v^\beta \\ T^{\langle\alpha\beta\rangle} = \left[h^{(\alpha}_\mu h^{\beta)}_\nu - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu}\right. \end{array}\right.$$

- The most general form of $\nabla_\mu u_\nu$ is

$$\boxed{\nabla_\mu u_\nu = -u_\mu \dot{u}_\nu + D_\mu u_\nu = -u_\mu \dot{u}_\nu + \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}}$$

$\theta = D_\mu u^\mu$ rate of expansion
$\sigma_{\mu\nu}$: shear (sym-tracefree)
$\omega_{\mu\nu}$: vorticity.
$\dot{u}^\mu = a^\mu = u^\nu \nabla_\nu u^\mu$ acceleration

• Generalized Hubble law



$$\eta_\perp^\nu = h^\nu_\mu \eta^\mu \equiv (\delta \ell)\, e^\nu$$

$\underline{e^\nu}$: $e^2 = +1 \quad u^\mu e_\mu = 0$
$\downarrow$
this implies that $e_\mu \dot{e}^\mu = 0$

$$u^\mu \nabla_\mu \eta_\perp^\nu = \delta\dot{\ell}\, e^\nu + \delta\ell\, \dot{e}^\nu = \eta_\perp^\mu \nabla_\mu u^\nu \qquad \text{(by symmetry)}$$

$\times e_\nu$: $\quad \delta\dot{\ell} = e_\nu \delta\ell e^\mu \nabla_\mu u^\nu$

$$= \delta\ell\, e_\nu e^\mu \Big[\tfrac{1}{3}\theta\, h_\mu^\nu + \sigma_\mu^\nu + \underset{\downarrow\, 0}{\omega_\mu^\nu} - \underset{\downarrow\, 0}{u_\mu a^\nu} \Big]$$

$$\boxed{\frac{\delta\dot{\ell}}{\delta\ell} = \frac{1}{3}\theta + \sigma_{\mu\nu} e^\mu e^\nu}$$

$\times h^\alpha{}_\nu$: $\quad \underbrace{h^\alpha_\nu e^\nu}_{e^\alpha} \cdot \delta\dot{\ell} + \delta\ell\, h^\alpha_\nu \dot{e}^\nu = \delta\ell\, h^\alpha_\nu e^\mu \big(\tfrac{1}{3}\theta h_\mu^\nu + \sigma_\mu{}^\nu + \omega_\mu^\nu - u_\mu a^\nu$

$\quad\quad \hookrightarrow \left(\tfrac{1}{3}\theta + \sigma_{\mu\nu} e^\mu e^\nu\right)\delta\ell$

$$\boxed{h^\alpha_\nu \dot{e}^\nu = \left[\sigma^\alpha_\mu - (\sigma_{\alpha\beta} e^\alpha e^\beta)\, h^\alpha_\mu - \omega^\alpha_\mu\right] e^\mu}$$

{ shear $\Rightarrow$ recession is diff. in diff. directions
{ shear + vorticity $\Rightarrow$ direction changes with time

• $T_{\mu\nu}$ conservation

The most general form of $T_{\mu\nu}$ is

$$T_{\mu\nu} = \rho u_\mu u_\nu + 2 q_{(\mu} u_{\nu)} + P h_{\mu\nu} + \pi_{\mu\nu}$$

with $\quad q_\mu u^\mu = 0 \qquad \pi_\nu{}^\mu = \pi_{\mu\nu} u^\nu = 0$

This can be inverted as

$$\begin{cases} \rho = T_{\mu\nu} u^\mu u^\nu \\ P = \frac{1}{3} T_{\mu\nu} h^{\mu\nu} \\ q^\mu = - T_{\alpha\beta} h^{\alpha\mu} u^\beta \\ \pi_{\mu\nu} = h^\alpha{}_{\langle\mu} h^\beta{}_{\nu\rangle} T_{\alpha\beta} \end{cases}$$

* Now, project $\nabla_\mu T^{\mu\nu} = 0$ on u_ν

$$0 = u_\nu \nabla_\mu T^{\mu\nu} = \underbrace{\nabla_\mu (T^{\mu\nu} u_\nu)}_{\to\, -\rho u^\mu - q^\mu} - T^{\mu\nu} \underbrace{(\nabla_\mu u_\nu)}_{\to\, \frac{1}{3}\theta h_{\mu\nu} + \sigma_{\mu\nu} - u_\mu \dot{u}_\nu \ \text{(sym!)}}$$

$$0 = -\left(\dot{\rho} + \rho\theta + \nabla_\mu q^\mu \right) - \left(\theta P + \pi_{\mu\nu}\sigma^{\mu\nu} + \dot{u}_\nu q^\nu \right)$$

$$\underline{\dot{\rho} + \theta(\rho + P) + \pi_{\mu\nu}\sigma^{\mu\nu} + \dot{u}_\nu q^\nu + \nabla_\mu q^\mu = 0}$$

Note that $\nabla_\mu q^\mu = \nabla_\mu (h^\mu{}_\alpha q^\alpha) = h^\mu{}_\alpha (\nabla_\mu q^\alpha) + q^\alpha \nabla_\mu h^\mu{}_\alpha$
$= D_\alpha q^\alpha + q^\alpha \dot{u}_\alpha$

so that

$$\boxed{\dot{\rho} + \theta(\rho + P) + \pi_{\mu\nu}\sigma^{\mu\nu} + 2\dot{u}_\nu q^\nu + D_\alpha q^\alpha = 0}$$

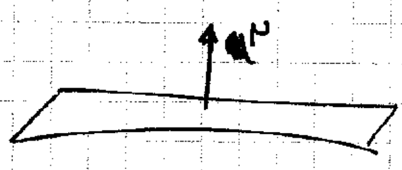
* Project on $h^\alpha{}_\nu$, $h^\alpha{}_\nu \nabla_\mu T^{\mu\nu} = 0$

$$0 = \nabla_\mu (T^{\mu\nu} h^\alpha{}_\nu) - T^{\mu\nu} \nabla_\mu (h^\alpha{}_\nu)$$
$$= \nabla_\mu (T^{\mu\nu} h^\alpha{}_\nu) - u^\alpha T^{\mu\nu} \nabla_\mu u_\nu - T^{\mu\nu} u_\nu \nabla_\mu u^\alpha$$

with the same algebra, we get

$$\boxed{\begin{aligned} \dot{q}^{\langle\mu\rangle} + D^\mu P + \nabla_\alpha \pi^{\mu\alpha} = -\frac{4}{3}\theta q^\mu - \sigma^\mu{}_\alpha q^\alpha - (\rho + P)\dot{u}^\mu \\ - a_\beta \pi^{\beta\mu} - \eta^{\mu\nu\rho} \omega_\nu q_\rho \end{aligned}}$$

with $\omega^\mu = \frac{1}{2} \eta^{\mu\alpha\beta} \omega_{\alpha\beta}$



$$h_{\mu\nu} = g_{\mu\nu} + u^\mu u^\nu$$

$$D_\mu T_\alpha{}^\beta = h_\mu^\nu h_\alpha^\lambda h_\rho^\beta \nabla_\nu T_\lambda{}^\rho \longrightarrow D_\mu h_{\alpha\beta} = 0$$

$$\begin{cases} h_{\mu\nu} \longrightarrow {}^{(3)}R_{\alpha\beta\gamma}{}^\delta \\ g_{\mu\nu} \longrightarrow R_{\alpha\beta\gamma}{}^\delta \end{cases}$$

You can check that

$$D_\alpha D_\beta \omega_\gamma = D_\alpha \left[h_\beta^\lambda h_\gamma^\rho \nabla_\lambda \omega_\rho \right]$$

So that

Gauss codacci

$$\left\| \begin{aligned} {}^3R_{\alpha\beta\gamma}{}^\delta &= h_\alpha^a h_\beta^b h_\gamma^c h^\delta_d R_{abc}{}^d - K_{ac}K_b{}^d + K_{bc}K_a{}^d \\ K_{ab} &= h_a^c \nabla_c u_b \end{aligned} \right.$$

Then

$$R_{abcd} h^{ac} h^{bd} = R\_\_ \,(g+uu)(g+uu)$$

$$= R + 2R_{ac} u^a u^c = 2G_{ac} u^a u^c$$

$$\Longrightarrow \boxed{2G_{ac} u^a u^c = {}^{(3)}R + (K^a{}_a)^2 - K_{ab}K^{ab}}$$

$$K_{ab} = h_a^c \left(\frac{1}{3}\theta h_{cb} + \sigma_{cb} + \omega_{cb} - \dot{u}_c u_b \right)$$

$$\boxed{K_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}}$$

To get the full set of equations, one uses the

- Ricci identities: $2\nabla_{[\mu}\nabla_{\nu]} u^\alpha = R_{\mu\nu}{}^{\alpha}{}_{\beta}\, u^\beta$
- Bianchi identities $\nabla_{[\mu} R_{\alpha\beta]\sigma\rho} = 0$

projected on u & h

I will not derive the whole set of equations between $\sigma_{\mu\nu}$, $\omega_{\mu\nu}$, a^μ, Θ, ρ, P, q^μ, $\pi^{\mu\nu}$ and $H_{\mu\nu}$, $E_{\mu\nu}$ but focus on two of them:

- Raychaudhuri equation
- Generalised Friedmann equation.

• **Raychaudhuri equation.**

$$\dot\Theta = u^\alpha \nabla_\alpha (\nabla_\mu u^\mu)$$
$$= u^\alpha \nabla_\mu \underbrace{\nabla_\alpha u^\mu} + \overbrace{R_{\alpha\mu}{}^{\mu}{}_{\beta}\, u^\beta u^\alpha}^{\to\, -R_{\alpha\beta}}$$

- insert the decomposition
- when terms like $u^\alpha \nabla_\mu \sigma_\alpha{}^\mu = \nabla_\mu(\sigma_\alpha{}^\mu u^\alpha) - \sigma_\alpha{}^\mu \nabla_\mu u^\alpha$
$= -\sigma_\alpha{}^\mu \sigma_\mu{}^\alpha \equiv -\sigma^2$

we get:
$$\boxed{\dot\Theta = -\frac{1}{3}\Theta^2 + \dot u^\mu \dot u_\mu + 2(\omega^2 - \sigma^2) + D_\mu \dot u^\mu - R_{\mu\nu} u^\mu u^\nu}$$

• **vorticity**

It can be shown that for a perfect fluid
$q^a = 0 = \pi^{ab}$ then $\omega^a_{\text{ini}} = 0 \Rightarrow \omega^a = 0$

• **Generalized Friedmann equation**

$$\boxed{{}^{(3)}R = -\frac{2}{3}\Theta^2 + 2\sigma^2 + 2G_{\mu\nu} u^\mu u^\nu}$$

- Beyond the cosmological Principle

Most solutions of Einstein equations have no symmetries and we have focused on a solution with maximal space symmetry, that is with 6 KV.

We can try to classify other solution according to their symmetries. Each symmetry will be associated to a KV.

- Consider the set of all KV that generates the isometries of a manifold. The group of isometries forms a Lie group (that is a differentiable manifold endowed with a group structure so that group operation is differentiable)
- The KV form the associated Lie algebra (It is a vector space under +) and the product is defined by the Lie derivative

$$\left(\mathcal{L}_{\xi_A}\xi_B\right)^\alpha = \frac{\partial \xi_B^\alpha}{\partial x^\sigma}\,\xi_A^\sigma - \frac{\partial \xi_A^\alpha}{\partial x^\sigma}\,\xi_B^\sigma \equiv [\xi_A, \xi_B]^\alpha$$

For KV $[\xi_A, \xi_B]$ is also a KV so that it can be decomposed on the basis of KV.

$$[\xi_A, \xi_B] = \underbrace{C^C_{AB}}_{\text{structure constant}}\,\xi_C$$

The translation along a KV can be interpreted as a mapping of the space on itself or as a motion of the space

<u>The orbit of a point p</u>: set of all points onto which p can be moved by the action of the isometries of space

<u>Invariant variety</u>: set of points moved into itself by the group of isometries

• <u>Group</u>: set with a map

$$G \times G \longrightarrow G \quad \text{(multiplication)}$$

- associative
- $\exists e \quad eg = ge$
- $\forall g \ \exists g^{-1} \quad g \cdot g^{-1} = g^{-1} g = e$

e.g. ① collection of diffeomorphism [infinite]
and multiplication is composition $\phi \circ \psi$

② collection of isometries (subgroup of ①). finite

- <u>Lie group</u>: of dim m

group which is also a m-dimensional manifold
such that $\iota(g) = g^{-1}$ and multiplication $f(g_1, g_2) = g_1 g_2$
are smooth (C^∞)

① is not.

② is $m \leq \frac{n(n+1)}{2}$

• Let G be a Lie group
$\forall h \in G \quad \Psi_h(g) = hg$ (left translation) is a diffeomorphism [1 to 1 with Ψ_h^{-1} C^∞]

$\Psi_h^*: V_p \rightarrow V_{(p)}$

<u>fixed points</u> are points left invariant by the isometries
(p identical to its orbit)
KV vanish on these points

The group is <u>transitive</u> on a surface S if it can map any point of S into any other point of S.
dim = $s \leq n$.

At each point, the dimension of the <u>isotropy group</u> (group leaving that point fixed) is q [generated by KV that vanish at this point]

$$q \leq \frac{1}{2}n(n-1)$$

the dimension of the <u>group of symmetries</u> is $r = s + q$

$$\boxed{0 \leq r \leq \frac{n(n+1)}{2}}$$

from (s, q) one can classify the type of symmetries and then find space from the C^A_{BC} of the Lie algebra of KV of dim r.

<u>In cosmology</u> $n = 4$. q can vary over space (not over an orbit)
it can be greater at some points (axes / centre of sym)
where s is less
r <u>same</u> everywhere

- <u>q = 3</u> isotropic around each point → FLRW
- <u>q = 1</u> Local rot. sym

q = 6	s = 4	M_4; dS, AdS
q = 3	s = 3	FLRW
	s = 4	Einstat
q = 1	s = 2	LTB
q = 1	s = 3	Bianchi (spatially homogeneous)

• Exemple I : Bianchi I ($s=3$)

simplest anisotropically expanding model:

$$ds^2 = -dt^2 + X^2(t)\,dx^2 + \dots Z^2(t)\,dz^2 . \qquad u^a = \delta^a_0$$

$S = (XYZ)^{1/3}$ average expansion.

• $\{t = ct\}$ are flat
• in general $q = 0 \Rightarrow r = 3$

• $\dot{u}^a = \omega^a = 0 \qquad \sigma_{ab} \neq 0$
It can be shown that $(S\,\sigma_{ab})^{\cdot} = 0 \Rightarrow \sigma_{ab} = \dfrac{\Sigma_{ab}}{S^3}$
so that $\sigma^2 = \dfrac{\Sigma^2}{S^6}$

The generalised Friedman eq gives $3\left(\dfrac{\dot{S}}{S}\right)^2 = \dfrac{\Sigma^2}{S^6} + \dfrac{M}{S^{3\gamma}}$
$P = (\gamma - 1)\rho$

→ shear decay but dominate in the past

with the ansatz $X = S(t)\,e^{\Sigma_1 W(t)}, \dots \quad W = \displaystyle\int \frac{dt}{S^3}$

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \qquad \Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2 = 2\Sigma^2$$

and $\Sigma_a = \dfrac{2\Sigma}{3} \sin \alpha_i \qquad \alpha_1 = \alpha \quad \alpha_2 = \alpha + \dfrac{2\pi}{3} \quad \alpha_3 = \alpha + \dfrac{4\pi}{3}$

eg. Dust ($\gamma = 1$)

$$S = \left(\frac{9}{2} M t^2 + \sqrt{3}\,\Sigma t\right)^{1/3} \qquad W = \frac{1}{\sqrt{3}\,\Sigma} \ln \left(\frac{t}{\frac{3}{2} M t + \sqrt{3}\Sigma}\right)$$

$$X = S(t)\left(\frac{t^2}{S^3}\right)^{\frac{2}{3}\sin\alpha_1} \dots$$

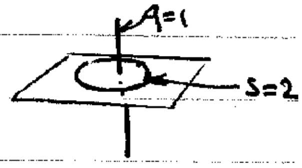
$t \to \infty \qquad \dfrac{t^2}{S^3} \to ct \qquad X, Y, Z \to c_i S \qquad$ isotropise. [redef x, y, z]

$t \to 0 \qquad X \to X_0\, t^{\frac{1}{3}(1 + 2\sin\alpha_1)} \dots \qquad$ 2 pos. expand 1. negative.

• **Example II**: LTB ($s=2$)

inhomogeneous - spherically sym - time dependent

$s=2,\ q=1 \Rightarrow r=3$



$P=0$ ($P\neq 0$: stellar collapse)

$$ds^2 = -dt^2 + X^2(t,r)\,dr^2 + Y^2(t,r)\,d\Omega^2$$

$\hookrightarrow$ angular diameter

Euler equations lead to

$$\begin{cases} X^2 = \dfrac{Y'^2}{1+2E(r)} \\ \dot{Y}^2 = \dfrac{2M(r)}{Y(t,r)} + 2E(r) \qquad (*) \\ 4\pi G\mu(t,r) = \dfrac{M'(r)}{Y^2 Y'} \end{cases}$$

(*) can be solved as

$$Y = \frac{M}{\mathcal{E}}\phi_0(t,r) \qquad t = t_B(r) + \frac{M(r)}{\mathcal{E}^{3/2}}\phi_0'(t,r)$$

$$\mathcal{E} = (2E, 1, -2E) \qquad \phi_0 = (\mathrm{ch}\,\eta - 1, \tfrac{1}{2}\eta^2, 1-\cos\eta) \text{ wh } E>0,0,<0$$

$\Longrightarrow$ 3 arbitrary functions
- $t_B(r)$ local time @ which $Y=0$
- $M(r)$ effective grav. mass within r
- $E(r)$ local curvature ($-kr^2$)

FLRW: $2E = -kr^2$, $M(t) = \frac{4\pi}{3}\mu(t)S^3 r^3$ $Y = S(t)\,r^3$, $t_B = 0$

IX. Problems and questions.

- curv.
- horizon
- structure : next lectures

- Dark sector

- cosm. constant : possible solution.

COURS III et IV : PERTURBATION THEORY

Tentative plan:

- ✓ - Newtonian perturbations
- ✓ - Gauge invariant perturbations
- ✓ - Some solutions
- ✓ - Power spectrum of matter fluctuations
- ✓ - observation of LSS
- - weak lensing
- - CMB
- - origin of perturbation: inflation &

0- INTRODUCTION

what can we learn?
how?
kind of observations.
origins?

3 main points:

How do perturbations evolve?
What are they origin?
link to observation?

I. Newtonian perturbations

Before we expose the general relativistic theory of cosmological perturbations, I will consider purely Newtonian perturbations.

- **Statique spacetime**

For a fluid of density ρ and pressure P we have

$$\begin{cases} \partial_t \rho + \nabla.(\rho \vec{v}) = 0 & \text{continuity} \\ \partial_t \vec{v} + (\vec{v}.\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}P - \vec{\nabla}\Phi & \text{euler} \end{cases}$$

If we neglect gravitation and linearize the system ($\rho = \bar{\rho} + \delta\rho$) we can infer a wave equation

$$\partial_t^2 \delta\rho - \Delta \delta P = 0$$

$6(0)$: si $v=0$ $\vec{\nabla}P = 0$ $\dot{\rho} = 0$

$6(1)$ $\begin{cases} \dot{\delta\rho} + \rho \nabla \delta \vec{v} = 0 \\ \delta \dot{v} = -\frac{1}{\rho} \nabla \delta P \end{cases}$

If we introduce the compressibility coefficient:

$\chi_a = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_a$ with $a = T$ if prop. is fast

S if eq. cant be reached

then:

$$\boxed{\partial_t^2 \delta\rho - c_s^2 \Delta \delta\rho = 0 \qquad c_s^2 = \rho \chi_S = \left(\frac{\delta P}{\delta \rho}\right)_S}$$

density perturbation propagate with a constant amplitude with velocity c_s: The sound speed

- **effect of gravity**

we have to add the Poisson equation $\Delta\Phi = 4\pi G \delta\rho$

Perturbing around a homogeneous background, the perturbation equation takes the form

$$\partial_t^2 \delta\rho - c_s^2 \Delta \delta\rho = 4\pi G \bar{\rho}\, \delta\rho$$

considering a plane wave equation [$\delta\rho \propto e^{i(\omega t - \vec{k}\vec{x})}$] leads to the dispersion equation

$$\omega^2 = c_s^2 \left(k^2 - \frac{4\pi G \bar{\rho}}{c_s^2}\right)$$

which takes the form

$$\boxed{\omega^2 = 4\pi^2 c_s^2 \left(\frac{1}{\lambda^2} - \frac{1}{\lambda_J^2}\right) \qquad \lambda_J \equiv c_s \sqrt{\frac{\pi}{G\bar{\rho}}}}$$

λ_J is the Jeans length and distinguishes 2 regimes

$$\begin{cases} \lambda > \lambda_J : \text{ gravity dominates } \omega^2 < 0 : \ \delta\rho \text{ grows exp.} \\ \lambda < \lambda_J : \text{ Pressure dominates } \omega^2 > 0 : \text{ waves} \end{cases}$$

- <u>Expanding spacetime</u>:

we can use the same equations but we have to take into account that

$$\vec{r}(t) = a(t)\,\vec{x}$$

so that $\vec{v}(t) = \dot{\vec{r}}(t) = H\vec{r} + \vec{u}$.

Previously time and space operators where considered independent so $\partial_t \rho(t,\vec{r})$ means $\partial_t \rho(t,\vec{r})|_{\vec{r}}$. Now $\vec{r}$ depends on t and we have to switch to $\partial_t \rho(t,\vec{r})|_{\vec{x}}$. It is clear that

$$\begin{cases} \nabla_{\vec{r}} \longrightarrow \frac{1}{a}\nabla_{\vec{x}} \\ \partial_t \rho|_{\vec{r}} \longrightarrow \partial_t \rho|_{\vec{x}} - H\vec{x}.\vec{\nabla}\rho \end{cases}$$

so that (use $\nabla_x . \vec{x} = 3$)

$$\boxed{\begin{aligned} &\dot{\rho}(\vec{x},t) + 3H\rho(\vec{x},t) + \frac{1}{a}\nabla_x[\rho\vec{u}] = 0 \\ &\dot{u} + Hu + \frac{1}{a}(u.\nabla_x)u = -\frac{1}{a}\nabla_x\Phi - \frac{1}{a\rho}\nabla_x P \\ &\Delta_x \Phi = 4\pi G\bar{\rho}a^2 \frac{\delta\rho}{\bar{\rho}} \end{aligned}}$$

Define the density contrast by

$$\rho = \bar\rho(t)\,[1+\delta(x,t)],$$

the previous set of equations reduce to

$$\boxed{\begin{aligned} &\dot\delta + \frac{1}{a}\nabla\left[(1+\delta)u\right] = 0 \\ &\dot u + 2H\dot\delta + \frac{1}{a}(u\cdot\nabla_x)u = -\frac{1}{a}\left[\nabla_x\Phi + \frac{1}{\rho}\nabla_x P\right] \\ &\Delta_x\Phi = 4\pi G\bar\rho a^2\delta \end{aligned}}$$

[we used $\dot\rho + 3H\rho = 0$]

Now ∇ will mean ∇_x always. We can derive a second order equation for δ:

$$\ddot\delta + 2H\dot\delta = \frac{1}{\rho a^2}\Delta P + \frac{1}{a^2}\nabla\left[(1+\delta)\nabla\Phi\right] + \frac{1}{a^2}\partial_i\partial_j\left[(1+\delta)u^i u^j\right]$$

- does not assume $\delta \ll 1$
- needs a description of the fluid to be closed.
- better description can be obtained from a Boltzmann approach

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\vec p}{ma^2}\cdot\nabla f - m\nabla\Phi\cdot\frac{\partial f}{\partial p} = 0 \qquad \vec p = ma\vec u$$

If one can solve this collisionless eq. then find moments:

$$\begin{cases} \rho = \int d^3p\, f(x,p,t) \\ \rho u = \int \frac{p}{am} f\, d^3p \\ \rho u_i u_j + \sigma_{ij} = \int \frac{p_i p_j}{a^2 m^2} f\, d^3p \end{cases}$$

• <u>linearized system:</u> $\delta \ll 1$

$$\boxed{\ddot\delta + 2H\dot\delta = \frac{c_s^2}{a^2}\Delta\delta + 4\pi G\bar\rho\delta}$$

similar to the static case but

1. Jeans length is time depend
2. expansion induces a damping.

• Growth of density perturbation

We decompose $\delta = D_+ \mathcal{E}_+(x) + D_-(t)\, \mathcal{E}_-(x)$ where $\mathcal{E}$'s are the initial perturbations

For pressureless matter $c_s^2 = 0$ so that

$$\ddot{D} + 2H\dot{D} - 4\pi G \bar{\rho}\, D = 0$$

Friedmann equation implies that $H^2(t) = \frac{8\pi G}{3} \bar{\rho}(t) \frac{1}{\Omega_m(t)}$ so that

$$\boxed{\ddot{D} + 2H\dot{D} - \frac{3}{2} H^2(t)\, \Omega_m(t)\, D = 0}$$

clearly when $\Omega_m = 1$; $a \propto t^{2/3}$ so that

$D_+ \propto t^{2/3} \propto a$ and $D_- \propto t^{-1} \propto a^{-3/2}$.

In more general cases one needs to solve the equation numerically. Let us use a as a time variable.

$$\cancel{\frac{dD}{da} = \frac{1}{aH}\dot{D} \quad \& \quad \frac{d^2D}{da^2} = \frac{H^2}{a^2}\ddot{D} + \frac{\dot{D}H}{a}\left(\frac{1}{H}\frac{dH}{da} - \frac{1}{a}\right)}$$

$$\frac{dD}{da} = \frac{1}{aH}\dot{D} \quad \& \quad \frac{d^2D}{da^2} = \frac{1}{a^2H^2}\ddot{D} - \frac{\dot{D}}{aH}\left(\frac{1}{H}\frac{dH}{da} + \frac{1}{a}\right)$$

so that (setting $a_0 = 1$) and using $\rho_m = \rho_{m_0} a^{-3}$:

$$\boxed{\frac{d^2D}{da^2} + \left(\frac{1}{H}\frac{dH}{da} + \frac{3}{a}\right)\frac{dD}{da} - \frac{3}{2}\frac{\Omega_{m0}}{a^5}\left(\frac{H}{H_0}\right)^{-2} D = 0}$$

From Friedman eq. $\left[\dot{H} = -4\pi G\rho + \frac{K}{a^2} \;;\; \ddot{H} = 4\pi G\rho(3H) - \frac{K}{a^2}H\right]$

we can show that

$$\boxed{D_- \propto H \quad ; \quad D_+ = \frac{5}{2}\frac{H}{H_0}\Omega_{m_0}\int_0^a \frac{da'}{[a' E(a')]^3}}$$

where the last formula is obtained by a variation of constant method and normalised such that when $K=0$ $\Omega_m = 1$ $D_+ = a$.

Indeed, this applies only to matter dominated universe (Λ, K still)

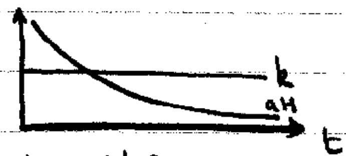
- for modes on scales < Hubble radius

→ e.g. Formation of LSS.

It gives a good description of the speed of growth of LSS with ~~time~~ given the cosmological parameter.

D_+ is tied to $H(a)$ also ⇒ low z sub-Hubble scales there is a rigidity perturbation-bgd.

<u>but</u>: $a \propto t^{\alpha}$, $H \propto \frac{\alpha}{t}$ $\quad k^{phys} = \frac{k}{a}$



$Ha = \alpha t^{\alpha-1} \searrow \; (\alpha < 1)$

All sub-Hubble modes where super-Hubble in the past

⟶ we need to study perturbations directly from GR.

II. Gauge invariant cosmological perturbations:

At linear level the most general metric close to FLRW is of the form

$$ds^2 = a^2(\eta)\left[-(1+2A)\,d\eta^2 + 2B_i\,dx^i d\eta + (\gamma_{ij}+h_{ij})\,dx^i dx^j\right]$$
$$\equiv (\bar{g}_{\mu\nu} + \delta g_{\mu\nu})\,dx^\mu dx^\nu$$

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - \delta g^{\mu\nu} \quad \& \quad \delta g^{\mu\nu} = \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\delta g_{\rho\sigma}.$$

We perform a SVT decomposition:

$$\begin{cases} B^i = D^i B + \bar{B}^i & \text{with } D^i\bar{B}_i = 0 \\ h_{ij} = 2C\gamma_{ij} + 2D_iD_jE + 2D_{(i}\bar{E}_{j)} + \bar{E}_{ij} & \text{with } D_i\bar{E}^{ij} = \bar{E}_i^{\,i} = 0 \end{cases}$$

Note that $D_iB = \gamma_{ij}D^jB$ etc...

Note also that the decomposition is not unique since we have e.g. $D_i B^i = \Delta B$ and one needs to specify boundary conditions

The 10 degrees of freedom of the metric are decomposed as

$$\left.\begin{array}{lll} 4 \text{ scalars: } & A, B, C, E & \rightarrow 4\, d^o \\ 2 \text{ vectors : } & \bar{B}^i \;\; \bar{E}^i & \rightarrow 4\, d^o \\ 1 \text{ tensor : } & \bar{E}^{ij} & \rightarrow 2\, d^o \end{array}\right\} 10\, d^o$$

- <u>Gauge problem:</u>

We want to compare two metrics:

$$\begin{cases} g_{\mu\nu} : \text{ "real" metric} \\ \bar{g}_{\mu\nu} : \text{ smooth FLRW "close" to } g_{\mu\nu} \end{cases}$$

In standard perturbation theory, spacetime is fixed and we compare the values of a given quantity at a given point of spacetime:

$$\delta Q(\vec{x}, t) = Q(\vec{x}, t) - \bar{Q}(\vec{x}, t)$$

Here the situation is more tricky because we are in fact comparing 2 spacetimes

$$(\mathcal{M}, g_{\mu\nu}) \longrightarrow (\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$$

This means that we have to introduce an arbitrary way of mapping one spacetime to another

There is no natural identification and it implies that some d^o of freedom are non physical and related to the freedom in the choice of the diffeomorphism φ

The form of the perturbed metric assumes that a coordinate system has been chosen

In particular one could ~~....~~ make a coordinate transformation $x^i \to y^i = x^i + \xi^i$ for a FLRW. The

$$g_{\mu\nu} \to g_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}$$

$$\bar{g}_{\mu\nu} dx^\mu dx^\nu = + a^2 [-d\eta^2 + \delta_{ij} dx^i dx^j]$$

$$= a^2 [-d\eta^2 \underbrace{- 2(\xi'_i - 2\mathcal{H}\xi_i)}_{2B_i} dy^i d\eta + (\delta_{ij} - 2 D_i \xi_j) dy^i dy^j]$$

The second metric seems to be the one of a perturbed FLRW but it is just the one of a pure FLRW in a (bad !) coordinate system

$\Rightarrow$ 4 degrees of freedom can be set to zero by such coordinate changes.

$\Rightarrow$ [we have to find perturbations that do not depend on the choice of coordinates system]

• Gauge invariant metric perturbation

Let us consider a change of coordinates $x^\mu \to x^\mu + \xi^\mu$, then $g_{\mu\nu} \to g_{\mu\nu} - \mathcal{L}_\xi g_{\mu\nu}$ $\quad \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$

we set $\xi^\mu = (T, L^i) \Rightarrow \xi_\mu = (-Ta^2, a^2 L_i)$ $\quad L_i = \delta_{ij} L^j$

$$\begin{cases} \mathcal{L}_\xi \bar{g}_{00} = 2\nabla_0 \xi_0 = 2\partial_0(-Ta^2) - 2\Gamma^\alpha_{00}\xi_\alpha = 2a^2[-T' - 2\mathcal{H}T] - 2\mathcal{H}(-a^2 T) \\ \qquad = -2a^2[T' + \mathcal{H}T] \\ \mathcal{L}_\xi \bar{g}_{0i} = \nabla_i \xi_0 + \nabla_0 \xi_i = \partial_i(-a^2 T) - \Gamma^\alpha_{i0}\xi_\alpha + \partial_0(a^2 L_i) - \Gamma^\alpha_{i0}\xi_\alpha \\ \qquad = a^2[-\partial_i T + L'_i] \\ \mathcal{L}_\xi \bar{g}_{ij} = \nabla_i \xi_j + \nabla_j \xi_i = \partial_i \xi_j + \partial_j \xi_i - 2\Gamma^\alpha_{ij}\xi_\alpha = a^2(D_i L_j + D_j L_i) - 2\Gamma^0_{ij}(-a^2 \\ \qquad = a^2[D_i L_j + D_j L_i + 2\mathcal{H}T\delta_{ij}] \end{cases}$$

At linear order

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} - \mathcal{L}_\xi \bar{g}_{\mu\nu}$$

so that

$$\begin{cases} A \longrightarrow A - T' - \mathcal{H}T \\ B_i \longrightarrow B_i + \partial_i T - L_i' \\ h_{ji} \longrightarrow h_{ij} - D_i L_j - D_j L_i - 2\mathcal{H}T\gamma_{ij} \end{cases}$$

using the SVT decomposition

$$A \rightarrow A - T' - \mathcal{H}T \qquad \bar{B}^i \rightarrow \bar{B}^i - \bar{L}^{i\prime} \qquad \bar{E}^{ij} \rightarrow \bar{E}^{ij}$$

$$B \rightarrow B + T - L' \qquad \bar{E}^i \rightarrow \bar{E}^i - \bar{L}^i$$

$$C \rightarrow C + \mathcal{H}T$$

$$E \rightarrow E - L$$

Clearly we can construct quantities which remain invariant under a change of coordinates: $\quad B - E' \rightarrow B - E' + T$

$$\boxed{\begin{aligned} &\Phi = A + \mathcal{H}(B - E') + (B - E')' \\ &\Psi = -C - \mathcal{H}(B - E') \\ &\Phi^i = \bar{E}^{i\prime} - \bar{B}^i \\ &\bar{E}^{ij} \end{aligned}}$$

we have absorbed the gauge freedom in a redef. of pert. variables
$\Rightarrow$ just $\underline{\underline{6}}$ (10-4) remain.

- Matter perturbation

the most general $\delta T_{\mu\nu}$ is

$$\delta T_{\mu\nu} = (\delta\rho + \delta P)\bar{u}_\mu \bar{u}_\nu + \delta P\, \bar{g}_{\mu\nu} + 2(\rho + P)\bar{u}_{(\mu}\delta u_{\nu)} + P\delta g_{\mu\nu} + a^2 P \pi_{\mu\nu}$$

$$u^\mu = \bar{u}^\mu + \delta u^\mu$$

Since $u^\mu u_\mu = -1$ & $\bar{u}^\mu = \frac{1}{a}\delta^\mu_0$ we get that

$$2\bar{u}^\mu \delta u_\mu + \delta g_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = 0 \Rightarrow \delta u_0 = -Aa$$

we set $\delta u^i \equiv v^i/a$ and thus

$$\boxed{\begin{array}{ll}\delta u^\mu = \frac{1}{a}(-A, v^i) & \delta u_\mu = a(-A, v_k + B_k) \\ v_i = D_i v + \bar{v}_i & \end{array}}$$

$\Pi_{\mu\nu}$ is decomposed as $\begin{cases}\pi_{ij} = \Delta_{ij}\bar{\pi} + D_{(i}\bar{\pi}_{j)} + \bar{\pi}_{ij} \\ \Delta_{ij} = D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\end{cases}$

under a chge of coord.

$$\begin{cases}\delta Q \longrightarrow \delta Q - \mathcal{L}_\xi \bar{Q} & \mathcal{L}_\xi \bar{Q} = \xi^\alpha \partial_\alpha \bar{Q} = T\bar{Q}' \\ \delta u^\mu \longrightarrow \delta u^\mu - \mathcal{L}_\xi \bar{u}^\mu & \mathcal{L}_\xi \bar{u}^\mu = \xi^\alpha \partial_\alpha \bar{u}^\mu - \bar{u}^\alpha \partial_\alpha \xi^\mu\end{cases}$$

e.g. $\begin{cases}\delta\rho \to \delta\rho - \rho' T \\ v_i \to v_i + L'_i\end{cases}$

As for the metric we can define gauge invariant matter pert.

$$\boxed{\begin{cases}\delta^N = \delta + \frac{\rho'}{\rho}(B - E') \\ \delta^F = \delta - \frac{\rho'}{\rho}\frac{C}{\mathcal{H}} \\ \delta^C = \delta + \frac{\rho'}{\rho}(v + B) \\ V = v + E' \\ \bar{V}_i = \bar{v}_i + \bar{B}_i\end{cases}}$$

$$\boxed{\begin{array}{l}\delta^F = \delta^N + \frac{\rho'}{\rho}\frac{\Psi}{\mathcal{H}} \\ \delta^C = \delta^N + \frac{\rho'}{\rho}V\end{array}}$$

Note : • we cannot tell what is the "correct" variable before we have explicitely said what is measured
• These variables have a priori no meaning.

• Gauge invariant pert equations

* $\bar{Q}=0 \Rightarrow \delta Q \to \delta Q - \mathcal{L}_\xi \bar{Q} = \delta Q$ (Stewart-Walker

* $\left.\begin{array}{l}\bar{G}_{\mu\nu}-\bar{T}_{\mu\nu}=0\\ \bar{\nabla}_\mu \bar{T}^{\mu\nu}=0\end{array}\right\} \longrightarrow$ perturbation eq. can be expressed in terms of GI variables

* S, V, T decouple.

* 2 strategies: 1. (A,... E...) write pert eq and gather terms
2. pick up a gauge
e.g. $B=E=\bar{B}_i=0$ then
$A=\Phi \quad C=-\Psi \quad \delta=\delta^N \quad v=V \quad \bar{v}_i=\bar{V}_i$
write the equations for $(\Phi, \Psi ..)$
then it can be obtained in any gauge

I will not detail the computation. one has to write $\begin{cases}\delta G_{\mu\nu} = \delta T_{\mu\nu} \\ \delta[\nabla_\mu T^{\mu\nu}] = 0\end{cases}$ for SVT and I pick up newt gauge

Ⓣ $\boxed{\bar{E}''_{kl} + 2\mathcal{H}\bar{E}'_{kl} + (2K-\Delta)\bar{E}_{kl} = \kappa a^2 P \bar{\pi}_{kl}}$

just one equation

Ⓥ :
$$\boxed{\begin{array}{ll}(\Delta+2K)\,\bar{\Phi}_i = -2\kappa a^2\rho(1+\omega)\bar{V}_i & (0i) \\ \bar{\Phi}'_i + 2\mathcal{H}\bar{\Phi}_i = \kappa P a^2 \bar{\pi}_i & (ij) \\ \bar{V}'_i + \mathcal{H}(1-3c_s^2)\bar{V}_i = -\frac{1}{2}\frac{\omega}{1+\omega}(\Delta+2K)\bar{\pi}_i & \text{euler}\end{array}}$$

(5) we decompose $\delta P = c_s^2 \delta\rho + P\Gamma$

$\hookrightarrow w\Gamma = \frac{1}{\rho}(\delta P - c_s^2 \delta\rho)$

$\Gamma = 0$ for adiabatic perturbations

$\Gamma \to \Gamma - \underbrace{(P' - c_s^2 \rho')}_{0} T = \Gamma$ GI

$$\boxed{\begin{array}{ll}
{}^0_0 + 3\mathcal{H}\,{}^0_i: & (\Delta + 3K)\Psi = \frac{\kappa}{2} a^2 \rho\, \delta^C \\
{}^i_j,\pi & \Psi - \Phi = \kappa a^2 P \bar{\pi} \\
{}^0_i & \Psi' + \mathcal{H}\Phi = -\frac{\kappa}{2} a^2 \rho(1+w) V \\
{}^i_i - 3\,{}^0_0 & \Psi'' + 3\mathcal{H}(1+c_s^2)\Psi' + \left[2\mathcal{H}' + (\mathcal{H}^2 - K)(1+3c_s^2)\right]\Psi - c_s^2 \Delta\Psi \\
 & \quad = -(\mathcal{H}^2 + 2\mathcal{H}' + K)\left[\frac{1}{2}\Gamma + (3\mathcal{H}^2 + 2\mathcal{H}')\bar{\pi} + \mathcal{H}\bar{\pi}' + \frac{1}{3}\Delta\bar{\pi}\right] \\
 & \quad\quad - 9 c_s^2 \mathcal{H}^2(\mathcal{H}^2 + K)\bar{\pi} \\
 & \left(\frac{\delta^N}{1+w}\right)' = -(\Delta V - 3\Psi') - 3\mathcal{H}\frac{w}{1+w}\Gamma \\
 & V' + \mathcal{H}(1 - 3c_s^2)V = -\Phi - \frac{c_s^2}{1+w}\delta^N - \frac{w}{1+w}\left[\Gamma + \frac{2}{3}(\Delta + 3K)\bar{\pi}\right]
\end{array}}$$

III - Some solutions

The study of the properties of these equations is vast. I just sketch some of them.

- **Vector modes**

  if $\bar{\pi}_i = 0$ then $\bar{\Phi}_i \propto a^{-2}$ $\bar{V}^i \propto a^{-(1-3c_s^2)}$

  $\downarrow$ quickly negligible $\hookrightarrow$ vorticity decay of NR

  we neglect them (but: $\vec{B}$ / TD ...)

- **Tensor modes**

  assume $a \propto \eta^\nu$ and set $x = k\eta$ then

$$\frac{d^2 E_{ij}}{dx^2} + \frac{2\nu}{x}\frac{dE_{ij}}{dx} + E_{ij} = 0$$

fluid with eq $w \neq 0$

$K = 0 \qquad W \neq 0 \qquad c_s^2 = w$

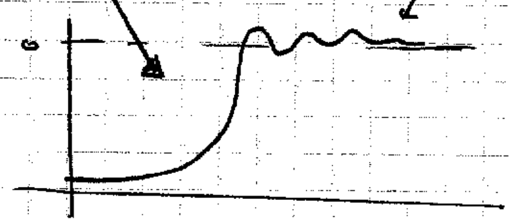
$a \propto \eta^\nu \qquad \nu = \frac{2}{1+3w}$

set $f = x^\nu \Phi$ & $x = k\eta$

$$\rightarrow \boxed{f'' + \frac{2}{x} f' + \left[w - \frac{\nu(\nu+1)}{x^2} \right] f = 0}$$

$$\Phi = x^{-\nu} \underbrace{\left[A\, j_{\nu+1/2}(c_s x) + B\, n_{\nu+1/2}(c_s x) \right]}_{Z_\nu} \qquad j_\nu = \sqrt{\frac{\pi}{2z}}\, J_{\nu+1/2}$$

	$c_s x \ll 1$	$c_s x \gg 1$
Φ	$\Phi_+ + \Phi_- x^{-1-2\nu}$	$\Phi_+ x^{-(1+\nu)} \cos(c_s x - \alpha_\nu)$



constant on large scale

decay + oscillate due to Pressure.

$$\rho \propto a^{-3(1+w)}$$

- $-k^2 \phi = \frac{\kappa}{2} a^2 \rho \delta^c \quad \downarrow \quad \delta^c \propto \cancel{x^{2-\nu}}\, Z_\nu$

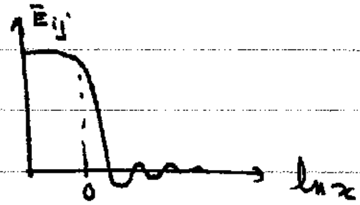
- $\Psi' + \mathcal{H} \Phi = -\frac{\kappa}{2} a^2 (1+w) V \longrightarrow kV \propto x^{1-\nu} \left[Z_\nu - \frac{c_s x}{\nu+1} Z_{\nu-1} \right]$

$\cancel{\delta^c \sim 2\Phi_+ x^2 \sim 2\Phi_+ x^{1-\nu} \cos(\ \ )}$.

$\sim -kV$

It has solutions in terms of Bessel functions.

$$E_{ij} = x^{1/2-\nu}\left[A_{ij} J_{\nu-1/2}(x) + \underbrace{B_{ij} N_{\nu-1/2}(x)}_{\text{diverge when } x\to 0 \text{ (decaying mode)}}\right]$$



- **scalar modes**

The system of equations can be rewritten as a single equation for Φ (single field) assuming $\Pi = 0$

$$\Phi'' + 3\mathcal{H}(1+c_s^2)\Phi' + \left[2\mathcal{H}' + (\mathcal{H}^2 - K)(1+3c_s^2)\right]\Phi - c_s^2\Delta\Phi = \frac{\kappa}{2} a^2 P \Pi$$

it can be rewritten as

$$\boxed{u_S'' - \left(\frac{\theta''}{\theta} + c_s^2\Delta\right) u_S = \frac{\kappa}{3}\frac{\theta}{\mathcal{H}} a^4 P \Pi} \quad u_S = \frac{2}{3}\frac{a^2\theta}{\mathcal{H}}\Phi$$

with $\theta = \frac{1}{a}\left(\frac{\rho}{\rho+P}\right)^{1/2}\left(1 - \frac{3K}{\kappa\rho a^2}\right)^{1/2} \Rightarrow \theta = \frac{\mathcal{H}}{a}\left[\frac{2}{3}(\mathcal{H}^2 - \mathcal{H}' + K)\right]^{-1/2}$

The interesting point is that on large scale ($k\eta \ll 1$) as long as Π negligible ($u = \theta$ is solution)

$$u_S = A\,\theta(\eta) + B\,\theta\int^{\eta}\frac{d\eta'}{\theta^2} + \mathcal{O}(k^2 c_s^2\,\theta/\theta'')$$

application w_1 | η_* | w_2 $\qquad n_i = \frac{p_i}{1-p_i} \qquad 1+w_i = \frac{2}{3p_i}$

$a = \eta^{n_i} = t^{p_i} \qquad K=0$

$$\Phi = \frac{\mathcal{H}}{a^2}\left[A_- + A_+\int a^2(1+w)\,d\eta\right]$$

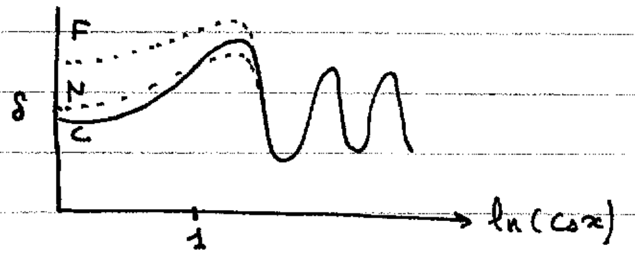
$$\Phi \sim A_+\frac{(1+w_1)p_1}{p_1+1} = \frac{2A_+}{3(1+p_1)} \qquad \eta < \eta_*$$

$$\sim \frac{2A_+}{3(1+p_2)} \qquad \eta \gg \eta_*$$

$$\boxed{\frac{\phi(\eta \gg \eta_*)}{\phi(\eta < \eta_*)} = \frac{1+p_1}{1+p_2} = \frac{9}{10} \text{ for Rad-mat}}$$

• gauge dependence

$w = c^t$



below sonic radius ($c_s x \gg 1$) gauge issue washed out.

$\Rightarrow$ important for fixing IC!

• $w = 0 = c_s^2$

$f = x^\nu \Phi \quad \nu = 2$

$$f'' + \frac{2}{x} f' - \frac{6}{x^2} f = 0 \Rightarrow \Phi = \Phi_- + \Phi_+ x^{-5}$$

$$\delta^c = -\frac{1}{6} x^2 \Phi$$

$a \propto \eta^2 \propto x^2$

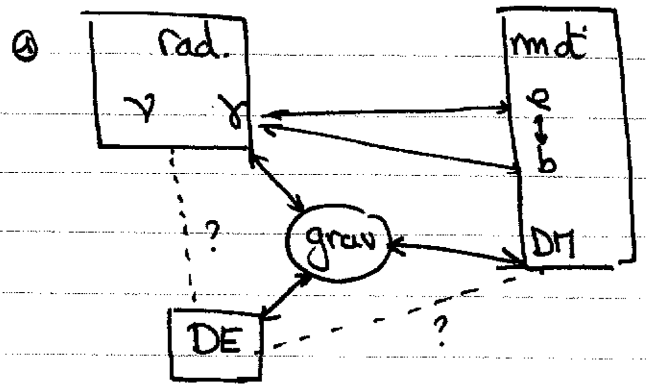
$$\Rightarrow \boxed{\delta_m^c \propto a} \quad \text{on all scales}$$

at most factor 10^4 in MDU! $\Rightarrow$ need initial pert of order 10^{-4} - 10^{-5}.

IV. POWER SPECTRUM OF DENSITY PERTURBATIONS

The general study of perturbation in the universe requires to include various aspects:

①



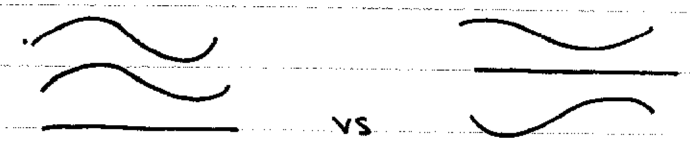
b, DM, γ, ν

DE (Λ) [illegible] no fluct.

② Initial conditions

die ça plustard

- on superHubble scales
- from a stochastic process → power spectrum



$\delta\rho_i \propto \delta\rho_j$ adiab vs $\delta\rho_i + \delta\rho_j = 0$ isocurv.

③ usually fluids can interact (e.g. Thomson scattering), so we do not expect to have $\nabla_\mu T_i^{\mu\nu} = 0 \quad \forall i$.

Instead we expect

$$\boxed{\nabla_\mu T_i^{\mu\nu} = Q_i^\mu}$$

force term and we need microphysics

Action-Reaction: $\boxed{\sum Q_i^\mu = 0}$

In the background, symmetries imply that

$$Q_\mu = -(aQ, 0)$$

so that the conservation equation becomes

$$\boxed{\rho_i' + 3\mathcal{H}(1+w_i)\rho_i = aQ_i}$$

and it follows (just derive P/ρ)

$$\boxed{w_i' = -\left[3\mathcal{H}(1+w_i) - \frac{Q_i}{\rho_i}\right](c_i^2 - w_i)}$$

At the perturbation level, I decompose Q^μ as

$$Q^\mu = Q u^\mu + F^\mu \quad \text{with } F^\mu u_\mu = 0$$

and then set $a f^\mu = \left(\delta^\mu_\nu + u^\mu u_\nu\right) F^\nu \Rightarrow f^0 = 0$

$$f^i = D^i f + \bar{f}^i$$

Again, in a gauge transformation

$$\delta Q \to \delta Q - Q' T$$

so that we define the GI quantity

$$\delta Q^N = \delta Q + Q'(B - E')$$

It is an (interesting) exercise to derive the generalisation of the conservation eq.

$$\boxed{\begin{aligned} \left(\frac{\delta^N}{1+w}\right)' &= -\left[\Delta V - 3\psi'\right] - 3\mathcal{H}\frac{w}{1+w}\Gamma \\ &\quad + \frac{a}{\rho+P}\left[\delta Q^N + Q(\phi - \delta^N)\right] \\ V' + \mathcal{H} V &= -\Phi - \frac{c_s^2}{1+w}\delta_a^c + \frac{w}{1+w}\left[\Gamma + \frac{2}{3}(\Delta + 3K)\bar{\pi}\right] \\ &\quad + \frac{a}{\rho+P}\left[f - a c_s^2 V\right] \end{aligned}}$$

For a 2 fluid melange

one can define

$$S_{ab} \equiv \left(\frac{\delta_a}{1+w_a} - \frac{\delta_b}{1+w_b} \right) \qquad V_{ab} \equiv v_a - v_b$$

by construction these quantities are gauge invariant.

$$\Omega \delta = \sum_a \Omega_a \delta_a \quad ; \quad \Omega (1+w) v = \sum_a (1+w_a) \Omega_a v_a$$

$$\boxed{\text{Note} \quad \frac{P}{\rho} = w = \frac{\sum P_i}{\sum \rho_i} = \frac{\sum w_i \rho_i}{\rho} = \frac{\sum w_i \Omega_i}{\Omega} \Rightarrow \Omega w = \sum w_i \Omega_i \text{ etc...}}$$

we decomposed $\begin{bmatrix} \delta_a & V_a \\ \delta_b & V_b \end{bmatrix} \longrightarrow \begin{bmatrix} S_{ab} & V_{ab} & \text{(relative)} \\ \delta & v & \text{(total)} \end{bmatrix}$

it can be inverted to get:

$$\left(\frac{\Omega_b}{1+w_a} + \frac{\Omega_a}{1+w_b} \right) \delta_a = \frac{\Omega}{1+w_b} \delta + \Omega_b S_{ab}.$$

From the 2 cons. eq. we get

$$S'_{ab} = - \Delta V_{ab} - 3\mathcal{H} \Gamma_{ab} \qquad \Gamma_{ab} = \frac{w_a}{1+w_a} \Gamma_a - (b)$$

on large scale (super-Hubble) for perfect fluid ($\Gamma_i = 0$) we have

$$\boxed{S_{ab} = \text{cst}}$$

The difference of euler equation gives a first order eq for V_{ab}.

- <u>Power spectrum for CDM + Rad</u> ($K=0$

we consider the system of CDM + rad

$y = \frac{a}{a_{eq}}$

$$\rightarrow \boxed{\Omega_m = \frac{y}{1+y} \qquad \Omega_r = \frac{1}{1+y}}$$

I concentrate on adiabatic perturbations

$$\delta_m = \frac{3(1+y)}{4+3y}\,\delta \qquad \delta_r = \frac{4}{3}\,\delta_m$$

<u>Eq. de Poisson</u>

$$\Delta\phi = 4\pi G \rho a^2 \delta^c \longrightarrow \boxed{\delta^c = -\frac{4}{3}\left(\frac{k}{k_{eq}}\right)^2 \frac{y^2}{1+y}\,\Phi}$$

where I have used the Friedmann eq as

$$\boxed{\mathcal{H}^2 = \mathcal{H}_{eq}^2\,\frac{1+y}{2y^2}}$$

and define $\boxed{k_{eq} = \mathcal{H}_{eq}}$ mode crossing Hubble radius at equality.

<u>Then</u>: $\delta^N = \delta^c - \frac{\rho'}{\rho} V = \delta^c + 3\mathcal{H}(1+w)V$ so that

$$\boxed{\delta^N = \delta^c - 2(\phi + y\dot{\phi})} \qquad \dot{} = \frac{d}{dy} \text{ here.}$$

<u>So deep in RDU</u>

adiabatique

$$\phi = \phi_i \qquad \dot{\phi}_i = 0 \qquad S = \dot{S} = 0$$

$$\delta^c = -\frac{2}{3} x_i^2\,\Phi \qquad x_i = k/\mathcal{H}_i$$

$$\delta^c_r = \delta_i = \frac{4}{3}\delta_m$$

we found that on <u>p. 13</u> for Φ

- $y_0 = \frac{a_0}{a_{eq}} = 1+z_{eq} \sim 2.4\ 10^4 h^2$

- $H^2 = \frac{8\pi G}{3}(\rho_r + \rho_m)$

$$\left.\begin{aligned} \mathcal{H}_{eq} &= a_{eq} H_{eq} \\ \mathcal{H}_{eq}^2 &= 2\Omega_{m_0} (1+z)_{eq} H_0^2 \end{aligned}\right\} k_{eq} = H_0 \sqrt{2\Omega_{m_0}(1+z_{eq})}$$

$$\boxed{k_{eq}^{-1} = \frac{14}{\Omega_{m_0} h^2} \mathrm{Mpc}}$$

<u>Basic syst for CDM + Rad</u>

$$\delta^{N'}_m = k^2 V_m + 3\Phi'$$

$$\delta^{N'}_r = \frac{4}{3} k^2 V_r + 4\Phi'$$

$$V'_m = -\mathcal{H} V_m - \phi$$

$$V'_r = -\phi - \frac{1}{4}\delta^N_r$$

$$-k^2\phi = \frac{3}{2}\mathcal{H}^2 \underbrace{\left[\Omega_m \delta^N_m + \Omega_r \delta^N_r - 3\mathcal{H}\left(\Omega_m V_m + \frac{4}{3}\Omega_r V_r\right)\right]}_{\Omega_m \delta_m^c + \Omega_r \delta_r^c}$$

Now, the evolution of the spectrum depends on a scale: k_{eq} and we must distinguish modes entering horizon in RDU or MDU

① Super-Hubble modes

we start from adiabatic I. cond.

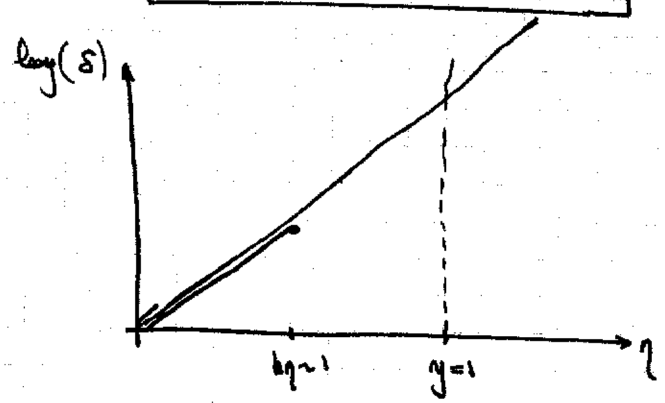
since

$$\left.\begin{aligned} V'_{rm} &= -\mathcal{H} V_{rm} - \frac{1}{4}\delta^c \\ \delta'_{rm} &= -\Delta V_{rm} \end{aligned}\right\} \quad \begin{aligned} &S = 0 \\ &V \propto 1/a \end{aligned}$$

Φ remains constant (up to a factor $\frac{9}{10}$)

$$\delta^c_m = -\frac{4}{3}\left(\frac{k}{k_{eq}}\right)^2 \frac{y^2}{1+y}\,\Phi \quad \begin{aligned} &\propto y^2 \Phi \quad (R) \quad \propto (\eta)^2 \Phi \\ &\propto y\,\Phi \quad (M) \quad \propto \eta^2 \Phi \end{aligned}$$

$$\boxed{k\eta \ll 1 \qquad \begin{aligned} &\Phi \sim c^t \\ &\delta^c_m \propto \eta^2 \end{aligned}}$$



② sub-Hubble mode

The behaviour depends on when the mode became sub-Hubble (R or M).

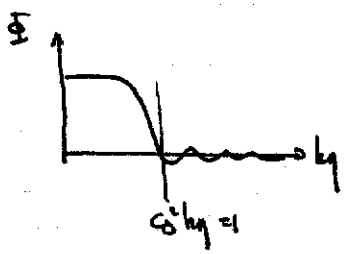
$\boxed{\text{During RDU}}$ $\mathcal{H} = 1/\eta$

$\Omega_r \gg \Omega_m$ so that $\Delta\Phi \propto \delta_r^c$

→ Φ is dictated by evol. of rad part

→ δ_m^N is then sol. of

$$\delta_m^{N''} + \mathcal{H}\delta_m^{N'} = 3\Phi'' + 3\mathcal{H}\Phi' - k^2\Phi$$



for pure rad.

$$\delta_m = A + B\ln(k\eta) + \delta^N_{\text{part}}$$

$\int G\, F(\Phi)$.

on large scale δ remain zero: m & r follow the same way

$\Phi \sim ct \quad \delta^c \propto \eta^2$

on small scale rad & matter behave differently

rad $\gg$ mat

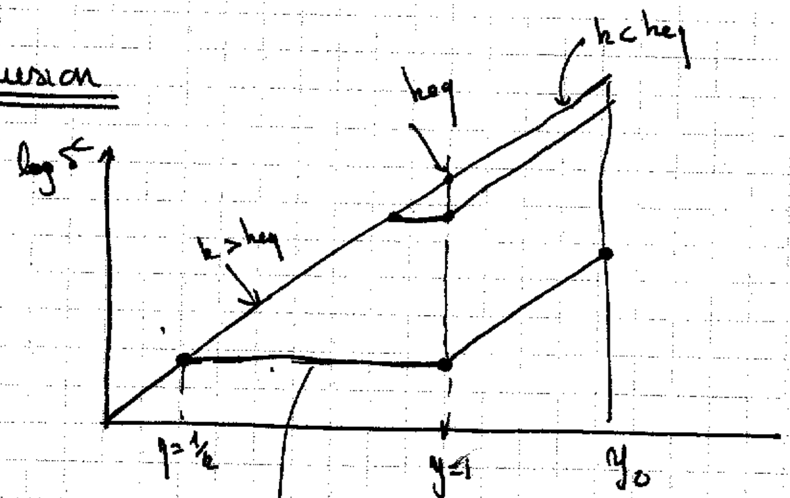
Φ decays

$$\boxed{\delta_m^c \sim \ln\eta}$$

$\boxed{\text{During RDU}}$ we neglect radiation we are back to the Newtonian equation. $D_+ \propto a$

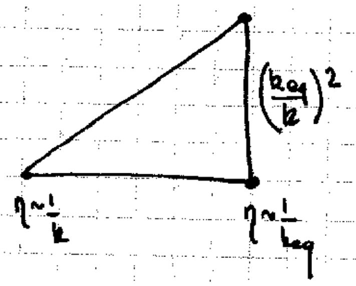
$$\boxed{\delta_m^c \propto a \sim \eta^2} \quad \text{whatever } k$$

conclusion



$k\eta = 1$

in fact a log but ~ constant

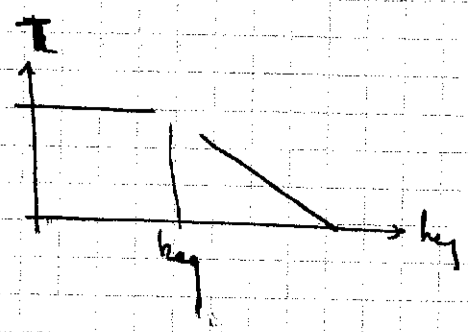


If we write

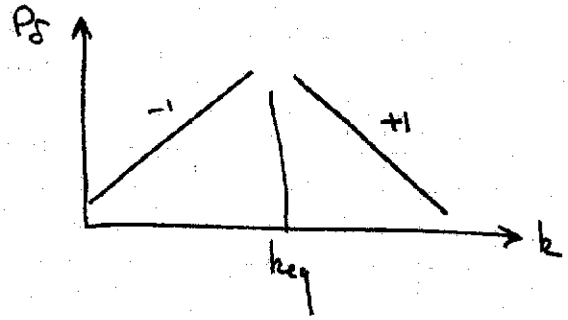
$$\delta_k = \delta_{k\,init}\, T(k)$$

then $T \sim 1 \quad k \ll k_{eq}$

$T \sim \left(\frac{k_{eq}}{k}\right)^2 \quad k \gg k_{eq}$



CI $\quad P_\Phi \sim k^{-3} \Leftrightarrow P_\delta \sim k^{-1}$



gives basic shape of power spectrum.

Sophistications

① baryons: coupled to radiation before decoupling.
in a fluid description we have to add a collision term

$$\begin{cases} V'_\gamma = -\frac{1}{4}\delta^N_\gamma - \Phi + \frac{1}{6}k^2\pi_\gamma + \boxed{a n_e \sigma_T (V_b - V_\gamma)} \\ V'_b = -\mathcal{H} V_b - \Phi - c_b^2 \delta_b^N + \boxed{\frac{4}{3}\frac{\rho_\gamma}{\rho_b} a n_e \sigma_T (V_\gamma - V_b)} \end{cases}$$

Simplification: Tight coupling approx.

$$V_b = V_\gamma$$

then Euler gives

$$\left[(1+R)V_\gamma\right]' = -\frac{1}{4}\delta^N_\gamma - (1+R)\Phi \qquad R = \frac{3}{4}\frac{\rho_b}{\rho_\gamma}$$

The b-γ eq can be reduced to $\qquad c_s^2 = \frac{1}{3}\frac{1}{1+R}$

$$\delta_\gamma^{N''} + \frac{R'}{1+R}\delta_\gamma^{N'} + k^2 c_s^2 \delta_\gamma^N = 4F(\Phi,\Psi)$$

WKB $\quad \omega_s = k c_s \gg \mathcal{H}^{-1}$

$$\begin{cases} \delta \sim \frac{1}{(1+R)^{1/4}} \begin{matrix}\cos \\ \sin\end{matrix}\, k r_s \\ r_s = \int_0^\eta c_s \, d\eta' \end{cases}$$

- These are baryonic oscillations
- φcal explanation

in power spectr. : oscillations depend by $\frac{\Omega_b}{\Omega_m}$

- CMB anisotropies

② $\underline{\pi_\gamma}$ no evol. eq. $\longrightarrow$ need kinetic approach

③ <u>NL regime</u>