Note on comparison to data

- we have computed $P(k)$ for CDM
- we observed galaxies

- $\delta_g \propto \delta_m$ $\qquad$ δ_m is a result of a stochastic process and is a random variable

$$\xi(r) = \langle \delta(x)\,\delta(x+r)\rangle \qquad \langle\rangle : \text{ensemble average.}$$

- ~~in~~dep of $|\vec{r}|$: homog. + isot. statistics. (from PC)

$$\delta(\vec{r}) = \int \frac{d^3k}{(2\pi)^{3/2}}\,\delta_{\vec{k}}\,e^{i\vec{k}.\vec{x}} \qquad \delta_k^* = \delta_{-k} \quad (\mathbb{R})$$

$$\langle \delta_k \delta_{k'}\rangle = \int \frac{d^3x}{(2\pi)^{3/2}}\frac{d^3x'}{(2\pi)^{3/2}}\,\langle \delta(x)\,\delta(x+r)\rangle\, e^{-i(k+k').x - ik'r}$$

$$\int dx \rightarrow \delta(k+k')$$

$$= \delta_D(k+k')\,P_\delta(k) \qquad \boxed{P_\delta = \int d^3\vec{r}\,\xi(r)\,e^{i\vec{k}.\vec{r}}}$$

$$\xi(r) = \int \frac{d^3k}{(2\pi)^{3/2}}\,P(k)\,e^{-i\vec{k}.\vec{r}}$$

$$= \int \frac{dk}{2\pi^2}\,k^2\,P_\delta(k)\,\frac{\sin kr}{kr}$$

V. Cosmic microwave background.

I will not derive here the whole CMB theory, which will require to work out the kinetic approach in detail. I will restrict myself to a fluid approach to grasp the main effects.

① Sachs-Wolfe Formula

consider a photon $x^\mu(\lambda)$: $k^\mu k_\mu = 0 \qquad k^\mu \nabla_\mu k^\nu = 0$

$g_{\mu\nu}$ is conformal to a static spacetime $g_{\mu\nu} = a^2(\eta)\, \hat{g}_{\mu\nu}$.

one can check that $\hat{k}^\mu = a^2 k^\mu$ satisfies

$$\hat{g}_{\mu\nu} \hat{k}^\mu \hat{k}^\nu = 0 \qquad \hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\nu = 0$$

recall
$u_0 \propto a$
$E \propto k^\mu u_\mu = k^0 u_0$
$\propto \frac{\hat{k}^0}{a^2} a \propto \frac{1}{a}$

Proof: Set $k^\mu = \alpha \hat{k}^\mu$

$$\hat{k}^\mu \nabla_\mu k^\nu = 0 \Rightarrow \hat{k}^0 \dot{\alpha} \hat{k}^\nu + \alpha \underbrace{\hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\nu}_{0} + \alpha \mathcal{H}\, \mathcal{T}^\nu_{\mu\rho} \hat{k}^\mu \hat{k}^\rho = 0$$

$$\mathcal{T}^\nu_{\mu\rho} = \hat{g}^{\nu\alpha} \left[\hat{g}_{\alpha\rho} \delta_{\nu 0} + \hat{g}_{\alpha\nu} \delta_{\mu 0} - \hat{g}_{\mu\rho} \delta_{\alpha 0} \right]$$

so $\dot{\alpha} + 2\alpha \mathcal{H} = 0 \Rightarrow \alpha = a^{-2}$

we decompose $\hat{k}^\mu = E\,(1+\Pi,\, e^i + \delta e^i)$ where E is a constant.

At the perturbation level we $(\Pi, \delta e^i)$ but because of null condition

$$\delta_{ij} e^i e^j = 1 \qquad \& \qquad e_j \delta e^j = A + \Pi - B_i e^i - \tfrac{1}{2} h_{ij} e^i e^j$$

Consider the $\nu = 0$ compnt of $\hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\nu = 0$

$$e^i \partial_i \Pi + (1+\Pi)\partial_0 \Pi + \Gamma^0_{00}(1+\Pi)^2 + 2\Gamma^0_{0i}(1+\Pi)(e^i + \delta e^i) + \Gamma^0_{ij}(e^i + \delta e^i)(e^j + \delta e^j) = 0$$

$$\underbrace{e^i \partial_i \Pi + \partial_0 \Pi} + A' + 2 D_i A\, e^i + \left[\tfrac{1}{2} h'_{ij} e^i e^j - D_{(i} B_{j)}\right] e^i e^j = 0$$

car seul $\Gamma^i_{jk} \neq 0$.

$$\boxed{\frac{d}{ds}\Pi = -A' - 2 e^i D_i A - \frac{1}{2} h'_{ij} e^i e^j + D_{(i} B_{j)} e^i e^j = 0}$$

The energy measured in a direction $\vec{e}$ will thus be

$$\frac{E_{obs}(\vec{e})}{E_{em}(x_E,\eta_E)} = \frac{(k^\mu u_\mu)_{obs}}{(k^\mu u_\mu)_{em}} \qquad \text{with } \vec{x}_E = x_0 + \vec{e}(\eta_0 - \eta_E)$$

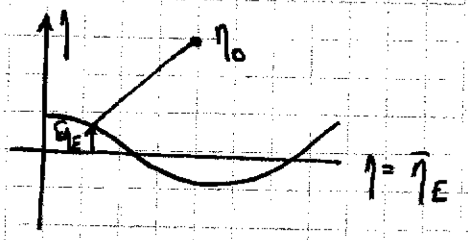
We thus work in a Born approximation in which all quantities are evaluated on a non-perturbed geodesics.

Remember that $u_\mu = a(-1-A, v_k + B_k)$. It implies that the observed temperature is related to the emission temperature

$$\frac{T_{obs}(\vec{e})}{T_{em}(x_E,\eta_E)} = \frac{a(\eta_E)}{a(\eta_0)}\left[1 + [\Psi + A - e^i(v_{bi} + B_i)]_E^0\right]$$

recombination is dictated by Thomson scattering so that collision length is prop to $\sigma_T n_e$, so that the last scattering surface can be approximated by $\{n_e = ct\}$

Note full kinetic approach not needed!



The emission time can be decomposed as $\eta_E = \bar{\eta}_E + \delta\eta_E$ since it differs because of density fluctuations.

Decomposing $T_E(x_E,\eta_E) = \bar{T}_E(\eta_E)\left[1 + \Theta_E(x_E,\eta_E)\right]$ "$\bar{T}_E(\eta_E) = \int T_E(x_E,\eta_E)d^3$"

and using that

$$\bar{T}_E(\eta_E)\, a(\eta_E) = a(\bar{\eta}_E)\, \bar{T}_E(\bar{\eta}_E)$$

idem with $T_0(\vec{e}) = \bar{T}_0(\eta_0)(1 + \Theta_0(\vec{e}))$

check that

we get that

$$\frac{\bar{T}_0(\eta_0)}{\bar{T}_E(\eta_E)}\left[1+\Theta_0(\vec{e})-\Theta_E(x_E,\eta_E)\right] = \frac{a(\eta_E)}{a(\eta_0)}\left\{1+\left[\Pi+A-e^i(v_{bi}+B_i)\right]_E^0\right\}$$

$$1+\Theta_0-\Theta_E = \underbrace{\frac{\bar{T}_E(\eta_E)a(\eta_E)}{\bar{T}_0(\eta_0)a(\eta_0)}}_{1 \text{ car } \bar{T}(\eta)a(\eta)=c^t}\left\{1+\left[\ \right]_E^0\right\}$$

so that

$$\boxed{\Theta_0(\vec{e}) = \Theta_E(x_E,\eta_E) + \left[\Pi+A-e^i(v_{bi}+B_i)\right]_E^0}$$

- This relates the emission teperature to the observed teperature.
- At this stage we cannot derive $\Theta_E(x_E,\eta_E)$. It requires to solve the full Boltzmann equation
- But it can be shown that $\eta_e[\rho_b,\rho_\gamma] \sim \eta_e[\rho_\gamma]$ so that the last scattering surface can be approximated by $\rho_\gamma = c^t$
- The Boltzmann-Stefan law $\rho_\gamma \propto T^4 \Rightarrow$

$$\left.\begin{array}{ccc} \rho_\gamma(x_E,\eta_E) & = & a\,T_E^4(x_E,\eta_E) \\ \| & & \downarrow \\ \rho_\gamma[1+\delta_\gamma(x_E,\eta_E)] & = & a\bar{T}_E(\eta_E)[1+4\,\Theta_E(x_E,\eta_E)] \end{array}\right\} \boxed{\Theta_E(x_E,\eta_E) = \frac{1}{4}\delta_\gamma(x_E,\eta_E)}$$

The last step is to express $[M]_E^o$ as

$$[M]_E^o = \int_E^o \frac{dM}{ds} = \int_E^o \underbrace{-A' - 2e^iD_iA}_{A' - 2\frac{dA}{ds}} - \frac{1}{2}h'_{ij}e^ie^j + D_{(i}B_{j)}e^ie^j \, ds$$

car $\frac{dA}{ds} = A' + e^i\partial_i A$

$$[M]_E^o = -2[A]_E^o + \int_E^o \left(A' - \frac{1}{2}h'_{ij}e^ie^j + D_iB_j\, e^ie^j\right) ds$$

Plugging in $\theta_o(\vec{e})$

$$\theta_o(\vec{e}) = \frac{1}{4}\delta\gamma(x_E, \eta_E) + 2[A]_E^o + \int_E^o \left(A' - \frac{1}{2}h'_{ij}e^ie^j + D_iB_j\, e^ie^j\right) ds$$
$$+ \left[A - e^i(v_{bi} + B_i)\right]_E^o$$

$$= \left(\frac{1}{4}\delta\gamma + 2A + e^i(v_{bi} + B_i) - A\right)(x_E, \eta_E) + \int_E^o (\,)\, ds$$

$+ f(o)$ ← monopole (can't be observed)

$$\boxed{\theta_o(\vec{e}) = \left[\frac{1}{4}\delta\gamma + A + e^i(v_{bi} + B_i)\right]_{|x_E, \eta_E} + \int_E^o \left(A' - \frac{1}{2}h'_{ij}e^ie^j + D_iB_j\, e^ie^j\right) ds}$$

with the decomposition of h_{ij}, we get

$$\Theta_0(\vec{e}) = \left[\frac{1}{4}\delta_\gamma + A + e^i(V_{bi}+B_i)\right]_{EM} + \int_E^0 (A' - C') - e^i e^j \left(D_i D_j E' + D_i E'_j - D_i B_i + \bar{E}'_{ij}\right) ds$$

use $\phi + \psi = A - C + (B - E')'$

$$\frac{1}{4}\delta_\gamma = \frac{1}{4}\delta^N_\gamma + \mathcal{H}(B-E')$$

$$A = \phi - \mathcal{H}(B-E') - (B-E')'$$

$$e^i D_i(V_b + B) = e^i D_i(V_b + E') + e^i D_i(B-E') = e^i D_i V_b + e^i D_i(B-E')$$

for the scalar term and you get

$$\Theta_0(\vec{e}) = \underbrace{\left[\frac{1}{4}\delta^N_\gamma + \Phi\right.}_{} + \left. e^i D_i V_b\right]_{em} + \int_E^0 (\phi' + \psi')\, d\eta$$

Here $\int \phi' ds = \int \phi'[x(\eta),\eta]\, d\eta$

$\underbrace{\Theta_{SW}}_{\text{denser - hotter grav. red}}$ $\underbrace{\Theta_{dop}}_{\text{red fine of baryons}}$ $\underbrace{\Theta_{ISW}}_{\text{cumulative z due to slowly facing.}}$

In conclusion

$$\Theta_0^{(S)} = \left[\frac{1}{4}\delta_\gamma^{k} + \Phi + e^i D_i V_b\right] + \int (\phi' + \psi')\, d\eta$$

$$\Theta_0^{(V)} = \left[e^i \bar{V}_{bi}\right] + \int e^i \bar{\Phi}'_i \, d\eta$$

$$\Theta_0^{(T)} = -\int e^i e^j \bar{E}'_{ij}\, d\eta$$

This relates the observed CMB anisotropies to the perturbation variables

↳ Simplified picture of CMB

(can be exploited easily numerically) :

- before decoupling : Tight coupling approx $V_\gamma = V_b = V$
- after decoupling : independent fluid - no interaction

- good description on large scale
- small scale : fails because $d \sim$ width of LSS. we cannot neglect the profile of $n_e(z)$.

→ need of a Boltzmann approach.

- **Angular power spectrum**

$$C(\theta) = \langle \Theta_0(\vec{e}_1)\, \Theta_0(\vec{e}_2) \rangle_{\vec{e}_1 \cdot \vec{e}_2 = \cos\theta}$$

$$= \sum_\ell \frac{2\ell+1}{4\pi}\, C_\ell\, P_\ell(\cos\theta)$$

If one decomposes $\Theta_0(\vec{e}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\vec{e})$

$$\Leftrightarrow \quad a_{\ell m} = \int d^2\vec{e}\, \Theta_0(\vec{x}_0, \eta_0, \vec{e})\, Y^*_{\ell m}(\vec{e})$$

$$\Rightarrow \quad (2\ell+1)\, C_\ell = \sum_m \langle a_{\ell m} a^*_{\ell m} \rangle$$

decompose $\Theta(x_0, \eta_0, \vec{e})$ as

$$\Theta(x_0, \eta_0, e) = \int \frac{d^3k}{(2\pi)^{3/2}}\, \hat{\Theta}(\eta_0, \vec{k}, \vec{e})$$ the phase $e^{i\vec{k}\vec{x}_0}$ absorbed $\hat{\Theta}$

it follows that

$$a_{\ell m} \overset{@ x_0, \eta_0}{=} \int d^2\vec{e}\, \frac{d^3k}{(2\pi)^{3/2}}\, \hat{\Theta}(\eta_0, \vec{k}, \vec{e})\, Y^*_{\ell m}(\vec{e})$$

We got that

$$\hat{\Theta}(\eta_0, \vec{k}, \vec{e}) = \left[\Theta_{SW}(\vec{k}) + ik\mu\, \hat{V}_b(\vec{k})\right] e^{ik\mu\Delta\eta_E} + \int (\hat{\phi}' + \hat{\psi}')\, e^{ik\mu\Delta\eta}\, d\eta$$

$\Delta\eta = \eta_0 - \bar{\eta}$

each term is a random variable.

$$X = X(k, \eta)\, a_{\vec{k}} \quad \text{with } \langle a_k a^*_{k'} \rangle = \delta(\vec{k} - \vec{k}')$$

$$\hat{\Theta}(\eta_0, k, \mu) = \left[\hat{\Theta}_{SW}(k) + V_b(k)\, \partial_{\Delta\eta_E}\right] e^{ik\mu\Delta\eta_E}$$
$$+ \int (\hat{\phi}' + \hat{\psi}')\, e^{ik\mu\Delta\eta}\, d\eta$$

Now use the devlpt:

$$e^{i\vec{p}\cdot\vec{x}} = 4\pi \sum_{\ell m} i^\ell j_\ell(pr) Y^*_{\ell m}(\hat{p}) Y_{\ell m}(\hat{x})$$

and you get

$$\hat{\Theta}(\eta_0, k, \mu) = 4\pi \sum_{\ell m} \hat{\Theta}_\ell^{(s)}(k) Y^*_{\ell m}(\hat{k}) Y_{\ell m}(e)$$

with

$$\boxed{\hat{\Theta}_\ell^{(s)}(k) = \hat{\Theta}_{SW}(k)\, j_\ell(k\Delta\eta_E) + \frac{\hat{V}_b}{k} j'_\ell(k\Delta\eta_E) + \int (\hat{\Phi}' + \hat{\Psi}')\, j_\ell(k\Delta\eta)\, d\eta}$$

from the def of C_ℓ ($\langle a_{\ell m} a^*_{\ell m}\rangle$) one gets, since $\langle a_k a_h\rangle$ dep

$$(2\ell+1) C_\ell^S = \sum_m \int d^2e_1\, d^2e_2 \frac{d^3k}{(2\pi)^3} \Theta(\eta_0, k, \mu_1)\, \Theta^*(\eta_0, k, \mu_2)\, Y_{\ell m}(e_1) Y^*_{\ell m}(e_2)$$

$$= \frac{2}{\pi} \int k^2 dk \sum \Theta_{\ell_1}^{(s)}(k)\, \Theta_{\ell_2}^{(s)*}(k) \int d^2k\, d^2e_1\, d^2e_2\, Y_{\ell_1 m_1}(\hat{k}_1)\, Y^*_{\ell_2 m_2}(\hat{k})\, Y_{\ell m}(e_1)\, Y^*_{\ell_1 m_1}(e_1)\, Y_{\ell_2 m_2}(e_2)\, Y^*_{\ell m}(e_2)$$

$\int \hat{k} \to \delta_{\ell_1\ell_2}\,\delta_{m_1 m_2}$

$\int e_1\, e_2 \to \delta_{\ell_1\ell}\,\delta_{m_1 m}$ et $\delta_{\ell\ell_2}\,\delta_{m m_2}$

$$\boxed{C_\ell^{(S)} = \frac{2}{\pi} \int |\hat{\Theta}_\ell^{(s)}(k)|^2\, k^2 dk}$$

- <u>petits ℓ - scalaire</u>

$$\Theta_{SW} = \left[\frac{1}{4}\,\delta^N_\gamma + \Phi\right]$$

$$S=0 \rightarrow \frac{1}{4}\delta^N_\gamma = \frac{1}{3}\delta^N_m = -\frac{2}{3}\Phi$$

$$\boxed{\Theta_{SW} = \frac{1}{3}\Phi}$$

Yesterday

$$\delta^N = \underbrace{\delta^c}_{k^2\eta^2} - 2(\Phi + \underbrace{\cancel{y\Phi}}_{0})$$

$$\left.\begin{array}{l}\delta^N = -2\Phi \\ \delta^N_r = \frac{4}{3}\,\delta^N \quad (y \gg 1)\end{array}\right\}\ \delta^N_r = -\frac{8}{3}\Phi$$

$$C_\ell = \frac{2}{\pi}\int j_\ell^2\left[k(\eta_0 - \eta_{LSS})\right]\frac{1}{9}\,P_\Phi^2(k)\,\frac{dk}{k}$$

$$\Phi = \frac{3\mathcal{R}}{5}$$

$$P_\mathcal{R} = \frac{25}{4}A_S^2(k) = \frac{25}{4}A_S^2(k_0)\,k^{n_s-1}$$

$$C_\ell \simeq \pi A_S^2(k_0)\left[\frac{\pi}{2}\,\frac{\Gamma(3-n_s)}{2^{3-n_s}}\,\frac{\Gamma\left(\ell + \frac{n_s-1}{2}\right)}{\Gamma^2\left(2-\frac{n_s}{2}\right)\Gamma\left(\ell + \frac{5-n_s}{2}\right)}\right]$$

with $\eta_{LSS} \ll \eta_0 \quad k \sim \frac{1}{\eta_0}$

$$\boxed{\ell(\ell+1)\,C_\ell^S = \pi A_S^2(k_0)\,\ell^{n_s-1}}$$

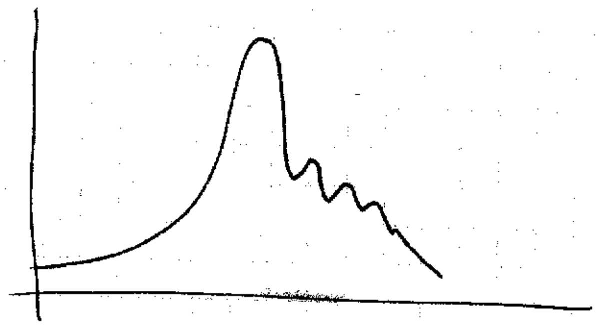
. Large Scale : plateau SW (1er pic)

+ ISW

- medium . BAO pic

. échelle $r_s(\eta_{LS}) / D_A(z_{LS}) \sim \frac{2\theta_0}{\sqrt{\Omega}}$

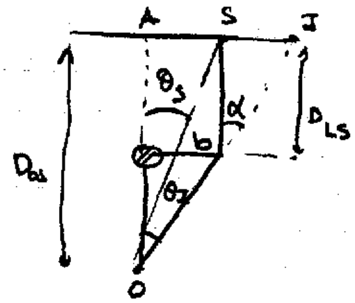
- silk damping



- Note Boltzmann
- polarisation

VI. weak lensing

- Any massive body deflects light (see R.G. course)



$AI = AS + SI$

$AI \to \theta_I D_{os}$, $AS \to \theta_S D_{os}$, $SI \to D_{LS}\,\alpha$ — angular distances

$$\boxed{\theta_I = \theta_S + \frac{D_{LS}}{D_{OS}}\alpha}$$

$\theta_I \equiv \theta$

$$\alpha = \frac{4GM}{bc^2} = 2\frac{R_{sch}}{b} \quad \text{and} \quad b = D_{OL}\,\theta$$

$$\boxed{\theta = \theta_S + 2\frac{R_{sch}}{\mathcal{D}}\frac{1}{\theta} \qquad \mathcal{D} = \frac{D_{OS}D_{OL}}{D_{LS}}} \quad (*)$$

- Einstein radius

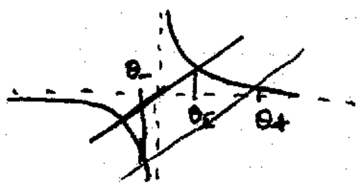
by symmetry $\theta_S = 0 \Rightarrow$ ring of angular diameter

$$\boxed{\theta_E = \sqrt{\frac{4GM}{c^2}\frac{D_{LS}}{D_{OS}D_{OL}}} \quad ; \quad R_E = D_{OL}\,\theta_E}$$

sources such that $\theta < \theta_E \to$ strong distortion (giant arc)

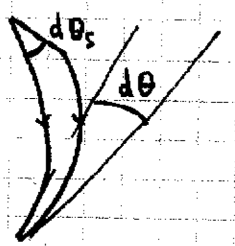
- multiple images

$$(*) \to \theta_S = \theta - \frac{\theta_E^2}{\theta}$$



This equation has 2 sol. $\theta_\pm = \frac{1}{2}\left(\theta_S \pm \sqrt{\theta_S^2 + 4\theta_E^2}\right)$ on each side of the source, one inside and the other outside the Einstein ring. and $\Delta\theta \geqslant 2\theta_E$

lensing is not associated to absorption or emission process, the surface brightness is conserved

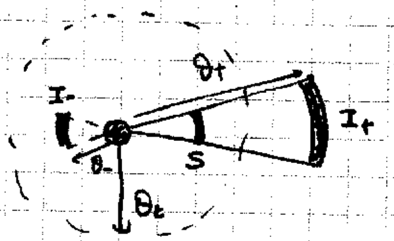


$$I(\theta) = I_s\left[\theta_s(\theta)\right]$$

For a spherical lens the surface el! $\propto \sin\theta\, d\theta \sim \theta\, d\theta$ so that the amplification is just given by the ratio of surface

$$\mu = \frac{\theta\, d\theta}{\theta_s\, d\theta_s}$$

plugging $\theta_s(\theta)$ we get $\mu_\pm = \left[1 - \left(\frac{\theta_E}{\theta_\pm}\right)^4\right]^{-1}$



$\theta_+ > \theta_-$

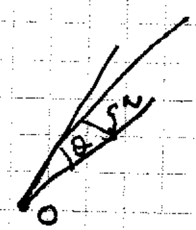
when $\theta_s \to 0$ $\theta_\pm \to \theta_E$ $\mu \to \infty$ divergence only for point sources!

Sachs Equation:

consider the propagation of a geodesic bundle of light in a perturbed spacetime.

$$X^\mu(\lambda) = \bar{X}^\mu(\lambda) + \xi^\mu(\lambda)$$

ref. geodesics

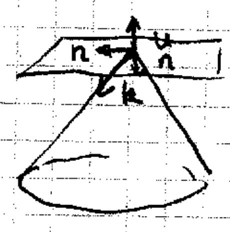


λ: affine para along $\bar{X}$

In o, consider the basis $\{k^\mu, n_1^\mu, n_2^\mu, u^\mu\}$

$$\frac{d\bar{x}^\mu}{d\lambda}$$

$k^2 = 0 \qquad u^2 = -1 \qquad n_a n_b = \delta_{ab} \qquad n u = n k = 0$



we construct the basis on any $\bar{x}(\lambda)$ by parallel transport

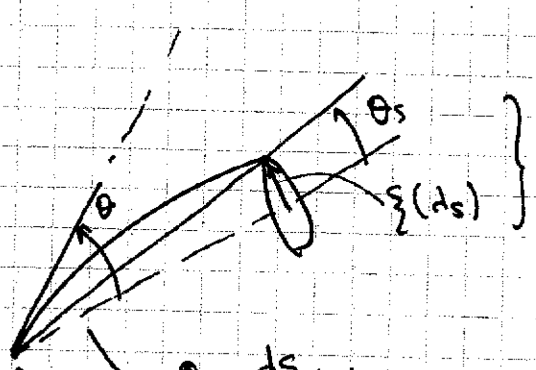
We decompose ξ^μ as

$$\xi^\mu = \xi_0 k^\mu + \Sigma_a \xi_a n_a^\mu + A\, u^\mu$$

note that

$$0 = \frac{d}{d\lambda}\left(\xi^\mu k_\mu\right) = -\frac{d}{d\lambda}(AE)$$

- Since $AE = 0$ in O $\rightarrow$ we can set $A = 0 \ \forall \lambda$
- we can also set $\xi_0 = 0$ because it describes the same geodesics.



$$\theta_S = \frac{\xi(\lambda_S)}{D_A(\lambda_S)}$$

$\theta = \frac{d\xi}{d\lambda}(0)$

The prop. eq. of ξ_μ derives from the geodesic equation

$$\frac{D}{D\lambda} \equiv k^\mu \nabla_\mu$$

$$\frac{D^2}{d\lambda^2}\xi^\mu = R^\mu{}_{\nu\alpha\beta}\, k^\nu k^\alpha \xi^\beta$$

we plug the dec. of ξ to get

(*) $\frac{d^2}{d\lambda^2}\xi = \mathcal{R}\cdot\xi$ $\qquad \mathcal{R} \equiv \mathcal{R}_a{}^b = R^\mu{}_{\nu\alpha\beta}\, k^\nu k^\alpha n^\mu_a n^\beta_b$

optical matrices $\qquad = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu \delta_a^b + C\ldots$

it is a linear eq. so that

(**) $\xi(\lambda) = \mathcal{D}\cdot\frac{d\xi}{d\lambda}\Big|_0$

(*) gives [recall $\xi(0)=0$] $\qquad \boxed{\frac{d^2}{d\lambda^2}\mathcal{D} = \mathcal{R}\cdot\mathcal{D}}$

$$\boxed{\mathcal{D}(0) = 0 \;/\; \frac{d}{d\lambda}\mathcal{D}(0) = I}$$

· (**) $\rightarrow$ $D_A(\lambda)\,\xi(\lambda) = \mathcal{D}\cdot\frac{d\xi}{d\lambda} = \mathcal{D}\theta$

$$\boxed{\xi^a = \frac{\mathcal{D}^a{}_b}{D_A}\,\theta^b}$$

no. hyp on optical matrix

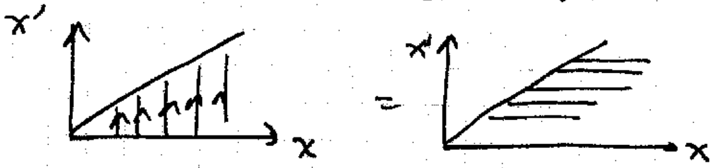
Sachs 1961

Now if we have $n(\chi)\,d\chi$ source distribution

$$\kappa(\vec\theta) = \int n(\chi)\,\kappa(\vec\theta,\chi)\,d\chi$$

$$\boxed{\begin{aligned}\kappa(\vec\theta) &= \frac{3}{2} H_0^2 \Omega_{m_0} \int_0^{\chi_{max}} g(\chi)\, f_K(\chi)\, \frac{\delta[f_K(\chi)\theta, \chi]}{a(\chi)}\, d\chi \\ g(\chi) &= \int_\chi^{\chi_H} n(\chi')\, \frac{f_K(\chi-\chi')}{f_K(\chi')}\end{aligned}}$$

Note $\int_0^\infty d\chi \int_0^\chi d\chi' = \int_0^\infty d\chi' \int_\chi^\infty d\chi$ and then $\chi \leftrightarrow \chi'$



if δ G. field κ also

$$\kappa(\vec\theta) = \sum \kappa_{lm} Y_{lm}(\vec\theta)$$

but in small angle (flat sky)

$$\kappa(\vec\theta) = \int \kappa(\ell)\, e^{i\vec\ell\vec\theta}\, \frac{d^2\ell}{2\pi}$$

$$\langle \kappa(\ell)\,\kappa^*(\ell')\rangle = P_\kappa(\ell)\,\delta^{(2)}(\vec\ell - \vec\ell')$$

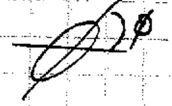
$$\boxed{P_\kappa(\ell) = \frac{9 H_0^4 \Omega_m^2}{4} \int \left[\frac{g(\chi)}{a(\chi)}\right]^2 P_\delta\left[\frac{\ell}{f_K(\chi)}, \chi\right] d\chi}$$

How do we measure that?

$$\left.\begin{aligned}\gamma_1 &= \frac{\partial_{11} - \partial_{22}}{2} \int \psi \\ \gamma_2 &= \partial_{12} \int \psi \\ \kappa &= \Delta_2 \int \psi .\end{aligned}\right\} \text{ not indep.} \qquad \text{(E mode)}$$

$$\left.\begin{aligned}\underbrace{\gamma(\vec\theta)}_{\gamma_1 + i\gamma_2} = \frac{1}{\pi} \int \underbrace{\mathcal{K}(\theta - \theta')}_{\frac{\theta_2^2 - \theta_1^2 - 2i\theta_1\theta_2}{\theta^4}} \kappa(\theta')\, d^2\theta'\end{aligned}\right\} \longrightarrow \boxed{P_\gamma = P_\kappa}$$

observe a galaxie $\longrightarrow$ ellipse $\quad \varepsilon = \varepsilon_1 + i\varepsilon_2 = \frac{1-r}{1+r} e^{2i\phi} \quad r = \frac{b}{a}$



$$\underbrace{\langle \varepsilon \rangle}_{\text{estimator of } \gamma} = \left\langle \frac{\gamma}{1-\kappa} \right\rangle = \langle \gamma \rangle \quad (\kappa \ll 1)$$

+ See Y. Mellier talk for all obs. pb.

+ Say smthg on NL regime.

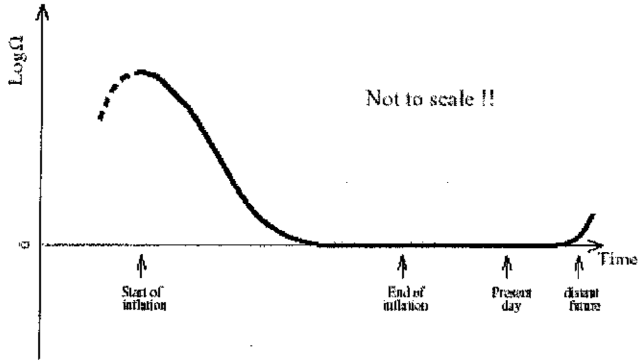
INTRODUCTION TO INFLATION

INFLATION: BASICS

The origin of the flatness problem is clear: $\Omega_K = -\frac{K}{a^2H^2}$

During the cosmological evolution *aH* decreases

Assume there is a primordial phase during which aH increases $|\tilde{\Omega}_K| \to 0$



Inflation = primordial phase of accelerated expansion

$aH \nearrow \qquad \ddot{a} > 0 \qquad \rho + 3P < 0$

FLATNESS PROBLEM

If the inflation phase is long enough then WK is brought very close to 0, explaining the flatness of our universe.

E-fold

We define the number of e-fold of inflation by $$N = \ln\left(\frac{a_{\mathrm{final}}}{a_{\mathrm{init}}}\right)$$

Solving the problem

If H is almost constant during this phase then

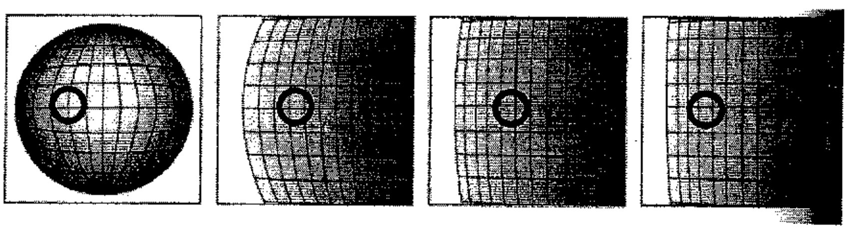
$\leq 10^{-60}$

$$\frac{\Omega_K(t_f)}{\Omega_K(t_i)} = \left(\frac{a_{\mathrm{final}}}{a_{\mathrm{init}}}\right)^2 = \exp(-\,2N)$$

Order 1

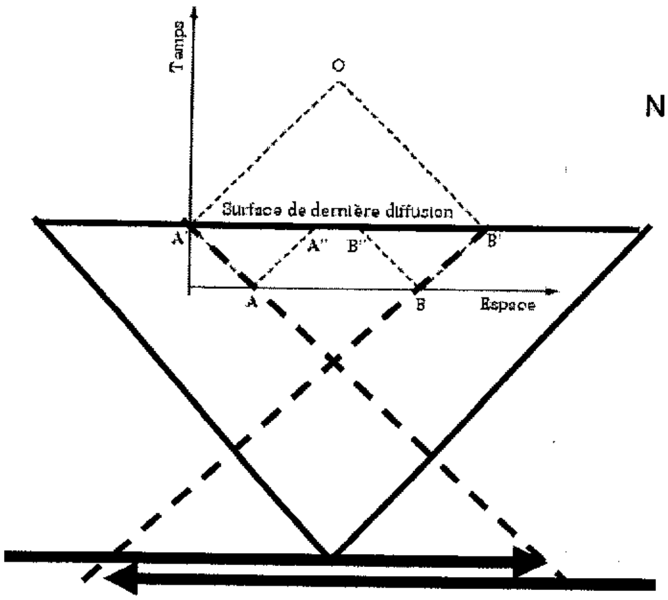
Thus, we need at least

$$N > 60 - 70$$

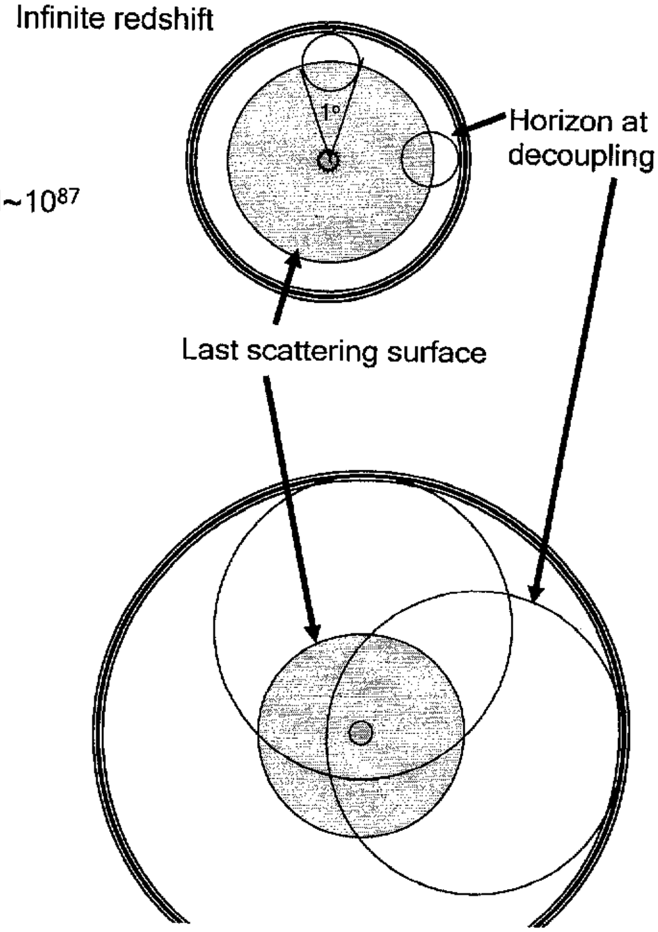


HORIZON PROBLEM

Without Inflation



$N \sim 10^{87}$



With Inflation

IMPLEMENTATION

We need matter satisfying ρ + 3P <0

Cosmological constant: de Sitter phase
exponential expansion

Scalar field:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi); \quad P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

so that

$$\rho + 3P = -2V(\phi) \times \left(1 - \frac{\dot{\phi}^2}{V}\right)$$

The field must be in slow-roll

SLOW-ROLL CONDITIONS

The 2 slow-roll conditions

$$\dot{\phi}^2 \ll V \qquad \ddot{\phi}^2 \ll 3H\dot{\phi}$$

Evolution equations   obtained from the Friedmann and Klain-Gordon equations

$$H^2 \simeq \frac{8\pi G}{3}V \qquad 3H\dot{\phi} \simeq -V'$$

The expansion will be quasi-exponential

Validity conditions

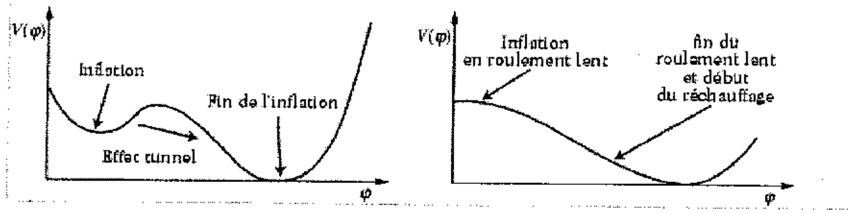
The slow-roll conditions are fulfilled if

$$(V'/V)^2 \ll 24\pi G, \qquad V''/V \ll 24\pi G$$

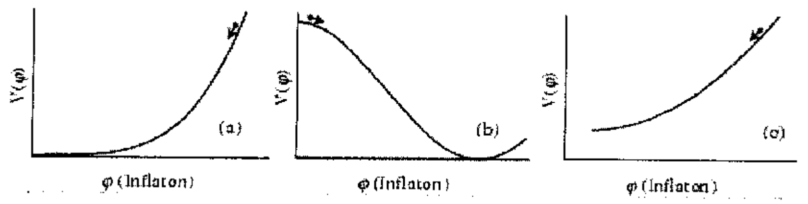
The potential must be flat

THE ZOO OF POTENTIALS

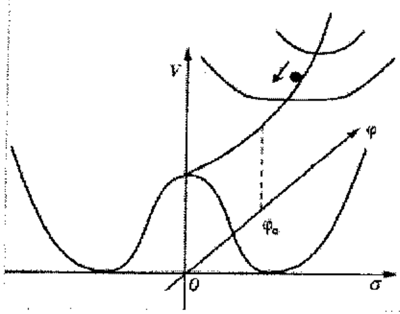
Old and new inflation



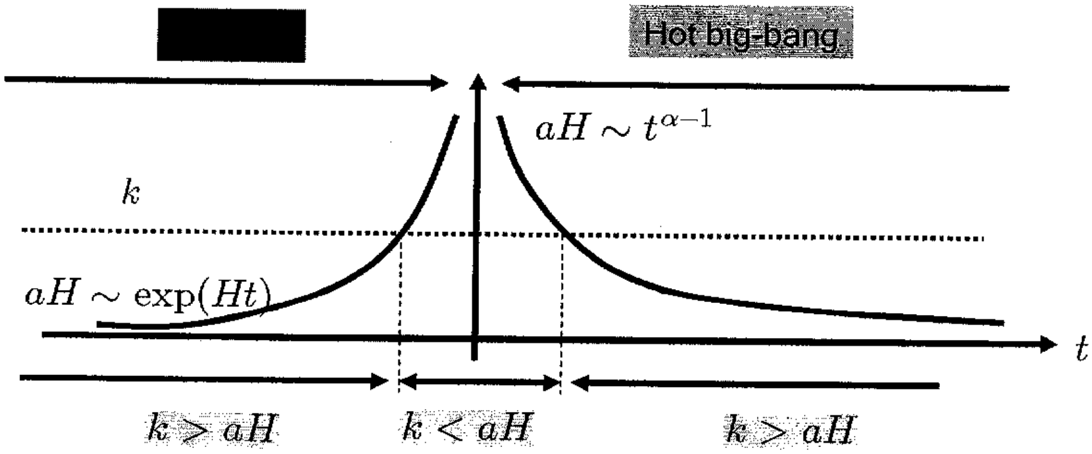
Large/small field



Hybrid inflation



ORIGIN OF FLUCTUATION: MODE EVOLUTION



The modes
* start sub-Hubble
* exit the Hubble radius during inflation
* enters the Hubble radius during the matter or radiation era

Test field in de Sitter

To grasp the origin of perturbation, consider a test massless scalar field in de Sitter

Its equation of evolution is the Klein-Gordon equation

$$\ddot{\chi} + 3H\dot{\chi} + \frac{k^2}{a^2}\chi = 0$$

$k > aH$ Friction negligible - harmonic oscillator.

$k < aH$ Gradient negligible - constant mode

Setting v=aχ, and using η=–exp(-Ht)/H the Klein-Gordon equation becomes

$$v'' + \left(k^2 - \frac{2}{\eta^2}\right)\chi = 0$$

Its solution is

$$v = A(k)\left(1 + \frac{1}{ik\eta}\right)\exp(-\,ik\eta) + B(k)c.c.$$

INITIAL CONDITION

$k\eta \gg 1$ The curvature is negligible

We can quantify as in Minkowski space

Initial conditions

$$v \to \frac{\exp(-ik\eta)}{\sqrt{2k}} \quad k\eta \to -\infty$$

Solution

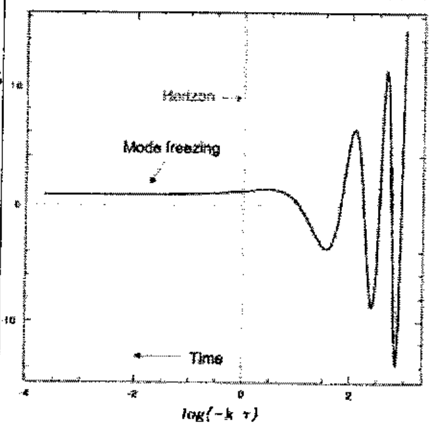
$$\chi = \frac{H\eta}{\sqrt{2k}}\left(1 + \frac{1}{ik\eta}\right)\exp(-ik\eta)$$

Frozen super-Hubble fluctuations

$$\chi_k \sim \frac{H}{\sqrt{2k^3}}$$

10^{-28} cm

amplitude ~$10^{-5} M_4$



Plane wave on small scales

$$\chi_k \underset{k\eta \gg 1}{\to} \frac{e^{-ik\eta}}{\sqrt{2k}}$$

10^{-33} cm

SOPHISTICATIONS

The inflaton also fluctuates

But its fluctuations are coupled to gravity

What is the variable to quantify?

Heuristically

Because of the initial fluctuations in the scalar field, inflation last more or less longer from one Hubble patch to another.

The curvature perturbation from one patch to the other is

$$\delta\mathcal{R} \sim H\delta t \sim H\delta\phi/\dot{\phi}$$

The fluctuation in the inflaton are of order

$$\delta\phi/\dot{\phi} \sim H/2\pi$$

$$\frac{\delta\rho}{\rho} \approx \delta\mathcal{R} \approx \frac{H^2}{2\pi\dot{\phi}}$$

Gravity waves

One produces GW with an amplitude of order $h \sim \frac{H}{2\pi M_p}$

SCALE OF INFLATION

On super-Hubble scales, the CMB anisotropy is of order

$$\frac{\delta T}{T} \sim \frac{1}{3}$$

The amplitude of the CMB temperature anisotropies allow to calibrate the spectrum

$$\frac{\delta T}{T} \sim 2 \times 10^{-6}\ K \longrightarrow \left(\frac{V}{\varepsilon}\right)^{1/4} \sim 5.7 \times 10^{16}\ \mathrm{GeV}$$

From WMAP data, we can infer the bound

$$\varepsilon < 0.032 \qquad \frac{H_{inf}}{M_p} < 1.4 \times 10^{-5}$$

$$\varepsilon = (V'/V)^2/16\pi G$$

GENERIC PREDICTIONS

1- universe is *flat*: =1
2- classical inhomogeneities are erased
   Justification of the *cosmological principle*

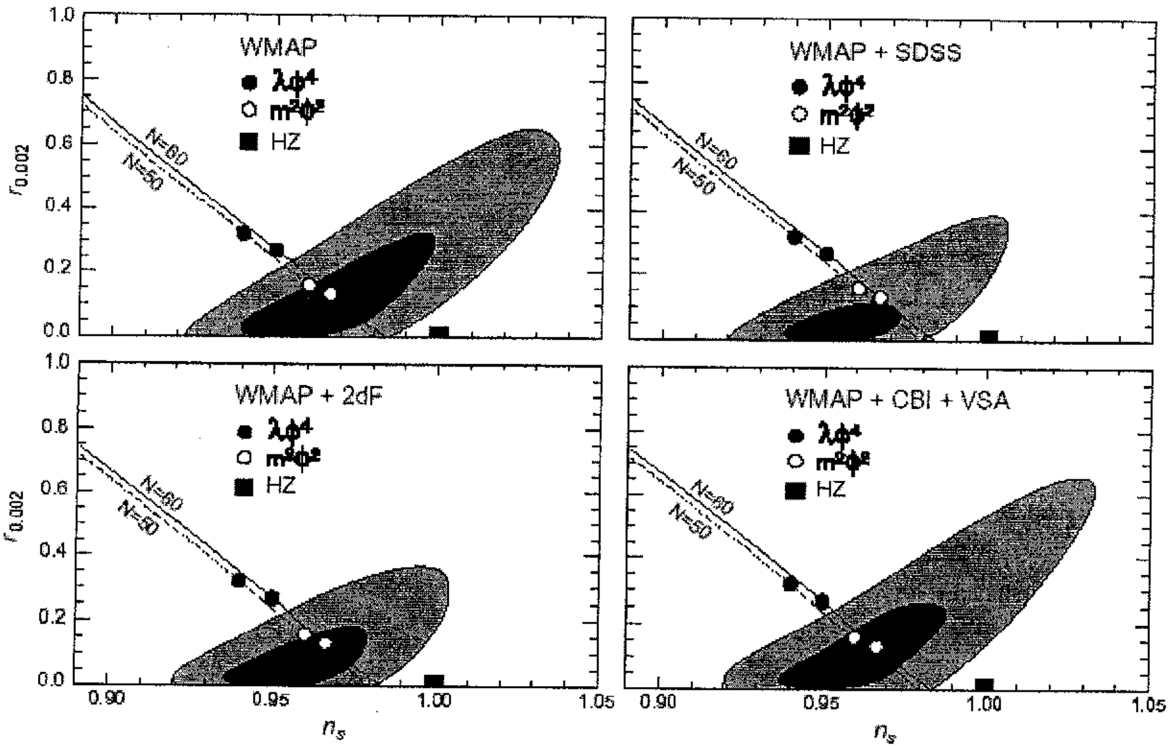
3- metric perturbations are Gaussian, almost scale invariant
   origin of the Harrison-Zel'dovich spectrum
4- no vector perturbation
5- tensor perturbations are almost scale invariant

6- there exists a *consistency relation* between T/S, n_t et n_s

extensions

- multi-field
   non-gaussianity, isocurvature modes

WMAP CONSTRAINTS

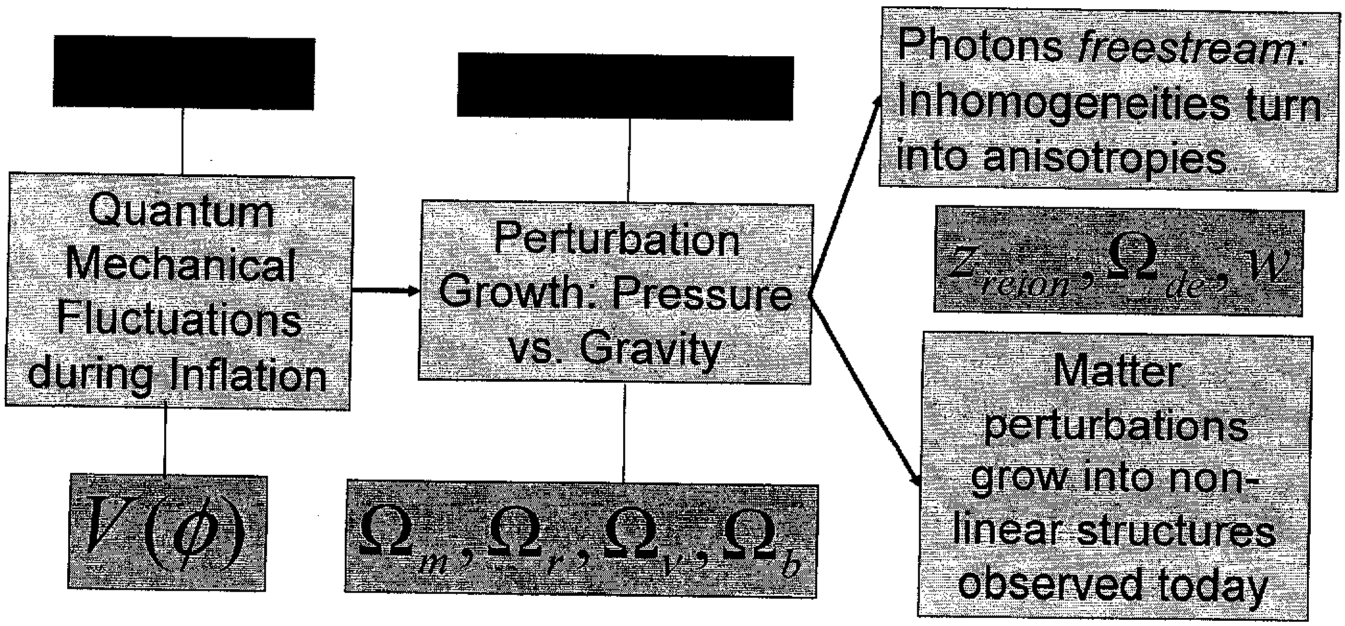


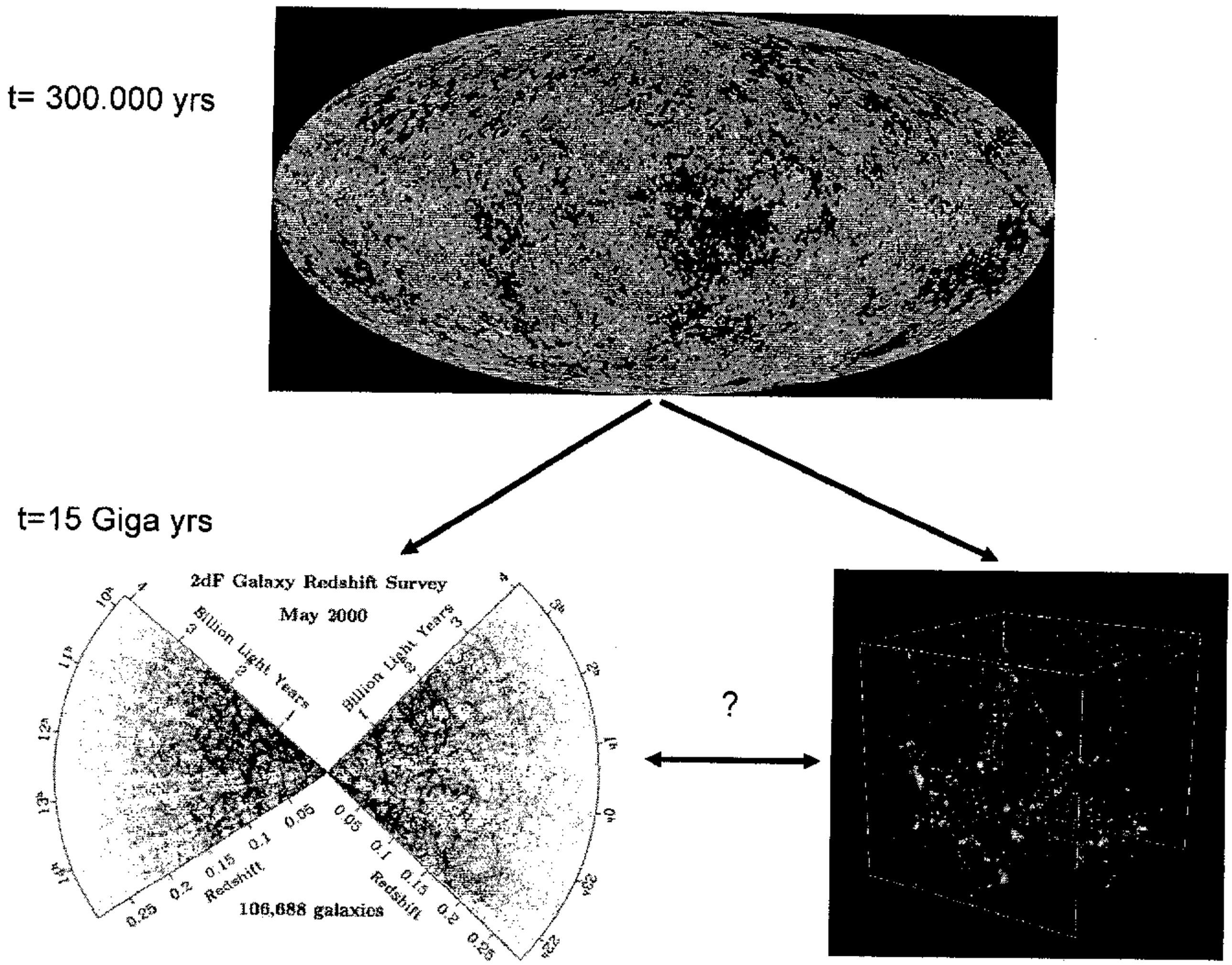
PART VII: CONCLUSIONS

Main topics

- *Status of the model*
- *Open issues and questions*

COHERENT PICTURE OF STRUCTURE FORMATION




t= 300.000 yrs
t=15 Giga yrs
2dF Galaxy Redshift Survey
May 2000
Billion Light Years
Billion Light Years
Redshift
Redshift
106,688 galaxies
?

Excellent standard model

The standard model is based on
- relativity
- electromagnetism
- weak interaction
- GUT - SUSY
- QFT
- string - quantum gravity

It explains
- the dynamics of the universe
- the origin of LSS

Tests
- numerous and successful
- more test of fundamental physics are needed
- what physics behind the parameters

Questions
- dark matter?
- dark energy ?
- inflaton ?
- constants ?

Laboratory
- constraints on extension of the standard models
- Model building