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Stability of solutions of the Einstein equations

Solutions of the Einstein-Vacuum equations tending to the Minkowski spacetime at infinity

Talk:

- Setting of the problem
- Questions - Solutions
- Solution by D. Christodoulou and S. Klainerman in 'The global nonlinear stability of the Minkowski space'
- Solution with more general initial data (B)
- Structures and ideas used in the proof

Solutions of the **Einstein-Vacuum (EV) equations**:

$$R_{\mu\nu} = 0 . \tag{1}$$

Spacetimes (M, g), where M is a four-dimensional, oriented, differentiable manifold and g is a Lorentzian metric obeying (1).

Is there any non-trivial, asymptotically flat initial data whose maximal development is complete?

Works by many authors:

Y. Choquet-Bruhat, R. Geroch, R. Penrose, S. Hawking,
D. Christodoulou, S. Klainerman, H. Lindblad,
I. Rodnianski, F. Nicolò, H. Friedrich
and more.

• **Y. Choquet-Bruhat (1952)**:
'Théorème d'existence pour certain systèmes d'equations aux dérivées partielles nonlinéaires':

- Cauchy problem for the Einstein equations,
- **local** in time, existence and uniqueness of solutions,
- reducing the Einstein equations to wave equations, introducing harmonic (or wave) coordinates.

Choquet-Bruhat proved the well-posedeness of the local Cauchy problem in these coordinates.

• **Y. Choquet-Bruhat and R. Geroch**, stating the **existence of a unique maximal future development for each given initial data set**.

$\Rightarrow$ **Question:** Is this maximal development **complete?**

• **R. Penrose**
gave the answer in his **incompleteness theorem**:

Consider **initial data**, where the initial Cauchy hypersurface H is non-compact and complete. If H contains a **closed trapped surface** S, the boundary of a compact domain in H, then the corresponding **maximal future development is incomplete**.

Closed trapped surface S**:** An infinitesimal displacement of S in M towards the future along the outgoing null geodesic congruence results in a pointwise decrease of the area element.

D. Christodoulou
A **closed trapped surface can form in the evolution, starting from initial data not containing any such surfaces.**

• **Theorem of Penrose** and its **extensions by S. Hawking and R. Penrose**

$\Rightarrow$ Question, formulated at the beginning.

Answer

Joint work of **D. Christodoulou** and **S. Klainerman** ([CK], 1993),
'The global nonlinear stability of the Minkowski space'.

Every asymptotically flat initial data which is globally close to the trivial data gives rise to a solution which is a complete spacetime tending to the Minkowski spacetime at infinity along any geodesic.

- No additional restriction on the data.
- No use of a preferred system of coordinates
- Relied on the invariant formulation of the E-V equations.
- **Precise description** of the **asymptotic behaviour at null infinity**.

H. Lindblad and I. Rodnianski:
'Global existence for the EV equations in wave coordinates'

- **Global stability** of **Minkowski** space for the **EV** equations in **harmonic (wave) coordinate gauge**

- for the set of **restricted data coinciding with the Schwarzschild solution in the neighbourhood of spacelike infinity**.

- Result contradicts beliefs that wave coordinates are 'unstable in the large' and provides an alternative approach to the stability problem

- Result is less precise as far as the asymptotic behaviour is concerned

- Focus on giving a solution in a physically interesting wave coordinate gauge

H. Lindblad and I. Rodnianski:
'The global stability of Minkowski space-time in harmonic gauge'

- Stability for **EV scalar field** equations
- Less decay of 'tail of the metric'

New Result [B]

More general asymptotically flat initial data with

less decay and

one less derivative than in [CK]

yielding a solution which is a complete spacetime, tending to the Minkowski spacetime at infinity along any geodesic.

$\Rightarrow$ Have **finite energy**

R. Bartnik's formulation of the positive mass theorem applies.

R. Bartnik

Positive mass theorem:

If we are given an asymptotically flat, connected, complete, 3-dimensional manifold (H, g) with

$$\| g_{ij} - \delta_{ij} \|_{2,2,-\frac{1}{2}} \leq \epsilon$$

and integrable scalar curvature $R \geq 0$.

Then the mass

$$m_{ADM} \geq 0$$

and $m_{ADM} = 0$ if and only if (H, g) is globally flat.

Initial data set: A triplet $(H, \bar{g}, k)$ with $(H, \bar{g})$ being a three-dimensional complete Riemannian manifold and k a two-covariant symmetric tensorfield on H, satisfying the **constraint equations**:

$$\begin{aligned} \nabla^i \, k_{ij} \; - \; \nabla_j \; trk \quad &= \quad 0 \\ R - \; \mid k \mid^2 \; + \; (trk)^2 \quad &= \quad 0 \, . \end{aligned}$$

Evolution equations:

$$
\begin{aligned}
\frac{\partial \bar{g}_{ij}}{\partial t} &= 2\Phi k_{ij} \\
\frac{\partial k_{ij}}{\partial t} &= \nabla_i \nabla_j \Phi \ - \ (R_{ij} \ + \ k_{ij} \ trk \ - \ 2k_{im}k_j^m)\Phi
\end{aligned}
$$

Constraint equations:

$$
\begin{aligned}
\nabla^i k_{ij} \ - \ \nabla_j \ trk &= 0 \\
R \ + \ (trk)^2 \ - \ |k|^2 &= 0
\end{aligned}
$$

A general **asymptotically flat initial data set** $(H, \bar{g}, k)$:
An initial data set such that

- the complement of a compact set in H is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$
- and there exists a coordinate system (x^1, x^2, x^3) in this complement relative to which the metric components

$$\begin{array}{ccc} \bar{g}_{ij} & \to & \delta_{ij} \\ k_{ij} & \to & 0 \end{array}$$

sufficiently rapidly as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$.

In [CK], consider the following
strongly asymptotically flat initial data set:
An initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighbourhood of infinity such that,
as $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$:

$$\bar{g}_{ij} = (1 + \frac{2M}{r}) \, \delta_{ij} + o_4 \, (r^{-\frac{3}{2}}) \tag{2}$$

$$k_{ij} = o_3 \, (r^{-\frac{5}{2}}) \, , \tag{3}$$

where M denotes the mass.

The strongly asymptotically flat initial data set has to satisfy a certain smallness assumption.

They introduce

$$
\begin{aligned}
Q(x_{(0)}, b) \quad = \quad & \sup_H \left(b^{-2} \, (d_0^2 \, + \, b^2)^3 \;\; | \, Ric \, |^2 \;\right) \\
& + b^{-3} \, \Big(\int_H \sum_{l=0}^{3} (d_0^2 \, + \, b^2)^{l+1} \;\; | \, \nabla^l k \, |^2 \\
& \quad + \int_H \sum_{l=0}^{1} (d_0^2 \, + \, b^2)^{l+3} \;\; | \, \nabla^l B \, |^2 \;\; \Big)
\end{aligned}
\tag{4}
$$

$d_0(x) = d(x_{(0)}, x)$: the Riemannian geodesic distance between the point x and a given point $x_{(0)}$ on H.

b : a positive constant.

∇^l : the l-covariant derivatives.

B (Bach tensor): the following symmetric, traceless 2-tensor

$$B_{ij} \; = \; \epsilon_j{}^{ab} \; \nabla_a \; (R_{ib} \; - \; \frac{1}{4} \; g_{ib} \; R) \; .$$

Global Smallness Assumption:

A strongly asymptotically flat initial data set is said to satisfy the **global smallness assumption** if the metric $\bar{g}$ is complete and there exists a sufficiently small positive ϵ such that

$$\inf_{x_{(0)} \in H, b \geq 0} \; Q(x_{(0)}, b) \; < \; \epsilon \; . \tag{5}$$

One version of the **main theorem in [CK]**:

Theorem 1. *Any strongly asymptotically flat, maximal, initial data set that satisfies the global smallness assumption (5), leads to a* **unique, globally hyperbolic, smooth and geodesically complete solution** *of the E-V equations foliated by a normal, maximal time foliation. This development is* **globally asymptotically flat**.

Proof for more general initial data in the following sense [B]:

We consider an asymptotically flat initial data set $(H_0, \bar{g}, k)$ for which there exists a coordinate system (x^1, x^2, x^3) in a neighbourhood of infinity such that with $r = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} \to \infty$, it is:

$$\bar{g}_{ij} = \delta_{ij} + o_3 \, (r^{-\frac{1}{2}}) \tag{6}$$

$$k_{ij} = o_2 \, (r^{-\frac{3}{2}}) \, . \tag{7}$$

Global smallness assumption:

$$
\begin{aligned}
Q(a,0) \quad = \quad & a^{-1} \Big(\int_{H_0} \Big(\mid k \mid^2 + (a^2 + d_0^2) \mid \nabla k \mid^2 \\
& \qquad + (a^2 + d_0^2)^2 \mid \nabla^2 k \mid^2 \Big) \, d\mu_{\bar{g}} \\
& + \int_{H_0} \Big((a^2 + d_0^2) \mid Ric \mid^2 \\
& \qquad + (a^2 + d_0^2)^2 \mid \nabla Ric \mid^2 \Big) \, d\mu_{\bar{g}} \Big) \\
< \quad & \epsilon \, . \qquad (8)
\end{aligned}
$$

a : positive scale factor.

Main Theorem [B]:

Theorem 2. *Any asymptotically flat, maximal initial data set satisfying the global smallness assumption, leads to a* **unique, globally hyperbolic, smooth and geodesically complete solution of the EV-equations**, *foliated by the level sets of a maximal time function. This development is* **globally asymptotically flat**.

- **Invariant formulation of the E-V equations**

- No use of a preferred coordinate system

- **Asymptotic behaviour** given in a **precise** way

- **Appropriate foliation of the spacetime**

- **Bianchi identity** for the **Weyl tensor $\mathbf{W}$**, having all the symmetry properties of the curvature tensor, in addition is traceless and satisfies the Bianchi equations

$$D_{[\epsilon} W_{\alpha\beta]\gamma\delta} = 0 \ .$$

- **Bel-Robinson tensor**:
  Associate to a Weyl field a tensorial quadratic form:
  - a 4-covariant tensorfield
  - being fully symmetric and trace-free.

$$Q_{\alpha\beta\gamma\delta} = \frac{1}{2} \left(W_{\alpha\rho\gamma\sigma} \ W_{\beta\ \ \delta}^{\ \rho\ \ \sigma} + \ {}^*W_{\alpha\rho\gamma\sigma} \ \ {}^*W_{\beta\ \ \delta}^{\ \rho\ \ \sigma} \right) \ .$$

  It satisfies the following positivity condition:

$$Q \ (X_1, \ X_2, \ X_3, \ X_4) \ \geq \ 0$$

  X_1, X_2, X_3, X_4 future-directed timelike vectors. For W satisfying the Bianchi equations:

$$D^\alpha \ Q_{\alpha\beta\gamma\delta} = 0 \ .$$

- A **general spacetime** has **no symmetries**, that is, the conformal isometry group is trivial.

  $\Rightarrow$ Use **Minkowski** as **background**.

  • **Spacetime** $\to$ **Minkowski** as $t \to \infty$. Minkowski having a large **conformal isometry group**. Define in the limit an action of a subgroup.

  • **Extend this action backwards in time** up to the initial hypersurface $\to$ obtain an **action** of the said subgroup **globally**.

  • Apply **Noether's principle (in a generalized way)**
  $\Rightarrow$ **Background vacuum solution**

- **Solution constructed** as the corresponding development of the initial data

Constructing a set of **quantities whose growth can be controlled in terms of the quantities themselves**.

Main structures of the spacetime used in the proof

Comparison argument with the **Minkowski** spacetime:

- **Canonical spacelike foliation**
- **Null structure**
- **Conformal group structure**

The (t, u) **foliations** of the spacetime define a codimension 2 foliation by 2-surfaces

$$S_{t,u} \ = \ H_t \ \cap \ C_u \ , \tag{9}$$

the intersection between H_t (foliation by t) and a u-null-hypersurface C_u (foliation by u).

Foliation by **time function** t with **lapse function**

$$\Phi \ (t, x) \ \ = \ \ (- \ \left\langle Dt, Dt \right\rangle)^{-\frac{1}{2}}$$

with D denoting the covariant differentiation on the spacetime M, and second fundamental form k.

Foliation by **optical function** u, a solution of the **Eikonal equation**:

$$g^{\alpha\beta} \ \frac{\partial u}{\partial x^{\alpha}} \ \frac{\partial u}{\partial x^{\beta}} \ = \ 0 \ .$$

Crucial Foliation:

The **asymptotic behaviour** of the **curvature tensor** R and the **Hessian** of t and u can only be **fully described** by decomposing them into **components tangent to** $S_{t,u}$.

Achieve this
$\Rightarrow$ by introducing
null pairs consisting of 2 future-directed null vectors e_4 and e_3 orthogonal to $S_{t,u}$ with e_4 tangent to C_u and

$$\left\langle e_4, e_3 \right\rangle \;=\; -\,2\ . \tag{10}$$

A null pair together with an orthonormal frame e_1, e_2 on $S_{t,u}$ forms a **null frame**.

The **null decomposition** of a tensor relative to a null frame e_4, e_3, e_2, e_1 is obtained by taking **contractions** with the vectorfields e_4, e_3.

Also define

$$\tau_-^2 \;:=\; 1 \;+\; u^2\ .$$

Null decomposition of the Riemann curvature tensor of an E-V spacetime:

$$R_{A3B3} = \underline{\alpha}_{AB} \quad (11)$$
$$R_{A334} = 2\ \underline{\beta}_A \quad (12)$$
$$R_{3434} = 4\ \rho \quad (13)$$
$$^*R_{3434} = 4\ \sigma \quad (14)$$
$$R_{A434} = 2\ \beta_A \quad (15)$$
$$R_{A4B4} = \alpha_{AB} \quad (16)$$

with

α, $\underline{\alpha}$: S-tangent, symmetric, traceless tensors
β, $\underline{\beta}$: S-tangent 1-forms
ρ, σ : scalars .

Obtained the following **properties for the null components of the curvature tensor** on each hypersurface H_t:

$$
\begin{aligned}
&\int_{H_t} \tau_-^2 \mid \underline{\alpha} \mid^2 \; + \; \int_{H_t} r^2 \mid \underline{\beta} \mid^2 \\
&+ \; \int_{H_t} r^2 \mid \rho \mid^2 \; + \; \int_{H_t} r^2 \mid \sigma \mid^2 \\
&+ \; \int_{H_t} r^2 \mid \beta \mid^2 \; + \; \int_{H_t} r^2 \mid \alpha \mid^2 \\
&+ \; \text{''} \int_{H_t} \text{ first derivatives ''} \qquad \leq \; \epsilon
\end{aligned}
$$

Components decaying like

$$
\begin{aligned}
\underline{\alpha} \quad &= \quad O\ (r^{-1}\ \tau_-^{-\frac{3}{2}}) \\
\underline{\beta} \quad &= \quad O\ (r^{-2}\ \tau_-^{-\frac{1}{2}}) \\
\rho,\ \sigma,\ \alpha,\ \beta \quad &= \quad o\ (r^{-\frac{5}{2}})
\end{aligned}
$$

Whereas in [CK] the null components have the **decay properties**:

$$
\begin{aligned}
\underline{\alpha} &= O\,(r^{-1}\ \tau_-^{-\frac{5}{2}}) \\
\underline{\beta} &= O\,(r^{-2}\ \tau_-^{-\frac{3}{2}}) \\
\rho &= O\,(r^{-3}) \\
\sigma &= O\,(r^{-3}\ \tau_-^{-\frac{1}{2}}) \\
\alpha,\ \beta &= o\,(r^{-\frac{7}{2}})
\end{aligned}
$$

[B]:

Control

One derivative of curvature (Ricci) in H. For Ric including corresponding weights according to (8):

$$\mathbf{Ric} \ \in \ \mathbf{W^{1,2}(H)}$$

Trace Lemma gives

$\Rightarrow$ Gauss curvature in the leaves of the u-foliation S:

$$\mathbf{K} \ \in \ \mathbf{L^4(S)}$$

[CK]:

Control

Two derivatives of curvature in $L^2(H)$. For Ric including weights as in (4):

$$\mathbf{Ric} \ \in \ \mathbf{L^\infty(H)}$$

$\Rightarrow$ also $K \ \in \ L^\infty(S)$

We also **control**

Two derivatives of the **second fundamental form** k

$\Rightarrow$ by Sobolev inequalities

$$\mathbf{k} \in \mathbf{L}^{\infty}(\mathbf{H})$$

$\Rightarrow$

$$(\text{components of } k) \in L^{\infty}(S)$$

Main Steps of the Proof

One Large Bootstrap - More Small Bootstraps

1. Estimate an appropriate quantity $\mathbf{Q_1(W)}$, integral over H_t involving Bel-Robinson tensor Q of Lie derivative of W.
   At time t: can be calculated by its value at $t = 0$ and an integral from 0 to t, both controlled.

2. **Weyl tensor** $\mathbf{W}$ verifying the Bianchi equations, controlled through $Q_1(W)$ by a **comparison argument**.

3. **Geometric quantities** determined from **curvature assumptions** using elliptic estimates, evolution equations, Sobolev inequalities, etc.

$\mathcal{Q}_1(W)$ is given as follows:

$$\mathcal{Q}_1(W) \quad = \quad Q_0 \; + \; Q_1$$

with Q_0 and Q_1 being the subsequent integrals,
and for $\bar{K} = K + T$,

$$\begin{aligned} Q_0(t) \quad &= \quad \int_{H_t} Q \; (W) \; (\bar{K}, T, T, T) \\ Q_1(t) \quad &= \quad \int_{H_t} Q \; (\hat{\mathcal{L}}_S W) \; (\bar{K}, T, T, T) \\ &\quad + \; \int_{H_t} Q \; (\hat{\mathcal{L}}_T W) \; (\bar{K}, \bar{K}, T, T) \; . \end{aligned}$$

Obtain the estimates of the angular derivatives of our curvature components directly from the Bianchi equations.

In the work [CK]: introduced rotational vectorfields to obtain the corresponding angular derivatives.

Here, no rotational vectorfields are needed.

Bootstrapping

Local $\Rightarrow$ Global

- **Bootstrap assumptions**: Initial assumptions on the main geometric quantities of the 2 foliations, i.e. $\{H_t\}$ and $\{C_u\}$.

- **Local existence theorem**

  $\rightarrow$ local existence

- **Bootstrap argument** together with evolution equations

  $\rightarrow$ global existence

To Point 3 - Estimating Geometric Quantities

Fundamental form χ of S relative to C:

$$\chi(X, Y) \;=\; g(D_X L, Y)$$

for any pair of vectors $X, Y \in T_p S$ and L generating vector-field of C.

Estimating χ from the propagation equation

$$\frac{\partial tr\chi}{\partial s} \;+\; \frac{1}{2}\,(tr\chi)^2 + \;|\,\hat{\chi}\,|^2 = \;0 \tag{17}$$

and the elliptic system on each section S_s of C

$$div\ \hat{\chi}_a \;=\; \frac{1}{2}\,d_a tr\chi \;+\; f_a \tag{18}$$

where

f_a involves curvature.

Assuming estimates for the spacetime **curvature** on the right hand side of (18)

$\Rightarrow$

To obtain estimates for the **quantities controlling the geometry** of C as described by its foliation $\{S_s\}$.

Closing the **bootstrap** arguments.

Energy and Linear Momentum

Energy and **linear momentum**

are

well-defined and **conserved**.

Definitions (ADM) in a hypersurface H of the spacetime:

Let $S_r = \{|x| = r\}$ be the coordinate sphere of radius r and dS_j the Euclidean oriented area element of S_r.

- Total Energy

$$E = \frac{1}{4} \lim_{r\to\infty} \int_{S_r} \sum_{i,j} (\partial_i \bar{g}_{ij} - \partial_j \bar{g}_{ii})\, dS_j \ ,$$

- Linear Momentum

$$P^i = -\frac{1}{2} \lim_{r\to\infty} \int_{S_r} (k_{ij} - \bar{g}_{ij}\ trk)\, dS_j \ ,$$

Open Question:

What is the **sharp critera** for **non-trivial asymptotically flat initial data sets** to give **rise to a maximal development that is complete?**