Geometric discretisation of GR

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- Only relations between several points carry information

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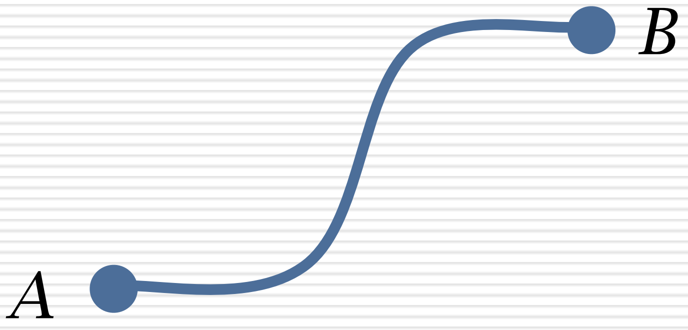

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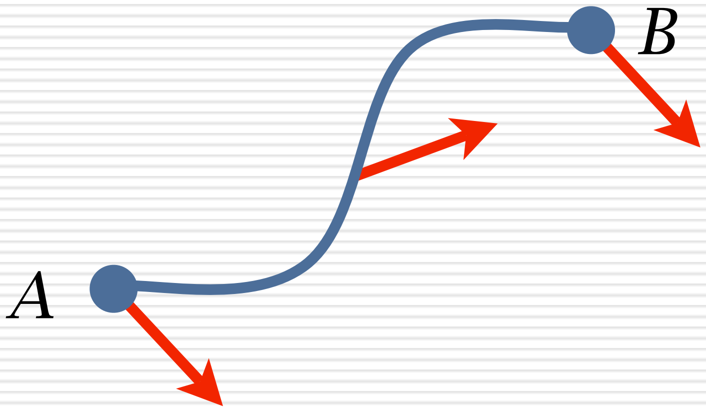

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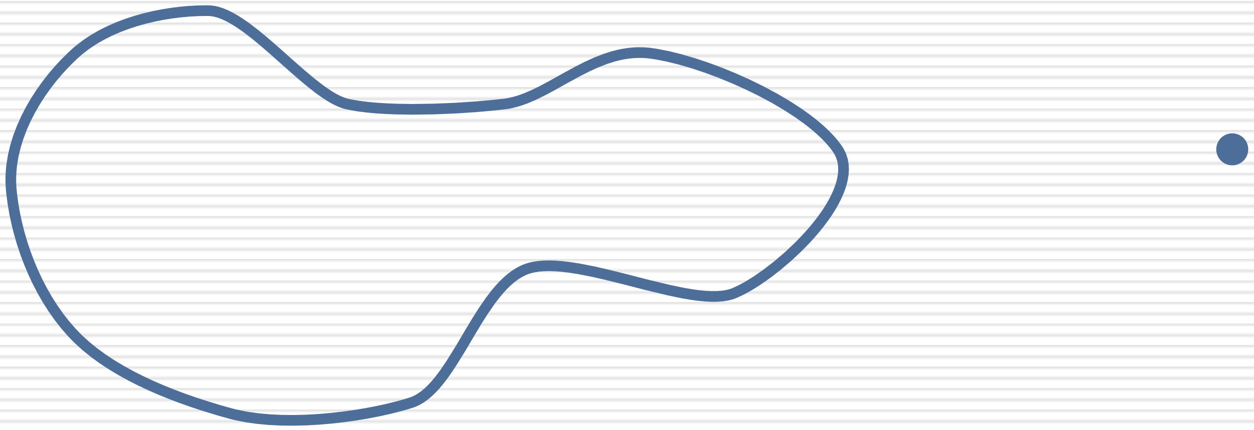

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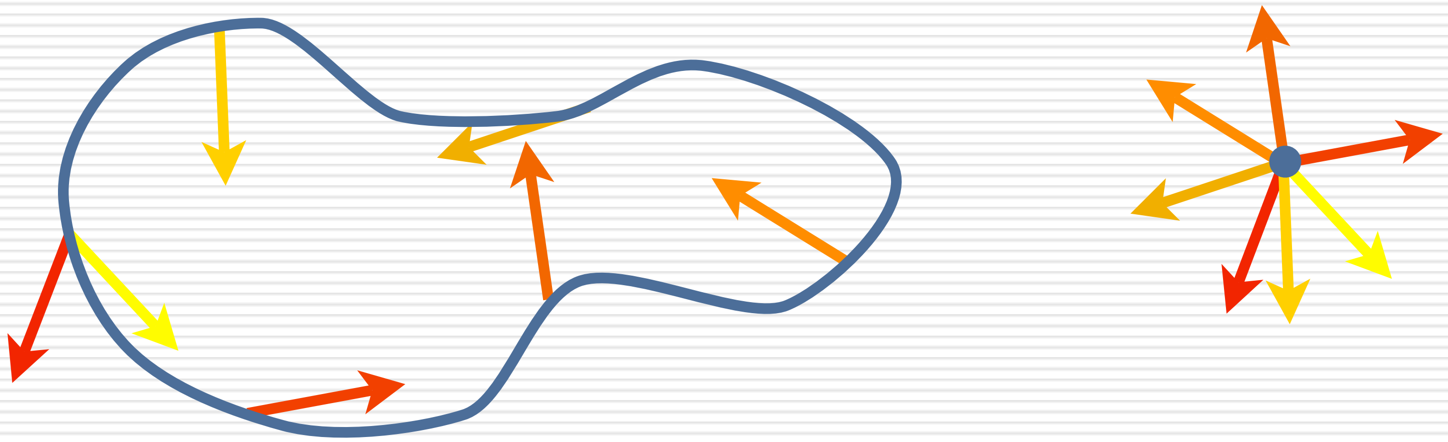

➡ Gauss curvature of a surface

Consequence

- localise objects not only on points
  - FD: tensor components as grid functions
- but also on lines, surfaces, volumes according to meaning

Maxwell theory

1-form: electric field

$$E = E_x dx + E_y dy + E_z dz$$

Maxwell theory

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line = 1-dim sub-mf

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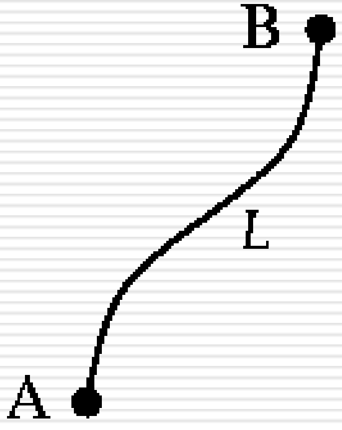
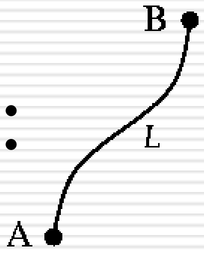


$$E = E_x dx + E_y dy + E_z dz$$

$E :$  $\mapsto \int_L E$

Work done on unit charge along L
(voltage between A and B)

Maxwell theory

2-form: magnetic induction

$$B = B_x \, dy \wedge dz + B_y \, dz \wedge dx + B_z \, dx \wedge dy$$

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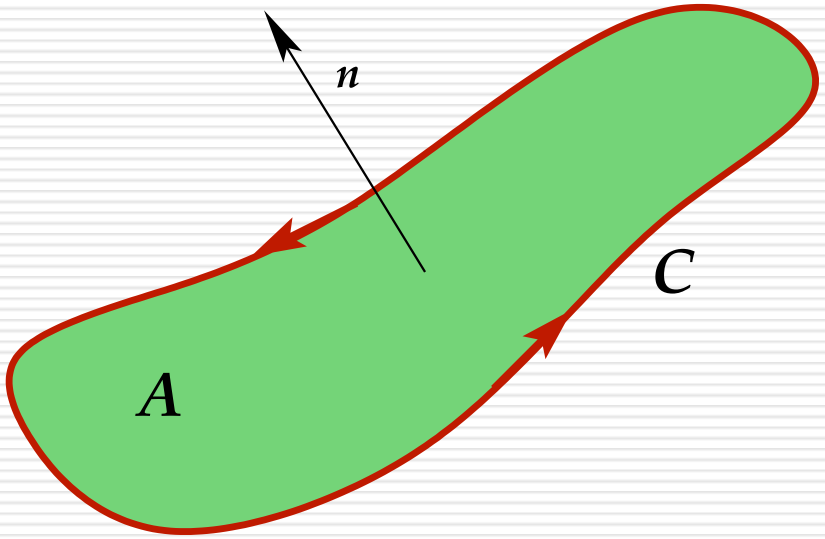

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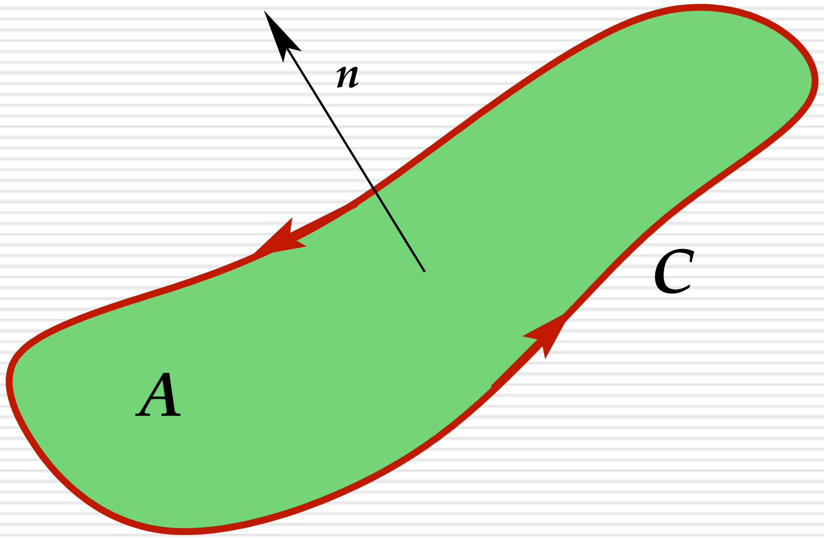
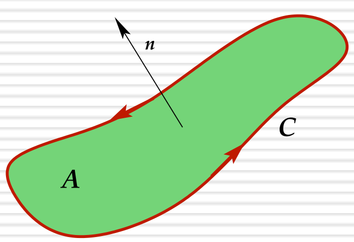


$B :$ 

$\mapsto \int_A B$

magnetic flux through surface A

Maxwell theory

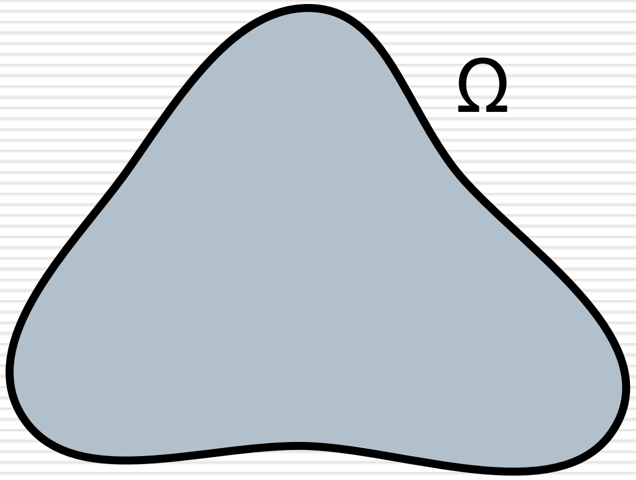
Maxwell equations:

$$\dot{B} = \mathrm{rot} E \iff \frac{d}{dt}\int_A B = \int_{\partial A} E$$

$$\dot{E} = -\mathrm{rot} B \iff \frac{d}{dt}\int_A E = -\int_{\partial A} B$$

Discretisation of Maxwell

$$\dot{B} = \mathrm{rot} E$$



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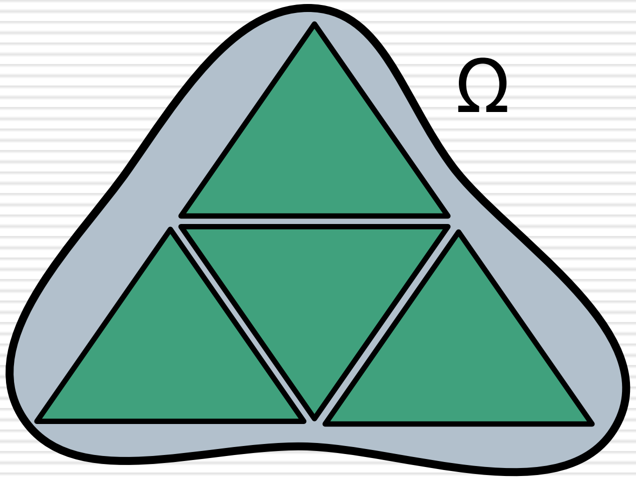

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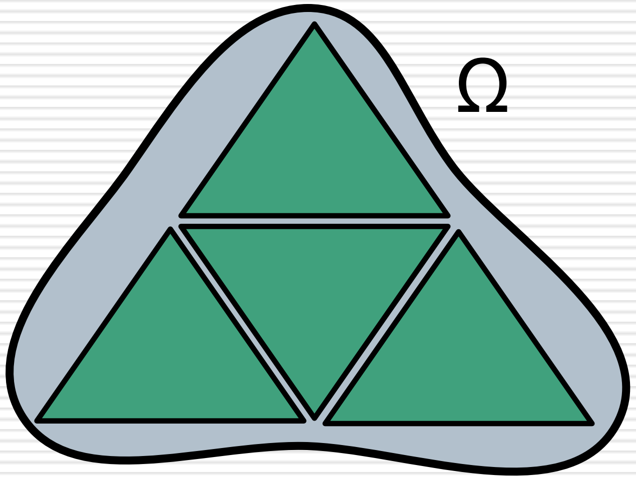

$$\frac{d}{dt} \sum_i \int_{\Delta_i} B = \sum_i \int_{\partial \Delta_i} E$$

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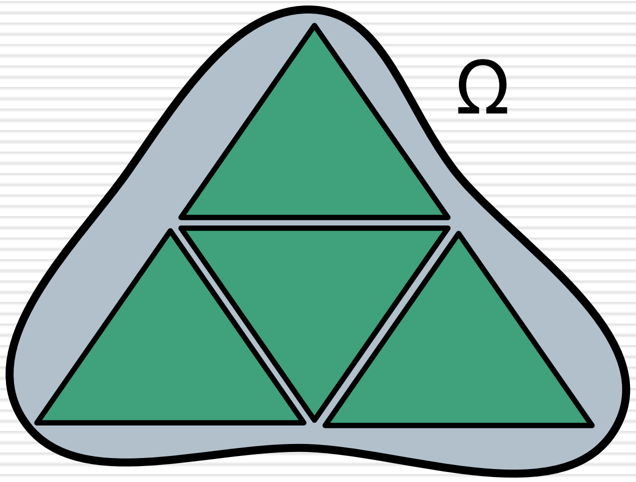

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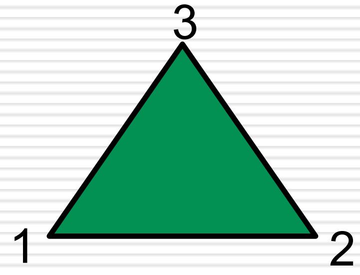

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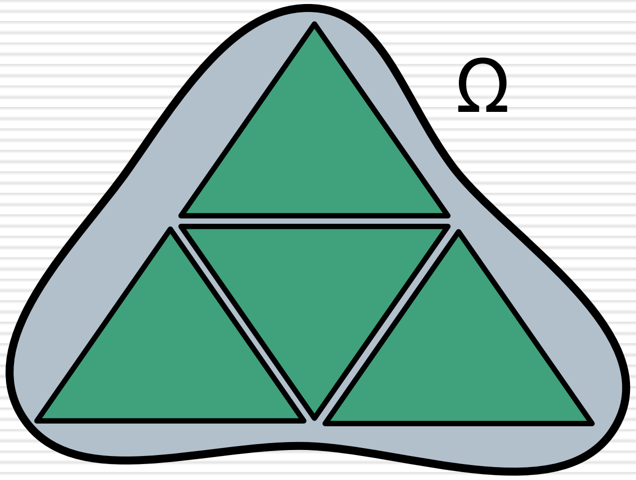

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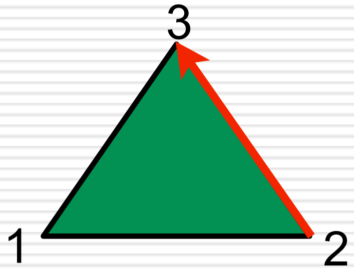

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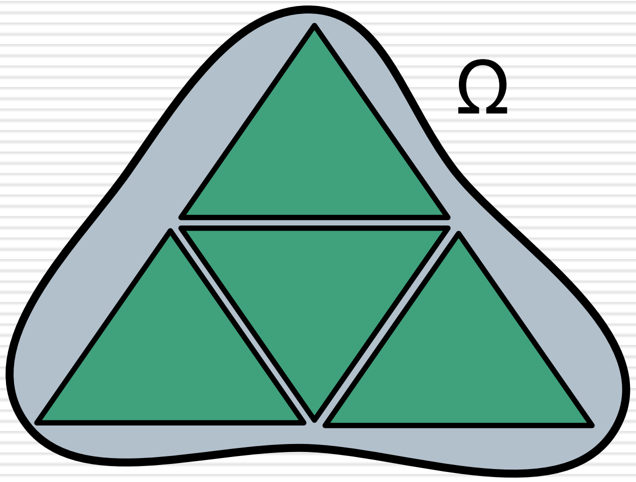

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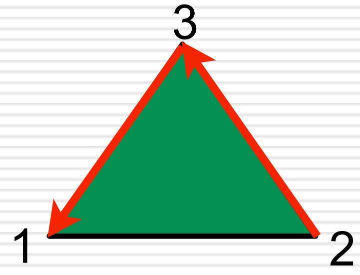

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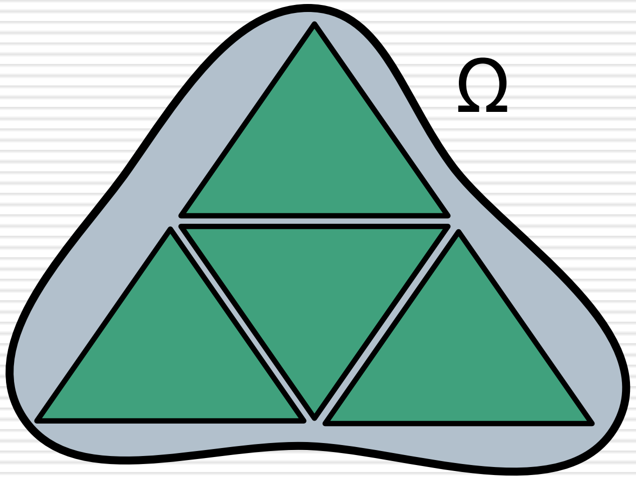

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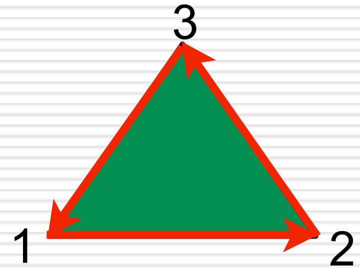

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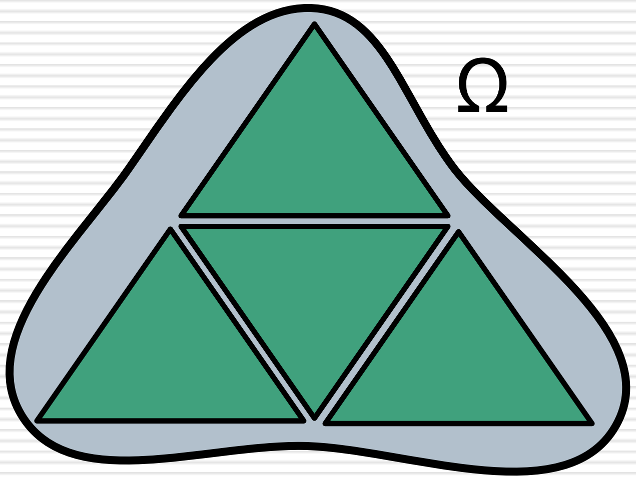

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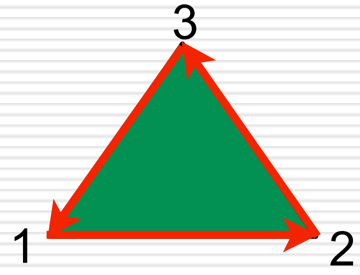

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$$\dot{b} = e_1 + e_2 + e_3$$

GR with differential forms

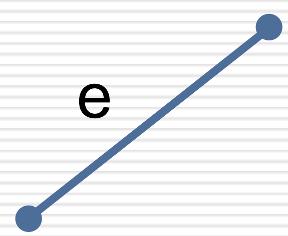
tetrad

$$(\boldsymbol{\theta}^0, \boldsymbol{\theta}^1, \boldsymbol{\theta}^2, \boldsymbol{\theta}^3)$$

GR with differential forms

tetrad

$$\boldsymbol{\theta}^i$$



$$l^2 = \eta_{ik}\boldsymbol{\theta}^i[e]\boldsymbol{\theta}^k[e]$$

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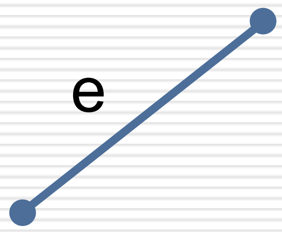

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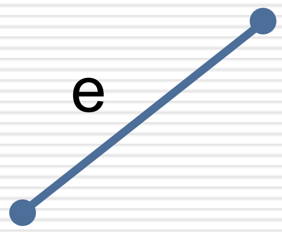

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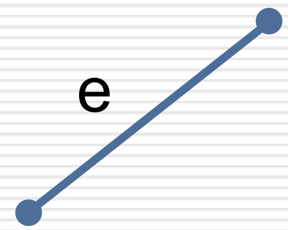

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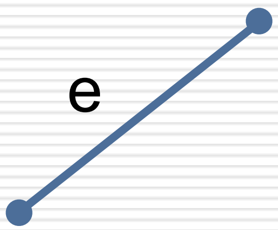

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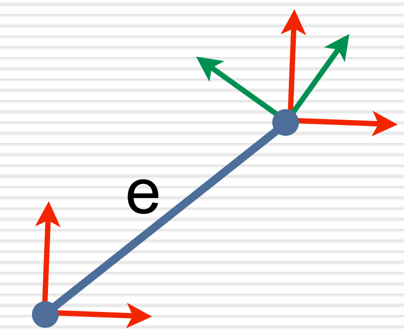

$\exp(\boldsymbol{\omega})$: holonomy along e

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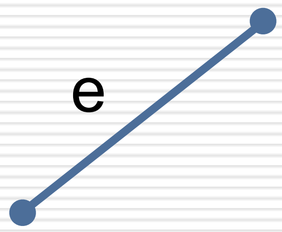

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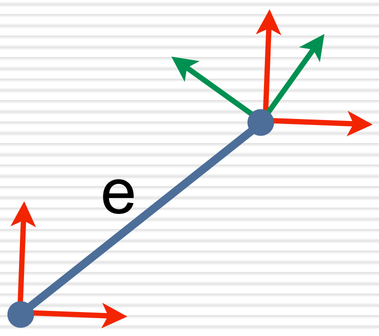

$\exp(\boldsymbol{\omega})$: holonomy along e

related by the torsion free condition

$$\mathrm{d}\boldsymbol{\theta}^i + \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\theta}^k = 0$$

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Einstein equation $$\mathrm{d}\mathbf{L}_i = \mathbf{S}_i$$

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- Marlene Vogel
- Christian Apeltauer

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(Novikov)

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discrete models of
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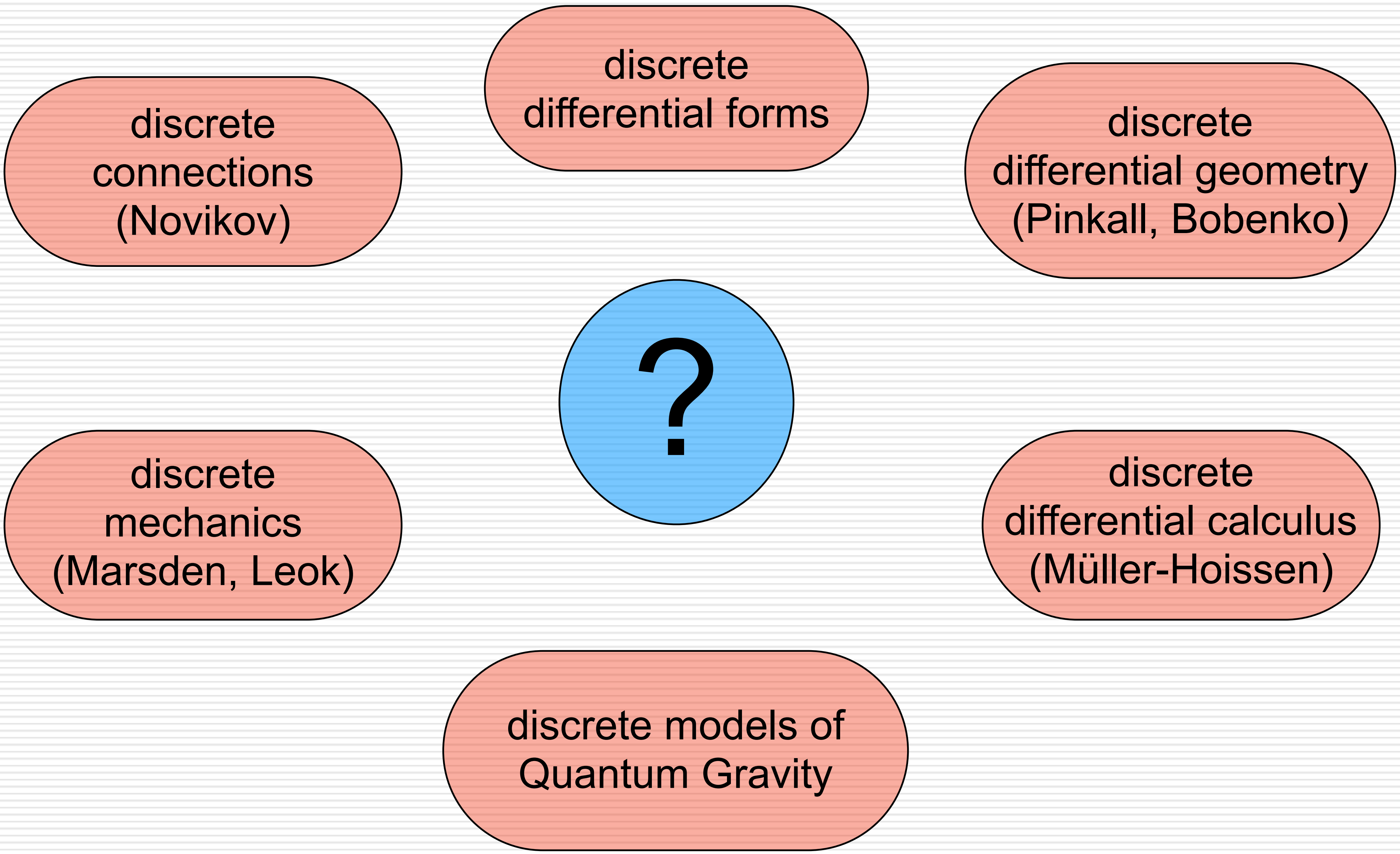
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