Binary black holes with helical Killing vector

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Outline

- Introduction
- Quasi-stationary approximation
- Projection formalism
- 2+1-decomposition
- Outlook

Introduction

- coalescing binary black holes strongest sources for gravitational radiation
- difficult relativistic problem (no symmetries), pure vacuum
- realistic initial values?
- Schild: quasi-circular orbits for two charged bodies in electrodynamics (incoming radiation)
- helical Killing vector, asymptotically $\partial_t + \Omega\partial_\phi$

Charges in Maxwell theory

- charges sources of the Maxwell equations, four-dimensional notation, Lorentz gauge, linear equations,

$$\Box A^\mu = j^\mu$$

  explicit expressions for the potentials in terms of retarded integrals over the charges

- charges move according to Lorentz force, therefore non-linear problem (radiation back-reaction)

- Schild: stationary approximation for the quasi-circular motion, incoming radiation compensates outgoing radiation (sequence of circular orbits)

Binary black holes

- pure vacuum problem, no symmetries
- 3+1 decomposition

$$ds^2 = -N^2 dt^2 + \gamma_{ab}(dx^a + \beta^a dt)(dx^b + \beta^b dt)$$

- 10 Einstein equations $R_{\mu\nu} = 0$ split in 6 time evolution and 4 constraint equations
- constraints constrain the initial values $\gamma_{ab}(t_0)$, $\gamma_{ab,t}(t_0)$, 4 equations plus 4 gauge freedoms for 12 quantities, underdetermined system, problem: not known how to encode physics in the initial data (gravitational radiation included?)

What has been done?

- post-Newtonian calculations:
  calculations to order 3pN, resummation of the perturbation series (Blanchet, Buonnano, Damour, Schäfer,…).

- initial data for binary black holes:
  numerical solution of the Lichnerowicz equations for Bowen-York initial data (conformally flat spatial metric), helical KV (Baumgarte, Cook, Shapiro,…)

- IWM spacetimes (Meudon):
  toy model for gravitation, theory without radiation: γ_{ab} conformally flat, binary IWM black holes with helical KV (non-regular horizon, Kerr not included).

Helical Killing vector

- asymptotically $\partial_{t'} + \Omega\partial_\phi$, choose $\xi = \partial_t$
- quotient space metric (Ehlers, Geroch)

$$ds^2 \quad = \quad -f(dt + k_a dx^a)(dt + k_b dx^b) + \frac{1}{f} h_{ab} dx^a dx^b$$

f: norm of the Killing vector, $\xi_a = -f(1, k_a)$

- Maxwell-type equation

$$\frac{1}{2} D_a (f^2 k^{ab}) \quad = \quad 0, \quad k_{ab} = k_{a,b} - k_{b,a}$$

- twist potential ($h = \det(h_{ab})$)

$$k^{ab} = \frac{1}{\sqrt{h} f^2} \epsilon^{abc} \partial_c b$$

- Ernst potential $\mathcal{E} = f + ib$

$$f D_a D^a \mathcal{E} = \frac{f}{\sqrt{h}} (\sqrt{h} h^{ab} \mathcal{E}_a)_b = D_a \mathcal{E} D^a \mathcal{E}$$

corresponds to the 4 constraint equations

- non-linear sigma model

$$R^{(3)}_{ab} = \frac{1}{2f^2} \Re(\mathcal{E}_a \bar{\mathcal{E}}_b)$$

- 3-dimensional gravity with sigma model 'matter' determined by the Ernst equation

Projection formalism

- advantage: less and simpler equations
- drawback: singular equations for f=0 (f changes sign at the light cylinder, numerical problems?)

Minkowski in rotating coordinates

- Minkowski: $f=1$, $b=0$,
  rotating coordinates $\phi' = \phi - \Omega t$

$$f' = 1 - \Omega^2\rho^2, \qquad b' = 2\Omega z$$

- $\rho < 1/\Omega$: $f > 0$,
  $\rho > 1/\Omega$: $f < 0$,
  $\rho = 1/\Omega$: light cylinder (observer rotates with c)

- transformed metric h_{ab} (rescaled with f), $h_{\phi\phi}$ invariant, rest

$$h'_{ab} \;=\; (1-\Omega^2\rho^2)h_{ab}$$

- signature change from $+3$ to -1 at the light cylinder. No signature change of 4d metric, but t and ϕ change roles.

- Ernst equation in non-rotating coordinates

$$f\Delta\mathcal{E} = (\nabla\mathcal{E})^2$$

- in rotating coordinates, Laplace operator replaced by $\mathcal{L}$

$$\mathcal{L}\mathcal{E} = \mathcal{E}_{\rho\rho} + \frac{1}{\rho}\mathcal{E}_{\rho} + \mathcal{E}_{zz} + \left(\frac{1}{\rho^2} - \Omega^2\right)\mathcal{E}_{\phi\phi}$$

elliptic equation inside the light cylinder, hyperbolic outside (if ϕ-dependent): symmetric positive system, unique solution (Torre), numerical studies (Whelan et al.)

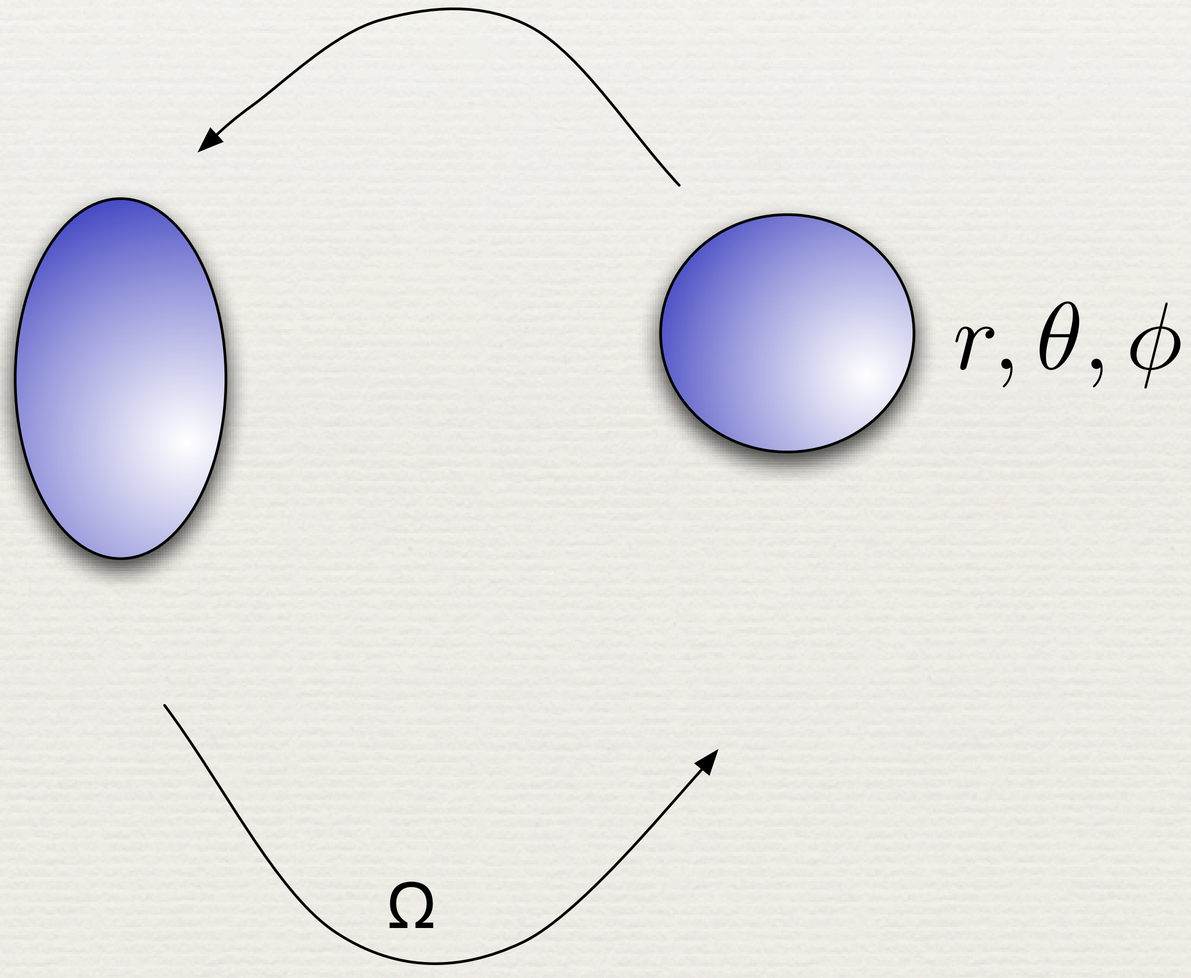
Horizons, 2+1 decomposition

- Killing horizon ($f = 0$) gives local concept
- $2 + 1$ decomposition: foliation by spheres

$$h_{ab}dx^a dx^b = s_{\alpha\beta}(dx^\alpha + \mathcal{B}^\alpha dr)(dx^\beta + \mathcal{B}^\beta dr) + \mathcal{A}^2 dr^2$$

- 3 parabolic equations ('constraint'), 3 elliptic equations
- singularities: horizon, light cylinder, infinity

2+1 decomposition



- horizon: regular singularity (expansion in $t = r - R$ with θ, ϕ dependent coefficients)

$$f \sim \mathcal{A}^2 \sim (r - R)^2,$$

  locally like Kerr black hole

- infinity not regular, formal expansion

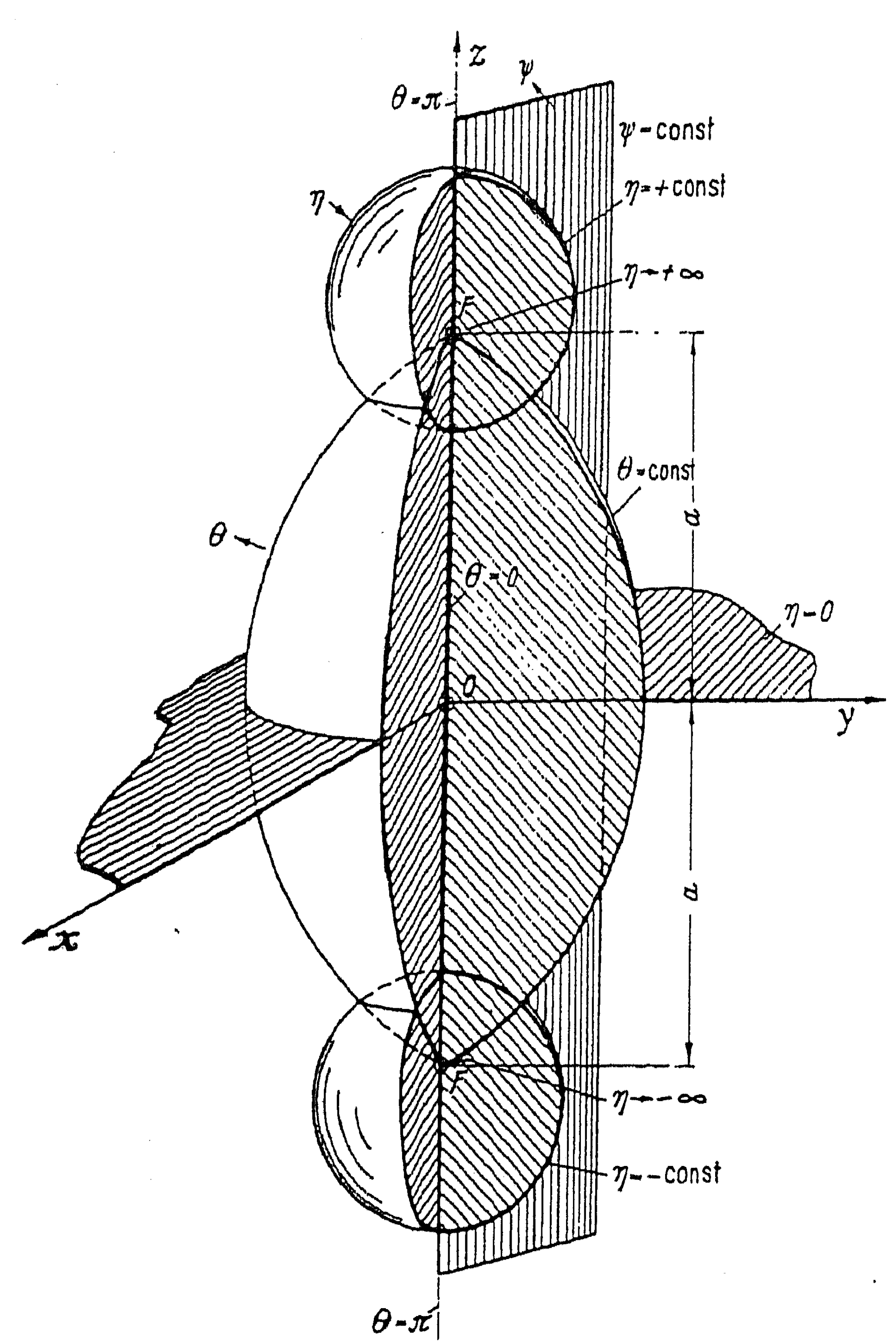
$$y = \sum_{n=1} \frac{y_n(r, \theta, \phi)}{r^n}$$

  r-dependence of y_n oscillatory terms

- light cylinder: regular singularity at surface with cylindrical topology, precise location unknown in coordinate system adapted to horizon ($f \sim \mathcal{A}^2$)

Outlook

- analytical task: global existence, asymptotic behavior
- numerical task: multi-domain spectral methods (Lorene)
- physical task: Killing vector approximate, only valid in finite region, matching to asymptotically flat spacetime



Bispherical coordinates

$$x = \frac{a \sin\theta \cos\psi}{\cosh\eta - \cos\theta},$$

$$y = \frac{a \sin\theta \sin\psi}{\cosh\eta - \cos\theta},$$

$$z = \frac{a \sinh\eta}{\cosh\eta - \cos\theta},$$

Laplace equation $\Delta F = 0$

$$F(\eta, \theta, \psi) = \sqrt{\cosh\eta - \cos\theta} \sum_{l,m} H_l(\eta) Y_{lm}(\theta, \psi)$$

$$H_l(\eta) = a_l e^{(l+\frac{1}{2})\eta} + b_l e^{-(l+\frac{1}{2})\eta}$$