HYPERBOLIC CONSERVATION LAWS
and
SPACETIMES WITH LIMITED REGULARITY

Philippe G. LeFloch
University of Paris 6 & CNRS

Interplay between nonlinear hyperbolic P.D.E.'s and geometry.
Fluids and metrics with limited regularity.

Three different topics :

- **Well-posedness theory for hyperbolic conservation laws on a Lorentzian background** (M. Ben-Artzi)
- **Injectivity radius estimates for Lorentzian manifolds with bounded curvature** (B.-L. Chen)
- **Existence of Gowdy-type matter spacetimes with bounded variation** (J.M. Stewart)

CONSERVATION LAWS ON A LORENTZIAN MANIFOLD

Joint work with M. Ben-Artzi, Jerusalem.

(M, g) : time-oriented, $(n+1)$-dimensional Lorentzian manifold with signature $(-, +, \dots, +)$.

Definition.

- *A* **flux** *on* M *: a vector field* $x \mapsto f_x(\bar{u}) \in T_x M$.
- **Time-like flux** *:* $g_{\alpha\beta}\, \partial_u f_x^\alpha(\bar{u})\, \partial_u f_x^\beta(\bar{u}) < 0, \qquad x \in M,\ \bar{u} \in \mathbb{R}$.
- **Conservation law :** $\nabla_\alpha \big(f^\alpha(u) \big) = 0, \quad u : M \to \mathbb{R}$ *being a scalar field.*
- **Geometry compatible** *:* $\nabla_\alpha f_x^\alpha(\bar{u}) = 0$ *for all* $\bar{u} \in \mathbb{R},\ x \in M$.

Remark.

- Nonlinear hyperbolic equation.
- A model for the dynamics of compressible fluids.
- Allow for shock waves and their interplay with the (fixed background) geometry.

Globally hyperbolic.

- Foliation by space-like, compact, oriented hypersurfaces

$$M = \bigcup_{t\in\mathbb{R}} \mathcal{H}_t.$$

n_t : future-oriented, unit normal vector field to $\mathcal{H}_t$
g_t : induced metric. X^{n_t} : normal component of X.

- Future of the Cauchy hypersurface $\mathcal{H}_0$

$$\mathcal{J}^+(\mathcal{H}_0) = \bigcup_{t\geq 0} \mathcal{H}_t.$$

- An initial data $u_0 : \mathcal{H}_0 \to \mathbb{R}$ being prescribed, we search for a weak solution $u = u(x) \in L^\infty(\mathcal{J}^+(\mathcal{H}_0))$ satisfying in a weak sense

$$u_{|\mathcal{H}_0} = u_0.$$

Discontinuous solutions in the sense of distributions. Non-uniqueness. Need an entropy criterion.

Definition.

- **Convex entropy flux** *:*
  $F = F_x(\bar{u})$ *if there exists* $U : \mathbb{R} \to \mathbb{R}$ *convex*

$$F_x(\bar{u}) = \int_0^{\bar{u}} \partial_u U(u') \, \partial_u f_x(u') \, du', \qquad x \in M, \; \bar{u} \in \mathbb{R}.$$

  Additional conservation laws for smooth solutions $\nabla_\alpha \big(F^\alpha(u) \big) = 0$.

- **Entropy solution** *of the geometry-compatible conservation law :*

  $u = u(x) \in L^\infty(\mathcal{J}^+(\mathcal{H}_0))$ *such that for all convex entropy flux* $F = F_x(\bar{u})$ *and smooth functions* $\theta \geq 0$

$$\int_{\mathcal{J}^+(\mathcal{H}_0)} F^\alpha(u) \, \nabla_\alpha \theta \, dV_g - \int_{\mathcal{H}_0} F^{n_0}(u_0) \, \theta_{\mathcal{H}_0} \, dV_{g_0} \geq 0.$$

Theorem. (Well-posedness theory for hyperbolic conservation laws on a Lorentzian manifold.)

There exists a unique entropy solution $u \in L^\infty(\mathcal{J}^+(\mathcal{H}_0))$:

- *the trace $u_{|\mathcal{H}_t} \in L^1(\mathcal{H}_t, g_t)$ exists for each t,*
- *for any convex entropy flux F the functions $\|F^{n_t}(u_{|\mathcal{H}_t})\|_{L^1(\mathcal{H}_t,g_t)}$ are non-increasing in time,*
- *for any two entropy solutions u, v,*

$$\|f^{n_t}(u_{|\mathcal{H}_t}) - f^{n_t}(v_{|\mathcal{H}_t})\|_{L^1(\mathcal{H}_t,g_t)} \approx \|u_{|\mathcal{H}_t} - v_{|\mathcal{H}_t}\|_{L^1(\mathcal{H}_t,g_t)}$$

is non-increasing in time.

Remarks.

- solutions are discontinuous (shock waves).
- the theory extends to the outer communication region of the Schwarzschild spacetime.

Work in progress.

- convergence of finite volume approximations (Riemann solvers, Godunov-type schemes).

INJECTIVITY RADIUS ESTIMATES FOR LORENTZIAN MANIFOLDS

Joint work with B.-L. Chen, Guang-Zhou.

Purpose.

- Investigate the geometry and regularity of $(n+1)$-dimensional Lorentzian manifolds (M, g).
- Exponential map $\exp_p$ at some point $p \in M$.
  – conjugate radius : largest ball on which $\exp_p$ is a local diffeomorphism
  – Injectivity radius : largest ball on which $\exp_p$ is a global diffeomorphism.
- Obtain lower bounds in terms of curvature and volume.

Results for Riemannian manifolds.
Cheeger, Gromov, Petersen, etc.

(M, g) : an n-dimensional Riemannian manifold
$\mathcal{B}(p, r)$: geodesic ball centered at $p \in M$.

$$\|\mathrm{Rm}_g\|_{\mathbf{L}^\infty(\mathcal{B}(p,1))} \leq K_0, \qquad \mathrm{Vol}_g(\mathcal{B}(p,1)) \geq v_0$$

- The injectivity radius is at least $i_0 = i_0(K_0, v_0, n) > 0$.
- Given $\varepsilon > 0$ and $0 < \gamma < 1$ there exist $C(\varepsilon, \gamma) > 0$ and some coordinates defined in $\mathcal{B}(p, r_0)$ in which

$$(1+\varepsilon)^{-1}\,\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\,\delta_{ij},$$
$$r\,\|\partial g\|_{\mathbf{C}^0(\mathcal{B}(p,r))} + r^{1+\gamma}\|\partial g\|_{\mathbf{C}^\gamma(\mathcal{B}(p,r))} \leq C(\varepsilon,\gamma), \qquad r \in (0, r_0].$$

Results for foliated Lorentzian manifolds.

- Anderson assumed

$$\|\mathrm{Rm}_g\|_{\mathbf{L}^\infty(\mathcal{B}(p,1))} \leq K_0$$

plus other structure conditions, and investigated the existence of "good" coordinates, and various issues of long-time evolution.

- Klainerman and Rodnianski relied instead on

$$\sup_{\Sigma \text{ spacelike}} \|\mathrm{Rm}_g\|_{\mathbf{L}^2(\mathcal{B}(p,1)\cap\Sigma)} \leq K_0,$$

and, in a series of papers, established estimates on the conjugacy radius and injectivity radius of null cones.

Aim.

- Purely *local* and fully *geometric* estimates, without assuming a system of coordinates or a foliation a priori.
- Injectivity radius estimates in arbitrary directions as well as in null cones.

Techniques.

- Use a "reference" Riemannian metric $\widehat{g}$, based on a vector-field or a vector at one point.
- Find a suitable generalization of *classical arguments* from Riemannian geometry: geodesics, Jacobi fields, comparison arguments, etc.
- Compare the behavior of g-geodesics and $\widehat{g}$-geodesics.

Reference Riemannian metric.

(M, g) : oriented $(n+1)$-dimensional Lorentzian manifold.

- $T_p \in T_p M$: future-oriented time-like unit vector field.
- Moving frame (orthonormal) : e_α $(\alpha = 0, 1, \ldots, n)$ consisting of $e_0 = T$ supplemented with spacelike vectors e_j $(j = 1, \ldots, n)$.
  e^α : dual frame. Lorentzian metric :
  $g = \eta_{\alpha\beta}\, e^\alpha \otimes e^\beta, \qquad \eta_{\alpha\beta}$: Minkowski.
- Riemannian metric :

$$\widehat{g} := \delta_{\alpha\beta}\, e^\alpha \otimes e^\beta, \qquad \delta_{\alpha\beta} : \text{ Euclidian}$$

  will be used to compute the norm $|A|_T$ of tensors on M.
- Special choice : Choose e_j in the orthogonal $\{e_0\}^\perp$.

All metrics equivalent if T varies in a compact subset of the future cone.

Injectivity radius with respect to a reference vector.

- If M is not geodesically complete, then $\exp_b$ is defined only on a neighborhood of the origin in T_pM.
- The metric g_p on T_pM is not positive definite and the norm of a non-zero vector may vanish. We need to rely on $\widehat{g}_p$ and consider the $\widehat{g}$-ball $B_{T_p}(0, r) \subset T_pM$.

Definition.

The injectivity radius with respect to the reference vector T_p

$$Inj_g(M, p, T_p)$$

is the largest radius r *such that* $\exp_p$ *is a global diffeomorphism from* $B_{T_p}(0, r)$ *to a neighborhood of* p.

First result : Lorentzian manifolds with a prescribed vector field.

$\Omega \subset M$: domain containing a point p and foliated by spacelike hypersurfaces with normal T, $\Omega = \bigcup_{t\in[-1,1]} \Sigma_t$, with lapse function : $n^2 := -g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$.

- $(A1)$: $|\log n| \leq K_0$ in Ω.
- $(A2)$: $|\mathcal{L}_T g|_T \leq K_1$ in Ω.
- $(A3)$: $|\mathrm{Rm}_g|_T \leq K_2$ in Ω.
- $(A4)$: $\mathrm{Vol}_{g_0}(\mathcal{B}_{\Sigma_0}(p,1)) \geq v_0$ (initial slice).

Theorem 1. *Let (M,g) be a Lorentzian manifold satisfying $(A1)$–$(A4)$ at some point p and for some vector field T. Then, there exists $i_0 > 0$ depending only upon the foliation bounds K_0, K_1, the curvature bound K_2, the volume bound v_0, and the dimension such that*

$$Inj_g(M,p,T_p) \geq i_0.$$

Second result : Lorentzian manifolds with a prescribed vector at one point.

No need to prescribe the whole vector field and foliation a priori.

- Given (M, g), $p \in M$, and a unit vector $T \in T_pM$, consider the reference metric $\widehat{g} := \langle\, ,\, \rangle_T$ on T_pM.
- Assume that $\exp_p$ is defined on $B_T(0, r_0) \subset T_pM$ (ball determined by $\widehat{g}$).
- Pull back : $g = \exp_p^\star g$ (still denoted by g) is defined on $B_T(0, r_0)$.
- g-parallel translate the vector T along the (straight) radial geodesics from the origin. Vector field still denoted by T and defined on $B_T(0, r_0)$.
- Use T and g to define a reference Riemannian metric $\widehat{g}$ on $B_T(0, r_0)$. Compute the norms $|A|_T$ on $B_T(0, r_0)$.

Investigate the geometry of the local covering

$$\exp_p : B_T(0, r_0) \to \mathcal{B}(p, r_0) := \exp_p(B_T(0, r)).$$

Theorem 2. (B.-L. Chen & P.G. LeFloch, 2006)
Let (M, g) be an $(n+1)$-dimensional Lorentzian manifold, and consider a point $p \in M$ together with a reference vector $T \in T_pM$. Assume that exp_p is defined on the ball $B_T(0, r_0) \subset T_pM$ and

$$|Rm_g|_T \leq r_0^{-2} \quad on\ B_T(0, r_0).$$

Then, there exists $c(n) \in (0, 1)$ depending only on the dimension of the manifold such that

$$Inj_g(M, p, T) \geq c(n)\, \frac{Vol_g(\mathcal{B}(p, c(n)\, r_0))}{r_0^{n+1}}\, r_0.$$

GOWDY MATTER SPACETIMES WITH BOUNDED VARIATION

Joint work with J.M. Stewart, Cambridge.

Spacetime. (M, g) : $(3+1)$-dimensional Lorentzian manifold satisfying Einstein field equations : $G_{\alpha\beta} = \kappa\, T_{\alpha\beta}$.

Perfect fluids. $T_{\alpha\beta} = (\mu + p)\, u_\alpha\, u_\beta + p\, g_{\alpha\beta}$

- energy density $\mu > 0$
- equation of state for the pressure
  $p = c_s^2\, \mu, \qquad 0 < c_s < 1, \qquad c_s$: sound speed
- light speed normalized $= 1$
- time-like, unit velocity vector u^α

Existence theory in the bounded variation class (BV) under symmetry assumptions.

Plane-symmetric Gowdy-type spacetimes with matter.

- Two linearly independent, commuting Killing fields X, Y and in coordinates

$$g = e^{2a}\,(-dt^2 + dx^2) + e^{2b}\,(e^{2c}\,dy^2 + e^{-2c}\,dz^2)$$

for some coefficients a, b, c depending on t, x. Work pioneered by Moncrief, Isenberg, Rendall, Chrusciel, etc.

- Velocity vector has only an x-component

$$u^\alpha = e^{-a}\gamma(1, v, 0, 0), \quad \gamma = (1 - v^2)^{-1/2}, \qquad |v| < 1$$

- From $T^{\alpha\beta}$ we define τ, S, Σ

$$\begin{aligned} T^{00} &= e^{-2a}\big((\mu + p)\gamma^2 - p\big) =: e^{-2a}\,\tau \\ T^{01} &= T^{10} = e^{-2a}\,(\mu + p)\,\gamma^2\,v =: e^{-2a}\,S \\ T^{11} &= e^{-2a}\,\big((\mu + p)\,\gamma^2\,v^2 + p\big) =: e^{-2a}\,\Sigma \end{aligned}$$

Evolution and constraint equations.

- Three evolution equations (second-order nonlinear wave equations)

$$a_{tt} - a_{xx} = b_t^2 - b_x^2 - c_t^2 + c_x^2 - \frac{\kappa}{2} e^{2a} (\mu + p)$$

$$b_{tt} - b_{xx} = -2\, b_t^2 + 2\, b_x^2 + \frac{\kappa}{2} e^{2a} (\mu - p)$$

$$c_{tt} - c_{xx} = -2\, b_t c_t + 2\, b_x c_x$$

- Two constraint equations (first-order in time)

$$2a_t b_t + 2a_x b_x + b_t^2 - 2b_{xx} - 3b_x^2 - c_t^2 - c_x^2 = \kappa e^{2a} \tau$$

$$-2a_t b_x - 2a_x b_t + 2b_{tx} + 2b_t b_x + 2c_t c_x = \kappa e^{2a} S$$

- From Bianchi identities we deduce the Euler equations

$$\tau_t + S_x = -\tau(a_t + 2b_t) - S(2a_x + 2b_x) - \Sigma a_t - 2pb_t,$$
$$S_t + \Sigma_x = -\tau a_x - S(2a_t + 2b_t) - \Sigma(a_x + 2b_x) + 2pb_x.$$

Special case : vacuum.

- Blow-up in sup norm in finite time.
- As long as the variable b remains bounded, the variables a and c remain bounded.
- Only expect existence for the Euler-Einstein equations until the geometry blows-up.

Special case : Relativistic Euler equations in the Minkowski space.

- Letting $a = b = \kappa = 0$ we obtain the fluid equations

$$\Big(\frac{1 + c_s^2\, v^2}{1 - v^2}\,\mu\Big)_t + \Big(\frac{1 + c_s^2}{1 - v^2}\,\mu\, v\Big)_x = 0,$$
$$\Big(\frac{1 + c_s^2}{1 - v^2}\,\mu\, v\Big)_t + \Big(\frac{v^2 + c_s^2}{1 - v^2}\,\mu\Big)_x = 0.$$

- Nonlinear hyperbolic equations, discontinuities in (μ, v). Work by Smoller, Temple, etc.

Theorem.(Initial-value problem for the Euler-Einstein equations in a plane-symmetric Gowdy spacetime).
Fix initial data $(a_t, a_x, b_t, b_x, c_t, c_x, \mu, v)(0)$ *satisfying the constraint equations and having locally bounded variation (BV). Then :*

- *There exists a solution* (a, b, c, μ, v) *which is defined for all* $x \in \mathbb{R}$ *on a maximal time interval* $t \in [0, T_{max})$.
- *It satisfies the constraint equations and* $(a_t, a_x, b_t, b_x, c_t, c_x, \mu, v)$ *has bounded variation at every time.*
- *The fluid variables satisfy entropy inequalities.*
- *When* $T_{max} < \infty$, *the sup norm of* (a, b, μ) *must blow-up at* $t = T_{max}$.

Remarks.

- Arbitrary large data, shock waves, Lipschitz continuous metric, Gravitational waves.
- Possible blow-up in the geometry a, b and matter concentration in μ.
- *Work in progress* : $T3$ Gowdy spacetimes in areal coordinates, and censorship conjecture for Euler-Einstein spacetimes.