The Conformally Reduced Einstein Equations

— an alternative approach to Scri?

joint work with: L. Buchman, O. Sarbach & R. Matzner

Scri$^+$ (future null infinity)

— the natural boundary for outgoing (gravitational) radiation

— can be chosen to coincide with a finite coordinate cylinder relative to a suitably defined conformal metric $\tilde{g}_{\mu\nu}$

$$\boxed{g_{\mu\nu} = \frac{1}{\Omega^2}\tilde{g}_{\mu\nu}, \quad \Omega\Big|_{\mathcal{I}^+} = 0}$$

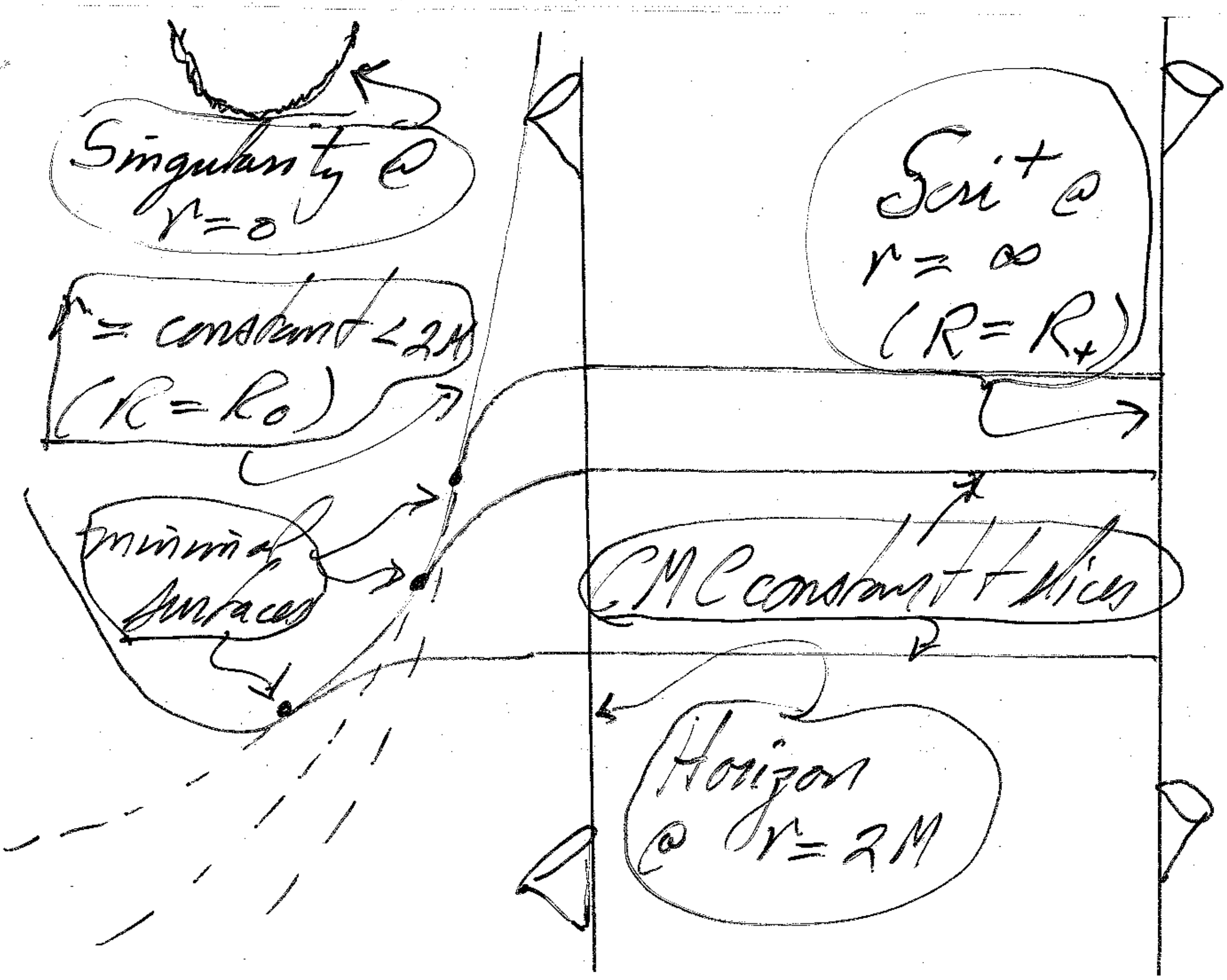
Relative to suitably chosen spacelike slices tending to Scri$^+$ ($\equiv \mathcal{I}^+$) a mono-chromatic wave makes only a finite # of wiggles

Original approach — the conformally regular field equations of H. Friedrich developed numerically by P. Hübner, S. Husa, C. Lechner, ...

Solve suitable hyperbolic equations for the conformal geometry, e.g., conformal curvature (Weyl tensor), connection, frame

- this is a complicated system of field equations for which, in particular, the constraints are not well understood at the analytical level (or the numerical one).

Question — can one solve more directly for the conformal metric $\tilde{g}_{\mu\nu}$ using a modified 3+1 formalism without demanding full conformal regularity?



A schematic picture of the Schwarzschild geometry

- CMC (constant mean curvature) constant time slices
- Scri$^+$ at a finite coordinate cylinder $R = R_+$
- inner boundary taken at minimal surfaces ~ $r < 2M$
- slices avoid singularity at $r = 0$.

Schwarzschild metric in CMC slicing
(following E. Malec, N. Ó Murchadha
or D. Brill et. al)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dT^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$
$$+ \frac{dr^2}{\left[1-\frac{2M}{r}+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2\right]} + \frac{2\left(\frac{C}{r^2}-\frac{Kr}{3}\right)dT\,dr}{\sqrt{1-\frac{2M}{r}+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}}$$

C, K are constants and

$$K = -\mathrm{tr}_g \underset{\sim}{K} = -(\text{mean curvature})$$

we want $K > 0$ (negative constant mean curvature)
and

$$C > (2M)^3 \frac{K}{3} > 0$$

For given C, K there is an $r_{min} < 2M$
s.t. $1 - \frac{2M}{r_{min}} + \left(\frac{K r_{min}}{3} - \frac{C}{r_{min}^2}\right)^2 = 0$
and $1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2 > 0$ on (r_{min}, ∞)

Introduce a (dimensionless) R defined on $(R_0, R_+) \longleftrightarrow (r_{min}, \infty)$ by setting

$$\ln\left(\frac{R}{R_0}\right) = \int_{r_{min}}^{r} \frac{dr'}{r'\sqrt{1 - \frac{2M}{r'} + \left(\frac{Kr'}{3} - \frac{C}{r'^2}\right)^2}}$$

with

$$\ln\left(\frac{R_+}{R_0}\right) = \int_{r_{min}}^{\infty} \frac{dr'}{r'\sqrt{1 - \frac{2M}{r'} + \left(\frac{Kr'}{3} - \frac{C}{r'^2}\right)^2}}$$

and (say) $R_+ = 1$ (which then fixes R_0 as a function of M, C, K).

One finds that the Schwarzschild metric then takes the form

$$ds^2 = \frac{r^2}{R^2}\left\{-\tilde{N}^2 dT^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(dR + \frac{R}{r}\left(\frac{C}{r^2} - \frac{Kr}{3}\right)dT\right)^2\right\}$$

$$:= \frac{1}{\Omega^2}\left\{-\tilde{N}^2 dT^2 + \delta_{ij}(dy^i + Y^i dT)(dy^j + Y^j dT)\right\}$$

where $\{y^i\} = \{R, \theta, \varphi\}$

$$\gamma_{ij}\, dy^i dy^j = dR^2 + R^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$Y^R = \frac{R}{r}\left(\frac{C}{r^2} - \frac{Kr}{3}\right), \quad Y^\theta = Y^\varphi = 0$$

$$\Omega = \frac{R}{r} = \text{conformal factor}$$

$$\tilde{N} = \frac{R}{r}\sqrt{1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2}$$

= unphysical (or 'conformal') lapse

with $N = \sqrt{1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2} = \frac{\tilde{N}}{\Omega}$

= physical lapse

Trace free part of the gravitational momentum $\pi^{tr\, ij}$ (where $\pi^{ij} = \pi^{tr\, ij} + \frac{1}{3} g^{ij} \pi_?$) is given by

$$\pi^{tr\, RR} = 2C \frac{R}{r^2} \sin\theta$$

$$\pi^{tr\, \theta\theta} = -C \frac{\sin\theta}{R r^2}, \quad \pi^{tr\, ij} = 0 \quad i \neq j$$

$$\pi^{tr\, \varphi\varphi} = \frac{-C}{R r^2 \sin\theta}$$

necessary for slicing desired!

When $tr_g K$ = constant the momentum constraint can be written

$$\nabla_j (\pi^{tr\, ij}) = 0 \ , \quad \pi^{tr\, ij} = \pi^{ij} - \frac{1}{3} \delta_i^j \, tr_g \pi$$

which is then equivalent to:

$$\tilde{\nabla}_j (\pi^{tr\, ij}) = 0 \qquad \tilde{\nabla}_j = \text{covariant derivative w.r.t.}$$

Could use a $\quad \gamma_{ij} = \Omega^2 g_{ij}$

flat conformal metric γ_{ij} (as in Schwarzschild) and use Bowen-York solutions

The Hamiltonian constraint now has the form

$$-4\Omega \tilde{\Delta}_\gamma \Omega + 6\gamma^{ij}\Omega_{,i}\Omega_{,j} - \Omega^2\, {}^{(3)}\tilde{R}(\gamma) - \frac{2}{3}K^2 + \frac{\pi^{tr\, ij}\pi^{tr}_{ij}}{(\mu_\gamma)^2}\Omega^6 = 0$$

where $K = -g^{ij}K_{ij}$, $\mu_\gamma = \sqrt{\det \gamma_{mn}}$

and is singular at Scri$^+$ where $\Omega =$

a solution Ω, regular at $\mathcal{L}^+$ must obey

$$\Omega\Big|_{\mathcal{L}^+} \overset{D}{=} 0, \quad \left(6\gamma^{ij}\Omega_{,i}\Omega_{,j} - \frac{2}{3}K^2\right)\Big|_{\mathcal{L}^+} =$$

We can normalize the choice of conformal metric $\tilde{\gamma}_{ij}$ (appealing to Yamabe's theorem) by requiring that

$${}^{(3)}\hat{R}(\gamma) = \text{constant}$$

The associated equation

$$\frac{\partial}{\partial t}\,{}^{(3)}\hat{R}(\gamma) \overset{\Delta}{=} 0$$

then leads to an elliptic equation for the quantity

$$\Gamma := \gamma^{mn}\gamma_{mn,t} - 2\tilde{\nabla}_l Y^l$$

given by

$$\frac{2}{3}\tilde{\Delta}_\gamma \Gamma + \frac{1}{3}{}^{(3)}\hat{R}(\gamma)\Gamma = \hat{\nabla}_i\hat{\nabla}_j\left[\frac{2\hat{N}}{\mu_\gamma}\pi^{tr\,ij}\right] - {}^{(3)}\hat{R}_{ij}(\gamma)\left[\frac{2\hat{N}}{\mu_\gamma}\pi^{tr\,ij}\right]$$

where $\hat{N}$ is first determined by solving the (singular) elliptic equation

$$0 = -\Omega^3 \tilde{\Delta}_\gamma \hat{N} + 3\Omega \gamma^{ij} \hat{N}_{,i} \Omega_{,j}$$
$$- \frac{3}{2} \hat{N} \gamma^{ij} \Omega_{,i} \Omega_{,j} + \frac{\hat{N}}{6} K^2 - \frac{\hat{N}}{4} \Omega^2 \, {}^{(3)}\hat{R}(\gamma)$$
$$+ \frac{5}{4} \frac{\hat{N}}{\mu_\gamma^2} \Omega^2 \gamma_{il} \gamma_{jm} \pi^{tr\, im} \pi^{tr\, jl}$$

which enforces the condition

$$\frac{\partial}{\partial t}\left(g^{ij} K_{ij}\right) = 0$$

A convenient choice of spatial gauge condition is that of (spatial) harmonic gauge:

$$V^k := \gamma^{ij}\left(\Gamma^k_{ij}(\gamma) - \hat{\Gamma}^k_{ij}(\hat{\gamma})\right) = 0$$

for a (fixed) conformal metric $\hat{\gamma}_{ij}$.

The condition

$$\frac{\partial V^k}{\partial t} = 0$$

then yields an elliptic equation for $Y^k \frac{\partial}{\partial x^k}$

$$0 = -\frac{1}{3}\Gamma V^{k} - \frac{1}{6}\gamma^{kl}\Gamma_{,l} - \frac{2\hat{N}}{\mu_\gamma}\pi^{\tilde{t}ij}\left(\Gamma^{k}_{ij}(\gamma) - \hat{\Gamma}^{k}_{ij}(\hat{\gamma})\right)$$
$$+ \tilde{\nabla}_l\left[\frac{2\hat{N}}{\mu_\gamma}\pi^{\tilde{t}kl}\right] + \tilde{\nabla}_j\left[\hat{\nabla}^{j}Y^{k} + \hat{\nabla}^{k}Y^{j}\right]$$
$$-\hat{\nabla}^{k}(\hat{\nabla}_j Y^{j}) - (\hat{\nabla}^{i}Y^{j} + \hat{\nabla}^{j}Y^{i})\left(\Gamma^{k}_{ij}(\gamma) - \hat{\Gamma}^{k}_{ij}(\hat{\gamma})\right)$$

Thus $\{\hat{N}, Y^{i}, \Omega, \Gamma\}$ are determined on each slice (i.e., "time step") by solving a system of (mostly linear) elliptic equations. Finally one evolves the $\{\gamma_{ij}, \pi^{\tilde{t}ij}\}$ data by the (conformal) "3+1" evolution equations

$$\partial_t \gamma_{ij} = \frac{2\hat{N}}{\mu_\gamma}\gamma_{il}\gamma_{jm}\pi^{\tilde{t}lm} + \frac{1}{3}\gamma_{ij}\Gamma + \gamma_{il}\gamma_{jm}\left(\hat{\nabla}^{l}Y^{m} + \hat{\nabla}^{m}Y^{l}\right)$$

$$\partial_t \pi^{\tilde{t}ij} = \left(Y^{m}\pi^{\tilde{t}ij}\right)_{,m} - Y^{i}_{,m}\pi^{\tilde{t}mj} - Y^{j}_{,m}\pi^{\tilde{t}im}$$
$$- \frac{2\hat{N}}{\mu_\gamma}\pi^{\tilde{t}im}\pi^{\tilde{t}jl}\gamma_{lm} + \mu_\gamma\left[\hat{\nabla}^{i}\hat{\nabla}^{j}\hat{N} - \frac{1}{3}\gamma^{ij}\hat{\Delta}_\gamma\hat{N}\right]$$
$$-\mu_\gamma\hat{N}\left[{}^{(3)}R^{ij}(\gamma) - \frac{1}{3}\gamma^{ij}\,{}^{(3)}R(\gamma)\right]$$
$$-\frac{2}{3}\frac{\hat{N}}{\Omega}K\pi^{\tilde{t}ij} - 2\mu_\gamma\hat{N}\left[\frac{\hat{\nabla}^{i}\hat{\nabla}^{j}\Omega}{\Omega} - \frac{1}{3}\gamma^{ij}\frac{\hat{\Delta}_\gamma\Omega}{\Omega}\right]$$

where $\hat{\Delta}_\gamma = \gamma^{mn}\hat{\nabla}_m\hat{\nabla}_n$

<u>Plan of Action</u>

[1.] Solve constraints (numerically) on CMC slices that extend to $\mathcal{I}^+$

(i) Caltech-Cornell SpEC code already adapted for this by Luisa Buchman

(ii) as a test case, generalize Bowen-York data sets to CMC

[2.] Solve (linear) elliptic equations for $\Gamma, \tilde{N}, Y^k$

(apparently less demanding than [1.])

[3.] write a code for the (conformal) 3+1 evolution equations for $\{\gamma_{ij}, \pi^{\text{tr}\,ij}\}$

(i) subject to $^{(3)}R(\gamma) = \text{const}$

$$V^k(\gamma) = \gamma^{ij}\left(\Gamma^k_{ij}(\gamma) - \hat{\Gamma}^k_{ij}(\hat{\gamma})\right) = 0$$

(ii) propagate 'true' degrees of freedom
i.e., conformal geometry

[4.] Use linearized Schwarzschild analysis as a guide to carrying out [3.] (nearly complete)

[5.] Prove a theorem showing above is mathematically sound