

The Conformally Reduced Einstein Equation

- an alternative approach to Scri?

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Scri⁺ (future null infinity)

- the natural boundary for outgoing (gravitational) radiation
- can be chosen to coincide with a finite coordinate cylinder relative to a suitably defined conformal metric $g_{\mu\nu}$

$$g_{\mu\nu} = \frac{1}{\Omega^2} g_{\mu\nu}, \quad \Omega|_{\mathcal{I}^+} = 0$$

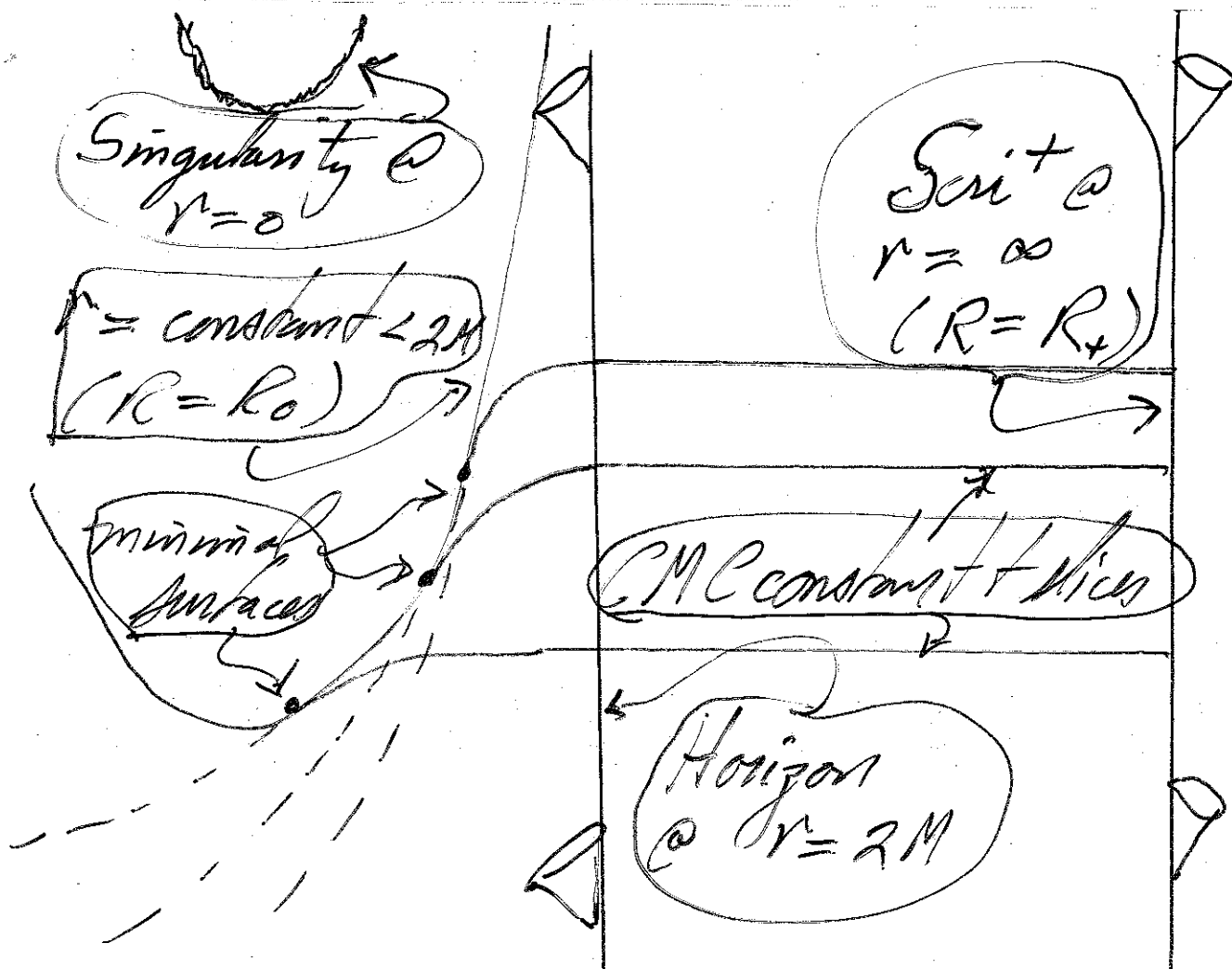
Relative to suitably chosen spacelike slices tending to Scri⁺ ($\equiv \mathcal{I}^+$) a monochromatic wave makes only a finite # of wiggles

Original approach - the conformally
regular field equations of H. Friedrich
developed numerically by P. Hübner,
S. Husa, C. Rechner, ...

Solve suitable hyperbolic equations
for the conformal geometry, e.g., conformal
curvature (Weyl tensor), connection,
frame

- This is a complicated system of PDE
equations for which, in particular,
the constraints are not well understood
at the analytical level (or the numerical
one).

Question - can one solve more directly for
the conformal metric $g_{\mu\nu}$ using a
modified 3+1 formalism without
demanding full conformal regularity?



A schematic picture of the Schwarzschild geometry

- CMC (constant mean curvature) constant time slices
- Scri+ at a finite coordinate cylinder $R = R_+$
- inner boundary taken at minimal surfaces $\sim r < 2M$
- slices avoid singularity at $r = 0$.

Schwarzschild metric in CMC slicing
 (following E. Malec, N. Ó Murchadha
 or D. Brill et. al)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{dr^2}{\left[1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2\right]} + \frac{2\left(\frac{C}{r^2} - \frac{Kr}{3}\right) dt dr}{\sqrt{1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2}}$$

C, K are constants and

$K = -\frac{1}{3} \text{tr} K = -$ (mean curvature
 one want $K > 0$ (negative constant
 and mean curvature,

$$C > (2M) \frac{K}{3} > 0$$

For given C, K there is an $r_{\min} < 2M$

s.t. $1 - \frac{2M}{r_{\min}} + \left(\frac{K r_{\min}}{3} - \frac{C}{r_{\min}^2}\right)^2 = 0$

and $1 - \frac{2M}{r} + \left(\frac{Kr}{3} - \frac{C}{r^2}\right)^2 > 0$ on (r_{\min}, ∞)

Introduce a (dimensionless) R
 defined on $(R_0, R_+) \leftrightarrow (r_{\min}, \infty)$
 by setting

$$\ln\left(\frac{R}{R_0}\right) = \int_{r_{\min}}^r \frac{dr'}{r' \sqrt{1 - \frac{2M}{r'} + \left(\frac{Kr'}{3} - \frac{C}{r'^2}\right)^2}}$$

with

$$\ln\left(\frac{R_+}{R_0}\right) = \int_{r_{\min}}^{\infty} \frac{dr'}{r' \sqrt{1 - \frac{2M}{r'} + \left(\frac{Kr'}{3} - \frac{C}{r'^2}\right)^2}}$$

and (say) $R_+ = 1$. (which
 then fixes R_0 as a function of M, C, K)
 One finds that the Schwarzschild metric
 then takes the form

$$ds^2 = \frac{r^2}{R^2} \left\{ -N^2 dt^2 + R^2 (d\theta^2 + \sin^2\theta d\phi^2) + \left(dR + \frac{R}{r} \left(\frac{C}{r^2} - \frac{K}{3} \right) dt \right)^2 \right\}$$

$$= \frac{1}{R^2} \left\{ -N^2 dt^2 + \delta_{ij} (dy^i + Y^i dt)(dy^j + Y^j dt) \right\}$$

where $\{y^i\} = \{R, \theta, \varphi\}$

$$g_{ij} dy^i dy^j = dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$Y^R = \frac{R}{r} \left(\frac{c}{r^2} - \frac{kr}{3} \right), \quad Y^\theta = Y^\varphi = 0$$

$$\frac{\Omega}{r} = \frac{R}{r} = \text{conformal factor}$$

$$N = \frac{R}{r} \sqrt{1 - \frac{2M}{r} + \left(\frac{kr}{3} - \frac{c}{r^2} \right)^2}$$

= unphysical (or conformal) lapse

$$\text{with } N = \sqrt{1 - \frac{2M}{r} + \left(\frac{kr}{3} - \frac{c}{r^2} \right)^2} = \frac{N}{\Omega}$$

= physical lapse

Trace free part of the gravitational momentum Π^{ij} (where $\Pi^j_j = \Pi^i_i + \frac{1}{3} g^{ij} \Pi_{ij}$) is given by

$$\Pi^{RR} = 2c \frac{R}{r^2} \sin \theta$$

$$\Pi^{\theta\theta} = -c \frac{\sin \theta}{R r^2}$$

$$\Pi^{\varphi\varphi} = -c \frac{\sin \theta}{R r^2}$$

$\Pi^{ij} = 0$ if $i \neq j$
 necessary for slicing derived!

When $t_{gr} = \text{constant}$ the momentum constraint can be rewritten

$$\nabla_j (\pi^{ij}) = 0, \quad \pi^{ij} = \pi_i^j - \frac{2}{3} \delta_i^j t_{gr}$$

which is then equivalent to:

$$\tilde{\nabla}_j (\pi^{ij}) = 0 \quad \tilde{\nabla} = \text{covariant derivative w.r.t.}$$

Could take a flat conformal metric $\delta_{ij} = \Omega^2 g_{ij}$

and use Bowen-York solutions

The Hamiltonian constraint now has the form

$$\begin{aligned} & -4\Omega \Delta \Omega + 6\delta^{ij} \partial_i \Omega \partial_j \Omega \\ & - \Omega^2 ({}^{(3)}R) - \frac{2}{3} K^2 + \frac{\pi^{ij} \pi_{ij}}{(\mu_8)^2} \Omega^6 \\ & = 0 \end{aligned}$$

where $K = -g^{ij} K_{ij}$, $\mu_8 = \sqrt{\det \delta_{mn}}$ and is singular at $\Omega = 0$ where $\Omega =$

A solution Ω , regular at ∂S^{i+} must obey

$$\Omega|_{\partial^{i+}} = 0, \quad \left(\delta^{ij} \Omega_{,i} \Omega_{,j} - \frac{2}{3} K^2 \right) |_{\partial^{i+}} = 0$$

We can normalize the choice of conformal metric δ_{ij} (appearing in Yamabe's theorem) by requiring the

$${}^{(3)}R(\delta) = \text{constant}$$

The associated equation

$$\frac{\partial}{\partial t} {}^{(3)}R(\delta) = 0$$

then leads to an elliptic equation for the quantity

$$\Gamma := \delta^{mn} \gamma_{,mn} - 2 \nabla_i \gamma^i$$

given by

$$\frac{2}{3} \Delta \gamma + \frac{1}{3} {}^{(3)}R(\delta) \gamma = \nabla_i \nabla_i \left[\frac{2N}{\mu} \pi^{ij} \right] - R_{ij}(\delta) \left[\frac{2N}{\mu} \pi^{ij} \right]$$

where N is first determined by solving the (singular) elliptic equation

$$0 = -\Omega^2 \Delta_\gamma N + 3\Omega \delta^{ij} N_{,i} \Omega_{,j} - \frac{3}{2} N \delta^{ij} \Omega_{,i} \Omega_{,j} + \frac{N}{\mu_8} K^2 - N \Omega^2 (13) R_{13} + \frac{5}{4} \frac{N}{\mu_8^2} \Omega^2 \delta^{is} \delta^{jm} \Pi^{t_{im}} \Pi^{t_{jd}}$$

which enforces the condition

$$\partial_t (g^{ij} K_{ij}) = 0$$

A convenient choice of spatial gauge condition is that of (spatial) harmonic gauge:

$$V^k := \delta^{ij} (\overset{\sim}{\Gamma}_{ij}^k(\alpha) - \overset{\wedge}{\Gamma}_{ij}^k(\alpha)) = 0$$

for a (fixed) conformal metric δ_{ij}

The condition

$$\frac{\partial V^k}{\partial t} = 0$$

then yields an elliptic equation for $\gamma^k_{\alpha^k}$

$$0 = -\frac{2}{3} \Gamma V^k - \frac{1}{6} \gamma^{klp} - \frac{2\hat{N}}{\mu_y} \pi^{tij} (\hat{\Gamma}_{ij}^k - \hat{\Gamma}_{ij}^k) + \hat{\nabla}_j \left[\frac{2\hat{N}}{\mu_y} \pi^{tkl} \right] + \hat{\nabla}_j \left[\hat{\nabla}^i \gamma^k + \hat{\nabla}^k \gamma^i \right] - \hat{\nabla}^k (\hat{\nabla}_j \gamma^i) - (\hat{\nabla}^i \gamma^j + \hat{\nabla}^j \gamma^i) (\hat{\Gamma}_{ij}^k - \hat{\Gamma}_{ij}^k)$$

Thus $\{\hat{N}, \gamma^i, \Omega, \Gamma\}$ are determined on each slice (i.e., time step) by solving a system of (mostly linear) elliptic equations. Finally one evolves the $\{\gamma_{ij}, \pi^{tij}\}$ data by the (conformal) 3+1 evolution equations

$$\partial_t \gamma_{ij} = \frac{2\hat{N}}{\mu_y} \gamma_{il} \gamma_{jm} \pi^{tlm} + \frac{1}{3} \gamma_{ij} \Gamma + \gamma_{il} \gamma_{jm} (\hat{\nabla}^l \gamma^m + \hat{\nabla}^m \gamma^l)$$

$$\begin{aligned} \partial_t \pi^{tij} = & (\gamma^m \pi^{tij})_{,m} - \gamma^i_{,m} \pi^{tmj} - \gamma^j_{,m} \pi^{tim} \\ & - \frac{2\hat{N}}{\mu_y} \pi^{tim} \pi^{tjl} \gamma_{lm} + \mu_y \left[\hat{\nabla}^i \hat{\nabla}^j \hat{N} - \frac{1}{3} \delta^{ij} \Delta_\gamma \hat{N} \right] \\ & - \mu_y \hat{N} \left[R^{(3)}(t) - \frac{1}{3} \delta^{ij} R^{(3)}(t) \right] \\ & - \frac{2}{3} \frac{\hat{N}}{\Omega} K \pi^{tij} - 2\mu_y \hat{N} \left[\frac{\hat{\nabla}^i \hat{\nabla}^j \Omega}{\Omega} - \frac{1}{3} \frac{\delta^{ij} \Delta_\gamma \Omega}{\Omega} \right] \end{aligned}$$

where $\Delta_\gamma = \gamma^{mn} \hat{\nabla}_m \hat{\nabla}_n$

Plan of Action

[1] Solve constraints (numerically) on CMC slices that extend to \mathcal{I}^+

(i) Caltech-Cornell SPEC code already adapted for this by Luisa Buchman

(ii) as a test case, generalize Bowen-York data set to CMC

[2] Solve (linear) elliptic equations for γ_{ij}, N, γ^k

(apparently less demanding than [1])

[3] write a code for the (central) 3+1 evolution equations for

$$\{\gamma_{ij}, \pi^{ij}\}$$

(i) subject to $(3)R(\gamma) = \text{const}$

$$V^k(\gamma) = \gamma^{ij} (\tilde{\Gamma}_{ij}^k(\gamma) - \tilde{\Gamma}_{ij}^k(\gamma^0)) = 0$$

(ii) propagate 'tree' degrees of freedom
i.e., conformal geometry

[4] Use linearized Schwarzschild
analysis as a guide to carrying
out [3] (nearly complete)

[5] Prove a theorem showing
above is mathematically sound