Solution of the Gravitational Wave Tensor Equation Using Spectral Methods

Jérôme Novak

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From Geometry to Numerics, November 21st 2006

Outline

1. Introduction
   - Maximally-constrained evolution scheme
   - Evolution Equation
   - Numerical Methods
2. Divergence-free evolution of a vector
   - Pure-spin vector spherical harmonics
   - Differential operators in terms of new potentials
   - New system for time evolution
3. Divergence-free evolution of a symmetric tensor
   - Method
   - Results

Introduction
Constrained evolution
Evolution Equation
Numerical Methods
Vector Evolution
Spherical Harmonics
PDEs
Time Evolution
Tensor Evolution
Method
Results
Summary

Outline

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   - Numerical Methods
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Flat metric and Dirac gauge
Following Bonazzola *et al.* (2004)

Conformal 3+1 (a.k.a BSSN) formulation, but use of f_{ij} (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of γ_{ij}, and $\mathcal{D}_i$ the associated covariant derivative.

Conformal factor Ψ

$$\tilde{\gamma}_{ij} := \Psi^{-4}\gamma_{ij} \text{ with } \Psi := \left(\frac{\gamma}{f}\right)^{1/12}, \text{ so } \det \tilde{\gamma}_{ij} = f$$

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

Generalization the gauge introduced by Dirac (1959) to any type of coordinates:

Divergence-free condition on $\tilde{\gamma}^{ij}$

$$\mathcal{D}_j\tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

+ Maximal slicing ($K = 0$)

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Einstein equations
Dirac gauge and maximal slicing

Constraint Equations

$$\begin{aligned} \Delta\Psi &= \mathcal{S}_{\rm Ham}, \\ \Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) &= \mathcal{S}_{\rm Mom}. \end{aligned}$$

Trace of dynamical equations

$$\Delta N = \mathcal{S}_K$$

Dynamical equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_\beta \frac{\partial h^{ij}}{\partial t} + \pounds_\beta \pounds_\beta h^{ij} = \mathcal{S}^{ij}_{\rm Dyn}$$

Introduction
Constrained evolution
Evolution Equation
Numerical Methods
Vector Evolution
Spherical Harmonics
PDEs
Time Evolution
Tensor Evolution
Method
Results
Summary

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Evolution Equation

Position of the problem

- Wave-like equation for a symmetric tensor: 6 components - 3 Dirac gauge conditions - $(\det \tilde{\gamma}^{ij} = 1)$ $\Rightarrow$2 degrees of freedom
- Work with $h = f_{ij} h^{ij}$ which has a given value: the condition $(\det \tilde{\gamma}^{ij} = 1)$ - non-linear condition is imposed with an iteration on h;
- the evolution operator appearing is not, in general, hyperbolic (complex eigenvalues); with the Dirac gauge, it is (result by I. Cordero).

Simplified numerical problem:

- solve a flat wave equation for a symmetric tensor $\Box h^{ij} = S^{ij}$,
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Introduction
Constrained evolution
Evolution Equation
Numerical Methods
Vector Evolution
Spherical Harmonics
PDEs
Time Evolution
Tensor Evolution
Method
Results
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Introduction
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- Evolution Equation
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Vector Evolution
- Spherical Harmonics
- PDEs
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Tensor Evolution
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- Results

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Introduction
Constrained evolution
Evolution Equation
Numerical Methods

Vector Evolution
Spherical Harmonics
PDEs
Time Evolution

Tensor Evolution
Method
Results

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Introduction
Constrained evolution
Evolution Equation
Numerical Methods

Vector Evolution
Spherical Harmonics
PDEs
Time Evolution

Tensor Evolution
Method
Results

Summary

Use of spherical coordinates:

- The radial part of a scalar field ϕ is decomposed on a set of orthonormal polynomials (here Chebyshev);
- The angular part is decomposed on a set of spherical harmonics $Y_\ell^m(\theta, \varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

$\Delta\phi = \sigma$	$\Box\phi = \sigma$
$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\phi_{\ell m}(r) = \sigma_{\ell m}(r)$	$\left[1 - \frac{\delta t^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\right]\phi^{J+1}_{\ell m} = \sigma^J_{\ell m}$
Accuracy on the solution $\sim 10^{-13}$ (exponential decay)	Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell, m)$ the operator inversion $\iff$ inversion of a $\sim 30 \times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

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Solutions of Poisson and wave equations

numerical library LORENE http://www.lorene.obspm.fr

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Vector spherical harmonics

Following *e.g.* Thorne (1980)

A 3D vector field $\boldsymbol{V}$ can be decomposed onto a set of vector spherical harmonics

$$\boldsymbol{V} = \sum_{\ell,m} R_{\ell m}(r)\boldsymbol{Y}^R_{\ell m}(\theta,\varphi) + E_{\ell m}(r)\boldsymbol{Y}^E_{\ell m}(\theta,\varphi) + B_{\ell m}(r)\boldsymbol{Y}^B_{\ell m}(\theta,\varphi),$$

- pure spin vector harmonics,
- orthonormal set of regular angular functions,
- not eigenfunctions of vector angular Laplacian

$$\boldsymbol{Y}^R_{\ell m} \propto Y_{\ell m}\boldsymbol{r}, \quad \text{(longitudinal)}$$
$$\boldsymbol{Y}^E_{\ell m} \propto \boldsymbol{\mathcal{D}} Y_{\ell m}, \quad \text{(transverse)}$$
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$V^r = \sum R_{\ell m}(r) Y_{\ell m}(\theta,\varphi)$, and we define two other potentials

$$V^\theta = \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi},$$
$$V^\varphi = \frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta};$$

$$\eta(r,\theta,\varphi) = \sum_{\ell,m} E_{\ell m}(r) Y_{\ell m},$$
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Differential operators in terms of new potentials

Flat wave operator $\Box V^i = S^i$ (divergence-free case)

$$
\begin{aligned}
-\frac{\partial^2 V^r}{\partial t^2} + \Delta V^r + \frac{2}{r}\frac{\partial V^r}{\partial r} + \frac{2V^r}{r^2} &= S^r, \\
-\frac{\partial^2 \eta}{\partial t^2} + \Delta \eta + \frac{2}{r}\frac{\partial V^r}{\partial r} &= \eta_S, \\
-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu &= \mu_S.
\end{aligned}
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Divergence-free condition $D_i V^i = 0$

$$
\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0
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... thus μ does not depend on the divergence of V.

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Differential operators in terms of new potentials

Flat wave operator $\Box V^i = S^i$ (divergence-free case)

$$
\begin{aligned}
-\frac{\partial^2 V^r}{\partial t^2} + \Delta V^r + \frac{2}{r}\frac{\partial V^r}{\partial r} + \frac{2V^r}{r^2} &= S^r, \\
-\frac{\partial^2 \eta}{\partial t^2} + \Delta \eta + \frac{2}{r}\frac{\partial V^r}{\partial r} &= \eta_S, \\
-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu &= \mu_S.
\end{aligned}
$$

Divergence-free condition $\mathcal{D}_i V^i = 0$

$$
\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0
$$

... thus μ does not depend on the divergence of $\boldsymbol{V}$.

Helmholtz decomposition

Any vector field $\boldsymbol{V}$ on $\mathbb{R}^3$, twice continuously differentiable and with rapid enough decay at infinity can be uniquely written as

$$\boldsymbol{V} = \tilde{\boldsymbol{V}} + \boldsymbol{\mathcal{D}}\phi, \text{ with } \mathcal{D}_i \tilde{V}^i = 0.$$

from $\boldsymbol{\mathcal{D}} \times \boldsymbol{V} = \boldsymbol{\mathcal{D}} \times \tilde{\boldsymbol{V}}$, one gets

$$\mu_V = \mu_{\tilde{V}} \text{ (twice: } r\text{- and } \eta\text{- components)},$$

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⇒the quantities

$$A = \frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r}$$

and μ are not sensitive to the gradient part of a vector.

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Evolution equations

ensuring divergence-free condition...

From the definition of A and the expression of the wave operator for a vector, one gets for the source ($\Box V^i = S^i$)

$$A_S = \frac{\partial \eta_S}{\partial r} + \frac{\eta_S}{r} - \frac{S^r}{r},$$

and

$$\Box A(V) = A_S$$

once A is known, one can reconstruct the vector V^i from

$$\frac{\partial \eta}{\partial r} + \frac{\eta}{r} - \frac{V^r}{r} = A_V,$$

$$\frac{\partial V^r}{\partial r} + \frac{2V^r}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0 \text{ divergence-free condition.}$$

and μ (since $\Box \mu = \mu_S$).

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Integration procedure

1. from S^i compute A_S and μ_S,
2. solve the equation for μ,
3. solve the equation for A,
4. solve the coupled system given by the divergence-free condition and the definition of A to get V^r and η,
5. reconstruct V^i from V^r, η and μ.

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Tensor spherical harmonics

A 3D symmetric tensor field $\boldsymbol{h}$ can be decomposed onto a set of tensor pure spin spherical harmonics and one can get 6 scalar potentials to represent the tensor:

$\boldsymbol{T}^{L_0}$	$\boldsymbol{T}^{T_0}$	$\boldsymbol{T}^{E_1}$	$\boldsymbol{T}^{B_1}$	$\boldsymbol{T}^{E_2}$	$\boldsymbol{T}^{B_2}$
h^{rr}	$\tau = h^{\theta\theta} + h^{\varphi\varphi}$	η	μ	W	X

with the following relations:

$$h^{r\theta} = \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi},$$

$$h^{r\varphi} = \frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta},$$

$$\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} = \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial W}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial \varphi^2} - 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial X}{\partial \varphi}\right),$$

$$h^{\theta\varphi} = \frac{\partial^2 X}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial X}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 X}{\partial \varphi^2} + 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial W}{\partial \varphi}\right).$$

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\frac{h^{\theta\theta} - h^{\varphi\varphi}}{2} &= \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{\tan\theta}\frac{\partial W}{\partial \theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 W}{\partial \varphi^2} - 2\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial X}{\partial \varphi}\right),\\
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\end{aligned}
$$

Differential operators

Divergence-free condition $H^i = \mathcal{D}_j h^{ij} = 0$

$$H^r = \frac{\partial h^{rr}}{\partial r} + \frac{2h^{rr}}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta - \frac{\tau}{r} = 0,$$

$$H^\eta = \frac{\partial \eta}{\partial r} + \frac{3\eta}{r} + \left(\Delta_{\theta\varphi} + 2\right)\frac{W}{r} + \frac{\tau}{2r} = 0,$$

$$H^\mu = \frac{\partial \mu}{\partial r} + \frac{3\mu}{r} + \left(\Delta_{\theta\varphi} + 2\right) X = 0;$$

"Electric type" potentials

h^{rr}, τ, η, W

"Magnetic type"

μ, X

$\Rightarrow$ two groups of coupled equations for the wave operator.

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Divergence-free part of a symmetric tensor

As for the Helmholtz decomposition:

$$h^{ij} = \tilde{h}^{ij} + \mathcal{D}^i V^j + \mathcal{D}^j V^i$$

... but no possibility to use the curl operator on a symmetric tensor!

3 degrees of freedom for $\tilde{h}$

$$\begin{aligned} A &= \frac{\partial X}{\partial r} - \frac{\mu}{r}, \\ B &= \frac{\partial W}{\partial r} - \frac{1}{2r}\Delta_{\theta\varphi} W - \frac{\eta}{r} + \frac{\tau}{4r}, \\ C &= \frac{\partial \tau}{\partial r} - \frac{2h^{rr}}{r} - 2\Delta_{\theta\varphi}\left(\frac{\partial W}{\partial r} + \frac{W}{r}\right) \end{aligned}$$

Wave equation $\Box h^{ij} = S^{ij}$

$$\begin{aligned} \Box A &= A_S, \\ \Box B + \frac{C}{2r^2} &= B_S, \\ \Box C - \frac{2C}{r^2} - \frac{8\Delta_{\theta\varphi}B}{r^2} &= C_S. \end{aligned}$$

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Divergence-free evolution

Define ℓ by ℓ

$$\tilde{B}_{\ell m} = 2B_{\ell m} + \frac{C_{\ell m}}{2(\ell+1)},$$

$$\tilde{C}_{\ell m} = 2B_{\ell m} - \frac{C_{\ell m}}{2\ell};$$

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In the case where $f_{ij}h^{ij} = h$ is given ($h^{rr} = h - \tau$):

1. compute A_S and $\tilde{B}_S$,
2. solve wave equations for A and $\tilde{B}$ (a wave operator shifted in ℓ),
3. solve the system composed of
   - definition of A
   - $H^\mu = 0$ (Dirac gauge)

   on the one hand, and
   - definition of $\tilde{B}$
   - $H^r = 0$
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   on the other hand,
4. recover the tensor components.

Tensor Wave Equation
Jérôme Novak
Introduction
Constrained evolution
Evolution Equation
Numerical Methods
Vector Evolution
Spherical Harmonics
PDEs
Time Evolution
Tensor Evolution
Method
Results
Summary

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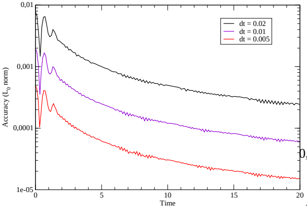
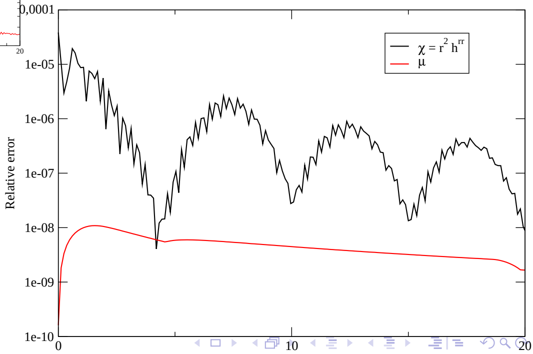
Initial data: Gaussian profile for h^{rr} and μ, with $\ell = 2$ and $\ell = 3$ modes. Evolution compared to the method of Bonazzola *et al.* (2004)

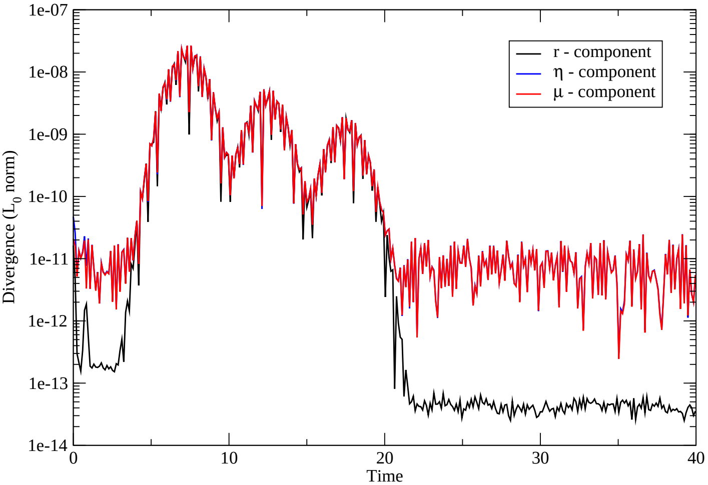


$\Box h^{ij} = 0$, with
$\mathcal{D}_j h^{ij} = 0$ and
$\det f^{ij} + h^{ij} = 1$
$dt = 0.02$, $R = 20$.
4 domains with 33 points in each.



Numerical Tests

Is the solution divergence-free?



Summary and outlook

- Algorithm to solve the tensor wave equation, ensuring the divergence-free condition,
- For a given value of the trace, solve only for two scalar wave equations,
- Designed for spectral methods in spherical coordinates (gain in CPU).
- Test it with the full Einstein equations,
- Take into account the full linear operator (with the "shift advection"),
- Evolution of one black hole,
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References

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Inversion formulas

$$\Delta_{\theta\varphi}\eta = \left(\frac{\partial h^{r\theta}}{\partial\theta} + \frac{h^{r\theta}}{\tan\theta} + \frac{1}{\sin\theta}\frac{\partial h^{r\varphi}}{\partial\varphi}\right)$$

$$\Delta_{\theta\varphi}\mu = \left(\frac{\partial h^{r\varphi}}{\partial\theta} + \frac{h^{r\varphi}}{\tan\theta} - \frac{1}{\sin\theta}\frac{\partial h^{r\theta}}{\partial\varphi}\right),$$

$$\Delta_{\theta\varphi}\left(\Delta_{\theta\varphi}+2\right)W = \frac{\partial^2 P}{\partial\theta^2} + \frac{3}{\tan\theta}\frac{\partial P}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 P}{\partial\varphi^2} - 2P + \frac{2}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial h^{\theta\varphi}}{\partial\theta} + \frac{h^{\theta\varphi}}{\tan\theta}\right),$$

$$\Delta_{\theta\varphi}\left(\Delta_{\theta\varphi}+2\right)X = \frac{\partial^2 h^{\theta\varphi}}{\partial\theta^2} + \frac{3}{\tan\theta}\frac{\partial h^{\theta\varphi}}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 h^{\theta\varphi}}{\partial\varphi^2} - 2h^{\theta\varphi} - \frac{2}{\sin\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial P}{\partial\theta} + \frac{P}{\tan\theta}\right).$$