Applications of gluing constructions in General Relativity

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What is "Gluing"?

Gluing refers to a class of constructions in geometric analysis for combining known solutions of nonlinear partial differential equations to obtain new solutions. This is often done with a topological modification of the underlying manifold on which the solution lives; the simplest example is the "connected sum" operation.

The underlying connected sum can lead to two distinct constructions which are depicted in the cartoon on the following slide

- Figure 1: "Wormhole" construction. There is only one summand, the underlying topology is altered by adding a neck connecting two points.
- Figure 2: The connected sum of two distinct disconnected summands (notation: $\Sigma_1 \# \Sigma_2$).

Gluing is a standard technique in geometric analysis

Examples where it has played an important role include:

- Existence of anti-self-dual connections on 4-manifolds (Taubes)
- Donaldson & Seiberg-Witten invariants (Taubes, Kronheimer, Morgan, Mrowka)
- Psuedo-holomorphic curves and Gromov-Witten invariants (Gromov, Tian, Ruan, Taubes Parker, Ionel)
- Manifolds with exceptional holonomy (Joyce)
- Metrics of constant scalar curvature (Schoen, Joyce, Mazzeo, Pacard, Pollack, Mazzieri)
- Surfaces of constant mean curvature in $\mathbb{R}^3$ (Kapouleas, Mazzeo, Pacard, Pollack)
- Minimal surfaces (Kapouleas, Mazzeo, Pacard, Traizet)
- Special Lagrangian submanifolds (Joyce, Lee, Butscher, Haskins, Kapouleas)
- Kähler manifolds with constant scalar curvature & extremal Kähler metrics (Arrezo, Pacard, Singer)

General remarks regarding gluing constructions

- Gluing is a "perturbation" technique and as such it usually involves a hypothesis concerning the surjectivity of the linearization of the relevant equations about the known solutions ("nondegeneracy").
- In all the examples listed, the relevant equations are *elliptic*.
- Often a gluing construction has a free parameter (e.g. "neck size"). In the limit, as the parameter tends to zero, the construction yields either the original known solutions or a singular version of these.
- Prior to applications in GR, all known gluing constructions involved a *global perturbation*. Away from the neck (where the connected sum takes place) one could prove that the new solution was only a small deformation of the original ones.
  - The presence of this global perturbation is a reflection of the underlying equations satisfying a unique continuation property.

Initial data for the Cauchy problem in General Relativity

- To formulate a gluing result for solutions of the Einstein field equations

$$Ric(g) - \frac{1}{2}R(g)g = T$$

(which are, up to a choice of gauge, hyperbolic) we begin with solutions to the corresponding system of constraint equations.

- The initial data on an n-dimensional manifold Σ consists of
  - a Riemannian metric $\bar{\gamma}$
  - a symmetric 2-tensor $\bar{K}$
  - $\mathcal{F}$ a collection of initial data for the non-gravitational fields.

Einstein constraint equations

- In terms of this data $(\bar{\gamma}, \bar{K}, \mathcal{F})$, the Einstein constraint equations are

$$\begin{aligned} \mathrm{div}_{\bar{\gamma}} \bar{K} - \nabla(\mathrm{tr}\, \bar{K}) &= J(\bar{\gamma}, \mathcal{F}) && \text{(Momentum constraint)} \\ R(\bar{\gamma}) - |\bar{K}|^2_{\bar{\gamma}} + (\mathrm{tr}\, \bar{K})^2 &= 2\rho(\bar{\gamma}, \mathcal{F}) && \text{(Hamiltonian constraint)} \\ C(\bar{\gamma}, \mathcal{F}) &= 0 && \text{(Non-gravitational constraints)} \end{aligned}$$

This is a highly underdetermined system of equations.

- For vacuum data ($\rho = 0 = J$ and no non-gravitational constraints) in $3+1$ dimensions this is 4 equations for 12 unknowns.
- This observation foreshadows a surprising degree of flexibility in constructing solutions (cf. Corvino, Chruściel-Delay, Corvino-Schoen, Chruściel-Isenberg-Pollack). It is here that we see an absence of the unique continuation property for the Einstein constraint equations.

The conformal method (aprés Lichnerowicz, Choquet-Bruhat and York)

Split the initial data into two parts

- "conformal data": regard as being freely chosen.
- "determined data": found by solving the constraint equations, reformulated as a determined system of elliptic PDE.

General Criteria: For constant mean curvature (CMC) initial data, where $\tau = \mathrm{tr}_{\bar{\gamma}} \bar{K}$ is constant, we want the equations to be "semi-decoupled":

- First solve the nongravitational constraints.
- Then solve the conformally formulated momentum constraint.
- These solutions enter into the conformally formulated Hamiltonian constraint, which we solve for the remaining piece of determined data.

conformal and determined data (vacuum case)

For the gravitational (vacuum) data, the free "conformal data" consists of

- γ, a Riemannian metric on Σ, representing a chosen *conformal class of metrics* $[\gamma] = \{\tilde{\gamma} = \theta^{\frac{4}{n-2}}\gamma : \theta > 0\}$.
- $\sigma = \sigma_{ab}$, a symmetric tensor which is divergence-free and trace-free w.r.t. γ (σ is a transverse-traceless or TT-tensor).
- τ, a scalar function representing the mean curvature of the Cauchy surface Σ in the resulting spacetime.

The "determined data" consists of

- ϕ, a positive function
- $W = W^a$, a vector field

Reconstructed data (vacuum case)

Use (ϕ, W) to reconstruct an initial data set $(\bar{\gamma}, \bar{K})$ from the conformal data set (γ, σ, τ) via:

$$\begin{aligned} \bar{\gamma} &= \phi^{\frac{4}{n-2}}\gamma \\ \bar{K} &= \phi^{-2}(\sigma + \mathcal{D}W) + \frac{\tau}{n}\phi^{\frac{4}{n-2}}\gamma \end{aligned}$$

here the operator $\mathcal{D}$ is the conformal Killing operator relative to γ.

$(\bar{\gamma}, \bar{K})$ satisfy the vacuum constraint equations if and only if (ϕ, W) satisfy

$$\boxed{\begin{gathered} \mathrm{div}(\mathcal{D}W) = \tfrac{n}{n-1}\phi^{\frac{2n}{n-2}}\nabla\tau \\ c_n^{-1}\Delta_\gamma\phi - R(\gamma)\phi + \left(|\sigma + \mathcal{D}W|_\gamma^2\right)\phi^{-\frac{3n-2}{n-2}} - \frac{n-1}{n}\tau^2\phi^{\frac{n+2}{n-2}} = 0 \end{gathered}}$$

where $c_n = \frac{n-2}{4(n-1)}$.

Conformal gluing constructions (vacuum with CMC data)

Our initial gluing constructions for the constraint equations were in the context of the conformal method as described above. This allowed us to perform either a connected sum or a wormhole construction in either of the following circumstances:

- For compact summands, we require that $\bar{K} \neq 0$ and that there do not exist conformal Killing fields which vanish at the points about which we wish to glue. (This is our "nondegeneracy" condition)
- For asymptotically flat or asymptotically hyperbolic summands we do not require any nondegeneracy conditions

Subsequently we showed how to relax the globally CMC requirement and only required the data to be CMC near the gluing points. Since in this setting the system does not semi-decouple this requires an nondegeneracy assumption on the surjectivity of the full linearized system obtained by the conformal method. This may be verified to hold in the neighborhood of CMC data solutions.

Applications I

- There are no restrictions on the spatial topology of *asymptotically hyperbolic* solutions of the vacuum Einstein constraint equations.
- One may add black holes or wormholes to any spacetime with a CMC Cauchy surface (indicated by a marginally trapped surfaces)
  - Chruściel-Mazzeo verified the existence of spacetime developments whose event horizons have multiple connected components
- There are no restrictions on the spatial topology of *asymptotically flat* solutions of the vacuum Einstein constraint equations.
  - Requires the latter construction without the globally CMC hypothesis
- In subsequent work with Isenberg & Maxwell we extended the conformal CMC gluing construction to higher dimensions and non-vacuum data (e.g. Einstein-Maxwell, Yang-Mills, Vlasov, fluids)

Corvino gluing

The earliest applications of gluing constructions to GR were given in Justin Corvino's 2000 PhD thesis. He demonstrated a different type of construction, initially working with time symmetric, asymptotically flat vacuum data (i.e. asymptotically flat, scalar flat metrics) he

- Performed a gluing construction which replaces a neighborhood of infinity with an exact slice of Schwarzschild
- Worked directly with the underdetermined constraint equation $R(\gamma) = 0$
- Was able to perform his perturbation with *compact support* within a large annulus. i.e. the original asymptotically flat data was left completely unchanged on an arbitrarily large compact set.

This lead to the remarkable result

Theorem (J. Corvino (2000))

There exist a large class of globally hyperbolic vacuum spacetimes which are Schwarzschild at spatial infinity.

Local gluing constructions: initial data engineering

By combining the conformal (IMP) gluing construction with extensions of the Corvino technique due to Chruściel and Delay, we are able to establish a *local* gluing construction for the Einstein constraint equations.

Definition

Let $(\Sigma, \gamma.K)$ be a set of initial data satisfying the Einstein vacuum constraint equations, and let $p \in \Sigma$ and let U be an open set containing p. The data has *No KIDs in U* if there do not exist non trivial solutions (N, Y) to the formal adjoint of the linearized constraint equations:

$$0 = \begin{pmatrix} 2(\nabla_{(i}Y_{j)} - \nabla^l Y_l \gamma_{ij} - K_{ij}N + \mathrm{tr}\, K \; N\gamma_{ij}) \\ \\ \nabla^l Y_l K_{ij} - 2K^l{}_{(i}\nabla_{j)}Y_l + K^q{}_l \nabla_q Y^l g_{ij} - \Delta N \gamma_{ij} + \nabla_i \nabla_j N \\ +(\nabla^p K_{lp}\gamma_{ij} - \nabla_l K_{ij})Y^l - N\,\mathrm{Ric}(\gamma)_{ij} \\ +2NK^l{}_i K_{jl} - 2N(\mathrm{tr}\; K)K_{ij} \end{pmatrix}$$

in U.

Local gluing constructions (continued)

- KIDs in U are in one-to-one correspondence with Killing fields within the domain of dependence of U in the spacetime development of the data (Montcrief)
- Under generic perturbations KIDs are absent in every open $U \subset \Sigma$ (Bieg-Chruściel-Schoen)
- The "no KIDs" condition will serve as our nondegeneracy assumption

Theorem (Chruściel-Isenberg-Pollack (2005))

Let $(\Sigma_1, \gamma_1, K_1)$ and $(\Sigma_2, \gamma_2, K_2)$ be a pair of smooth initial data sets which satisfy the vacuum ($\rho = 0$ and $J = 0$) constraint equations. Let $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ be a pair of points, with open neighborhoods $p_1 \in U_1$ and $p_2 \in U_2$ in which the No KIDs condition is satisfied. There exists a smooth data set $(\Sigma_1 \# \Sigma_2, \hat{\gamma}, \hat{K})$ which satisfies the Einstein constraint equations everywhere, and which agrees with (γ_1, K_1) and (γ_2, K_2) away from $U_1 \cup U_2$. (the "neck" connecting Σ_1 and Σ_2).

Applications II (of the *local* gluing construction)

Remarks

- We have stated the connected sum version of the construction. One may also add (local) wormholes into given initial data sets.
- We can also allow for general non-vacuum data satisfying a strict dominant energy condition (which takes the place of the No KIDs assumption). The new glued solutions also satisfy the dominant energy condition but we do not control any additional equations which the non-gravitation fields may satisfy.

The main application of this construction thus far is the existence of spacetimes with no CMC slices:

Corollary

There exist vacuum, maximally extended, spacetimes with compact Cauchy surfaces, which contain no compact, spacelike hypersurfaces with constant mean curvature.

Remarks and work in progress

- The local gluing construction applies to generic initial data sets. Beyond the No KIDs assumption, no global conditions such as compactness, completeness, or asymptotic conditions are imposed.
- Results can likely be extended to non-vacuum models, e.g. Einstein-Maxwell, Yang-Mills, etc. The main issue is the assertion of a local Corvino type perturbation result for the relevant model.
- The prevalence of spacetimes with no CMC slices is largely open.
- Recently Mazzieri has provided a gluing construction for metrics of constant positive scalar curvature where, in dimensions $n > 3$, one may glue along isometrically embedded submanifolds of codimension $k \geq 3$. We are working to extend this construction to the constraint equations. This will lead to a flexible construction of "black string" spacetimes (generalizing the construction, by Emperan & Real, of a stationary $4 + 1$ spacetime whose horizon has topology $\mathbb{S}^1 \times \mathbb{S}^2$).