The best chirplet chain for the detection of gravitational wave chirps

Éric Chassande-Mottin

CNRS, AstroParticule et Cosmologie (Paris)
and Observatoire de la Côte d'Azur (Nice) FRANCE

with Archana Pai   Albert Einstein Institute (Golm) GERMANY

November 24, 2006

Outline

1. *Introduction*
   - Targeted search and matched filtering
   - Exploratory searches, why?
   - GW unmodeled chirps: motivations and generic model
2. *Methodologies for chirp detection: A review*
   - Time-frequency heuristics
   - Quadrature matched filtering
3. *Conciliate both viewpoints*
   - Chirplet chains, Phys. Rev. D73, 042003, 2006

Targeted search and matched filtering

notation: sampled signals, $\mathbf{x} \equiv \{x_k \equiv x(k/f_s), k = 0 \ldots N-1\}$

detection = decide which hypothesis fits the data

Targeted search and matched filtering

notation: sampled signals, $\mathbf{x} \equiv \{x_k \equiv x(k/f_s), k = 0 \ldots N-1\}$

detection = decide which hypothesis fits the data

$$(H_0) \quad x_k = n_k \quad \text{white Gaussian noise}$$

$$(H_1) \quad x_k = s_k + n_k \quad \text{signal+noise}$$

define a *statistic* $\lambda(\mathbf{x}) \lesseqgtr \eta \leftrightarrow$ choose H_0 or H_1, partition (here, of $\mathbb{R}^N$)

Targeted search and matched filtering

notation: sampled signals, $\mathbf{x} \equiv \{x_k \equiv x(k/f_s), k = 0 \dots N-1\}$

detection = decide which hypothesis fits the data

$$(H_0) \quad x_k = n_k \quad \text{white Gaussian noise}$$

$$(H_1) \quad x_k = s_k + n_k \quad \text{signal+noise}$$

define a *statistic* $\lambda(\mathbf{x}) \lessgtr \eta \leftrightarrow$ choose H_0 or H_1, partition (here, of $\mathbb{R}^N$)

which λ is best? criterion: Neymann-Pearson (NP), error prob.

minimize $\mathbb{P}(\lambda(\mathbf{x}) < \eta | H_1)$ for a given $\mathbb{P}(\lambda(\mathbf{x}) > \eta | H_0)$

$$\text{solution} = \text{likelihood ratio } \lambda(\mathbf{x}) = \frac{\mathbb{P}(\mathbf{x}|H_1)}{\mathbb{P}(\mathbf{x}|H_0)}$$

Targeted search and matched filtering

notation: sampled signals, $\mathbf{x} \equiv \{x_k \equiv x(k/f_s), k = 0 \dots N-1\}$

detection = decide which hypothesis fits the data

$$(H_0) \quad x_k = n_k \quad \text{white Gaussian noise}$$
$$(H_1) \quad x_k = s_k + n_k \quad \text{signal+noise}$$

define a *statistic* $\lambda(\mathbf{x}) \lessgtr \eta \leftrightarrow$ choose H_0 or H_1, partition (here, of $\mathbb{R}^N$)

which λ is best? criterion: Neymann-Pearson (NP), error prob.

minimize $\mathbb{P}(\lambda(\mathbf{x}) < \eta | H_1)$ for a given $\mathbb{P}(\lambda(\mathbf{x}) > \eta | H_0)$

solution = likelihood ratio $\lambda(\mathbf{x}) = \dfrac{\mathbb{P}(\mathbf{x}|H_1)}{\mathbb{P}(\mathbf{x}|H_0)}$

for our problem, $\log(\lambda(\mathbf{x})) \propto \|\mathbf{x} - \mathbf{s}\|_2^2 - \|\mathbf{x}\|_2^2$

simplify: $\ell(\mathbf{x}) = \langle \mathbf{x}, \mathbf{s} \rangle = \sum_{k=0}^{N-1} x_k s_k$

matched filter: correlation of the data with a template $\mathbf{s}$

Unknown parameters and bank of matched filters

when the signal $\mathbf{s}$ depends on unknown parameters $\mathbf{p}$ ...

$$\text{likelihood ratio: } \lambda(\mathbf{x};\mathbf{p}) = \frac{\mathbb{P}(\mathbf{x}|H_1,\mathbf{p})}{\mathbb{P}(\mathbf{x}|H_0)}$$

NP uniformly for all values of $\mathbf{p}$? no solution in general

Unknown parameters and bank of matched filters

when the signal **s** depends on unknown parameters **p** . . .

$$\text{likelihood ratio: } \lambda(\mathbf{x}; \mathbf{p}) = \frac{\mathbb{P}(\mathbf{x}|H_1, \mathbf{p})}{\mathbb{P}(\mathbf{x}|H_0)}$$

NP uniformly for all values of **p**? no solution in general

sub-optimal but works well: "generalized likelihood ratio test"

idea: replace **p** by an (ML) estimate $\hat{\mathbf{p}} = \mathrm{argmax}_{\mathbf{p}} \lambda(\mathbf{x}; \mathbf{p})$

two ways for doing this

1. if analytical expression $\hat{\mathbf{p}}(\mathbf{x})$ exists,
   replace $\ell(\mathbf{x}) = \lambda(\mathbf{x}; \hat{\mathbf{p}}(\mathbf{x}))$
2. if not, maximize numerically (exhaustive search):
   $\ell(\mathbf{x}) = \max_{\mathbf{p}} \lambda(\mathbf{x}; \mathbf{p})$

for our problem, this is a bank of matched filters

targeted search = matched filter bank obtained from Physics

Why exploratory searches?

- targeted search is sensitive, strength and also weakness
  - require reliable and precise model
  - does not incorporate model uncertainties
- data are precious (expensive!): get the most from them
  - look for speculative or unknown sources!
- moral: be exploratory! let the model be more "general" . . .
  - relax assumptions = increase robustness
- . . . but not too general! be "quasi-physical"
  - exclude non-feasible/unlikely candidates
  - use "good sense" assumption to restrict the model

note: *exploration* useful for detection, *not* for identification and interpretation which needs complete physical model!

GW unmodeled chirps: motivations

- *basic idea*: GW = system "radiates away its asymmetries"
  if *orbiting and slowly moving* $\rightarrow$ quasi-periodic GWs (=chirps)
  GW chirps are generic signatures of orbiting systems
- this information is *robust*: this remains true even we don't know the system dynamics in detailed.
- *consequence*: search for chirps in "general"

GW (unmodeled) chirps:generic model

- generic model for chirps
  GW chirps: $s(t) \equiv A \cos(\phi(t) + \varphi_0)$
  unknown amplitude A and initial phase φ_0

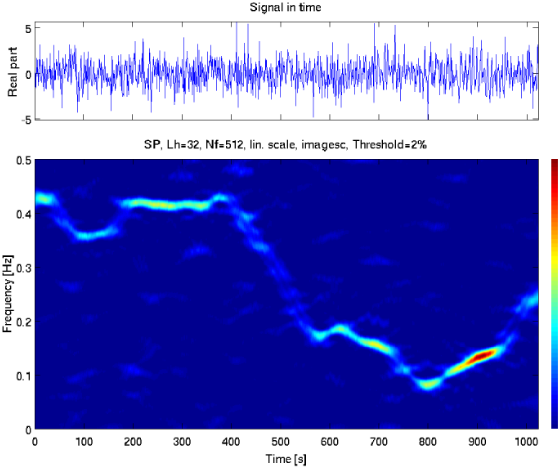
GW (unmodeled) chirps:generic model

- generic model for chirps
  GW chirps: $s(t) \equiv A\cos(\phi(t) + \varphi_0)$
  unknown amplitude A and initial phase φ_0
- unknown phase evolution $\phi(t)$
  exclude non-physical with "good sense" constraint:
  impose $|\dot{f}(t)| \leq F'$ and $|\ddot{f}(t)| \leq F''$ where $f(t) = (2\pi)^{-1}\dot{\phi}(t)$.

GW (unmodeled) chirps:generic model

- generic model for chirps
  GW chirps: $s(t) \equiv A\cos(\phi(t) + \varphi_0)$
  unknown amplitude A and initial phase φ_0
- unknown phase evolution $\phi(t)$
  exclude non-physical with "good sense" constraint:
  impose $|\dot{f}(t)| \leq F'$ and $|\ddot{f}(t)| \leq F''$ where $f(t) = (2\pi)^{-1}\dot{\phi}(t)$.
- typical duration $T \sim$ few sec in detector band

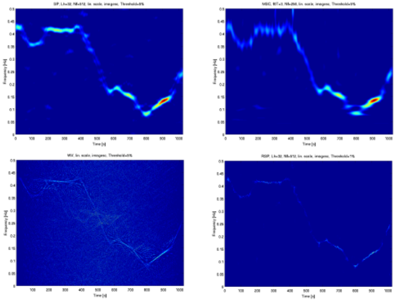
Chirps in the time-frequency plane (1)



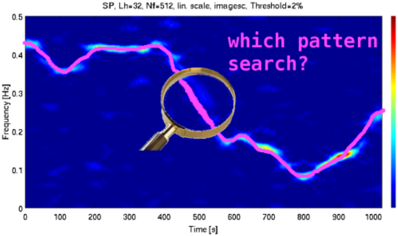
heuristic: chirp = "filiform" pattern in time-frequency plane

Two degrees of freedom (2)

which TF representation?
spectrogram, wavelets, Wigner-Ville, Cohen, reassignment, etc.



which pattern search?
Hough, “crazy climbers”, “snakes”, road tracker in satellite images, etc.



Multiple approaches. . . (3)

- Morvidone & Torrésani, *IJWMIP*, 2003
- Sylvestre, *Phys. Rev. D*, grqc/0210043
- Anderson & Balasubramanian, *Phys. Rev. D*, grqc/9905023
- Carmona, Hwang & Torrésani, *IEEE SP*, 1998
- Chassande-Mottin & Flandrin, *ACHA*, 1998
- Pinto et al., *Proc. of GWDAW*, 1997
- Innocent & Torrésani, *ACHA*, 1997

Chirps and quadrature matched filtering

let us apply generalized likelihood ratio test to chirps

we have 3 unknown parameters $\mathbf{p} = \{A, \varphi_0, \phi(t)\}$

two simple ones A, φ_0 = analytical replacement

$$\log(\max_{A,\phi_0} \lambda) \propto \left| \sum_{k=0}^{N-1} x_k \exp i\phi_k \right|^2 \equiv \ell(x, \phi) \lessgtr \eta$$

quadrature matched filtering

When chirp phase is not known. . . (2)

- last unknown parameter $\phi(t)$, how to do $\max_\phi \ell(x, \phi)$?

When chirp phase is not known... (2)

- last unknown parameter $\phi(t)$, how to do $\max_\phi \ell(x, \phi)$?
- analytical maximization impossible $\rightarrow$ numerical

When chirp phase is not known... (2)

- last unknown parameter $\phi(t)$, how to do $\max_\phi \ell(x, \phi)$?
- analytical maximization impossible $\rightarrow$ numerical
- sampling of the set of possible phase functions: template grids

When chirp phase is not known. . . (2)

- last unknown parameter $\phi(t)$, how to do $\max_{\phi} \ell(x, \phi)$?
- analytical maximization impossible $\rightarrow$ numerical
- sampling of the set of possible phase functions: template grids
- griding must be sufficiently tight, how to be sure?

When chirp phase is not known. . . (2)

- last unknown parameter $\phi(t)$, how to do $\max_{\phi} \ell(x, \phi)$?
- analytical maximization impossible $\rightarrow$ numerical
- sampling of the set of possible phase functions: template grids
- griding must be sufficiently tight, how to be sure?

we "receive" $x_k \hat{=} A \cos(\phi_k + \varphi_0)$ and we "search" with template ϕ^*_k

$$\text{distance:} \quad \Delta\ell(\phi, \phi^*) \equiv \frac{\ell(s, \phi) - \ell(s, \phi^*)}{\ell(s, \phi)}$$

the distance between two grid nodes should be small

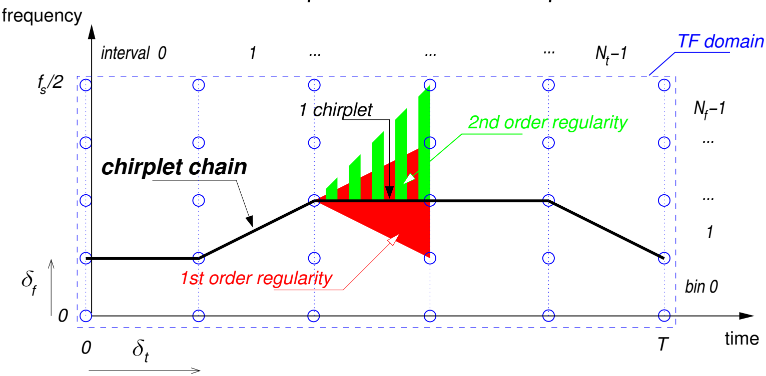
Conciliate viewpoints?

Does this method apply in general?

1. can we build a bank of matched filters for GW chirps?
2. with which templates?

Chirplet chains (CC), Phys. Rev. D73, 042003

CCs are piecewise linear chirps



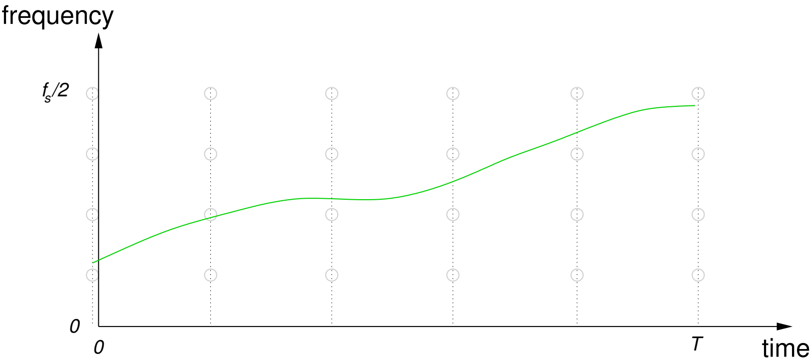
free parameters: N_t, N_f, N_r', N_r''

CCs form a tight template grid

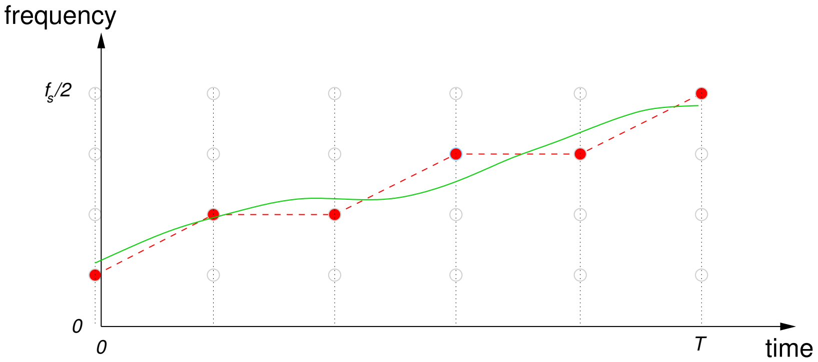
if N'_r and N''_r are large enough, for all smooth chirp ϕ, there exists a CC ϕ^* such that

$$\Delta\ell(\phi,\phi^*) \lesssim C\left[\frac{1}{2}\left(\frac{\sqrt{3F''T^3}}{N_t}\right)^2 + \frac{1}{2}\left(\frac{2N}{N_f}\right)\right]^2$$

CCs form a tight template grid



CCs form a tight template grid



CCs form a tight template grid

if N_r' and N_r'' are large enough, for all smooth chirp ϕ, there exists a CC ϕ^* such that

$$\Delta\ell(\phi,\phi^*) \lesssim C \left[\frac{1}{2}\left(\frac{\sqrt{3F''T^3}}{N_t}\right)^2 + \frac{1}{2}\left(\frac{2N}{N_f}\right)\right]^2$$

CC grid is tight!

CCs form a tight template grid

if N'_r and N''_r are large enough, for all smooth chirp ϕ, there exists a CC ϕ^* such that

$$\Delta\ell(\phi,\phi^*) \lesssim C \left[\frac{1}{2}\left(\frac{\sqrt{3F''T^3}}{N_t}\right)^2 + \frac{1}{2}\left(\frac{2N}{N_f}\right)\right]^2$$

CC grid is tight!

$\max_{\text{all GW chirps}}\{\ell\} \approx \max_{\text{all CCs}}\{\ell\}$

search over CCs? the number of CCs is finite!
. . . but *exponentially growing* with N_t (combinatorial)
CC grid is *too large* to be searched exhaustively!

best CC, step (1): maps to time-frequency

scalar products can be expressed in time or in frequency

$$\text{Parseval: } \int x(t)y^*(t)\,dt = \int X(f)Y^*(f)\,df$$

best CC, step (1): maps to time-frequency

scalar products can be expressed in time or in frequency

$$\text{Parseval: } \int x(t) y^*(t)\, dt = \int X(f) Y^*(f)\, df$$

or in time-frequency

$$\text{Moyal: } \left| \int x(t) y^*(t)\, dt \right|^2 = \iint W_x(t,f) W_y^*(t,f)\, dt\, df$$

best CC, step (1): maps to time-frequency

scalar products can be expressed in time or in frequency

$$\text{Parseval: } \int x(t)y^*(t)\,dt = \int X(f)Y^*(f)\,df$$

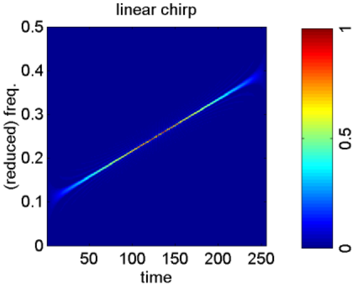
or in time-frequency

$$\text{Moyal: } \left|\int x(t)y^*(t)\,dt\right|^2 = \iint W_x(t,f)W_y^*(t,f)\,dt\,df$$

for discrete signals, discrete Wigner-Ville

$$\text{Moyal: } \quad \ell = \frac{1}{2N}\sum_n \sum_m W_x(n,m)W_e(n,m)$$

best CC, step (2): template WV is simple



W_e is almost Dirac $\approx \delta(m - m_n^{(cc)})$

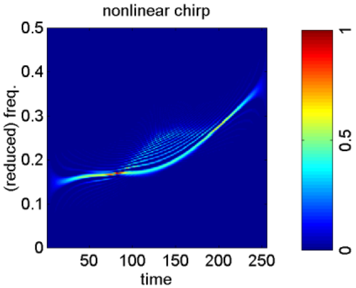
$$\ell \propto \sum_n \sum_m W_x(n,m) W_e(n,m)$$

path integral: $\ell \approx \sum_n W_x(n, m_n^{(cc)})$

$\max_\phi\{\ell\}$ is a longest TF path problem

dynamic programming solves this in polynomial time

best CC, step (2): template WV is simple



W_e is almost Dirac $\approx \delta(m - m_n^{(cc)})$

$\ell \propto \sum_n \sum_m W_x(n,m) W_e(n,m)$

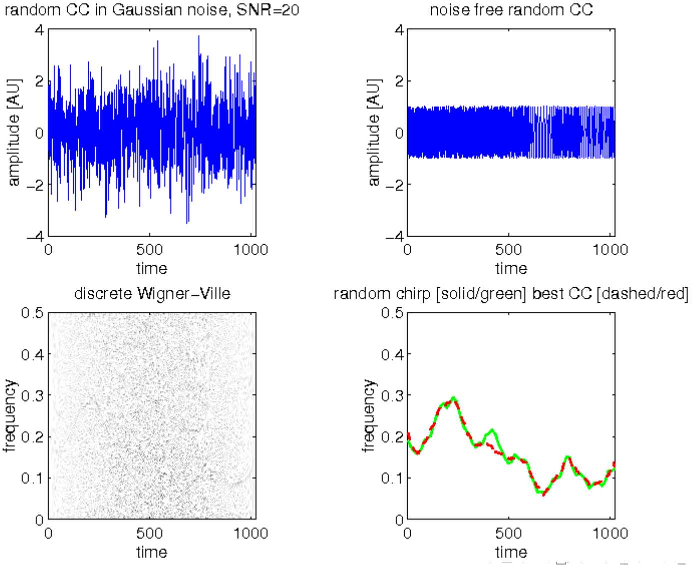
path integral: $\ell \approx \sum_n W_x(n, m_n^{(cc)})$

$\max_\phi\{\ell\}$ is a longest TF path problem

dynamic programming solves this in polynomial time

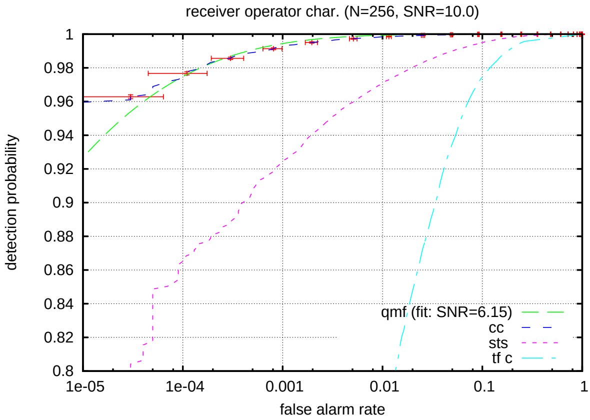
which TFR ? DWV which pattern search? largest path int. + DP

best CC: check



best CC: performance, ROCs (1)

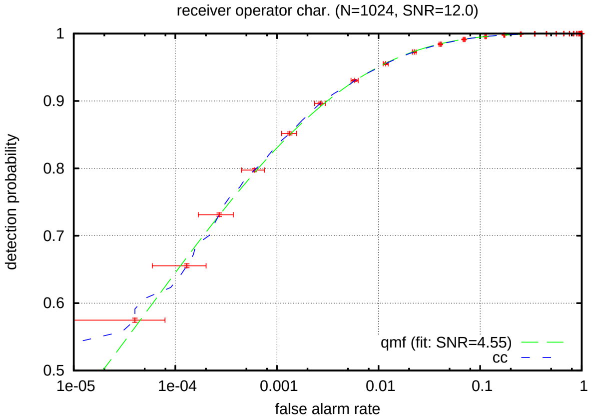
ROC: detection prob. *vs* false alarm



STS: Signal Track Search

TFC: TF Clusters

best CC: performance, ROCs (2)



"clairvoyant" observer knows incident chirp *a priori*

the SNR of "clairvoyant" observer is set such that ROC fits the other.

reduction factor in the sight distance wrt "clairvoyant" ≈ 2.6

Concluding remarks

- **best CC search**
- design a template grid which covers the entire set of ("regular") GW chirps
- use original time-frequency scheme to search efficiently through this grid
- robustness comes from the large size of this grid, *not* from specific property of time-frequency representation
- articles, codes and other resources available at `http://www.apc.univ-paris7.fr/~ecm`