Oscillations & Instabilities of Relativistic Stars

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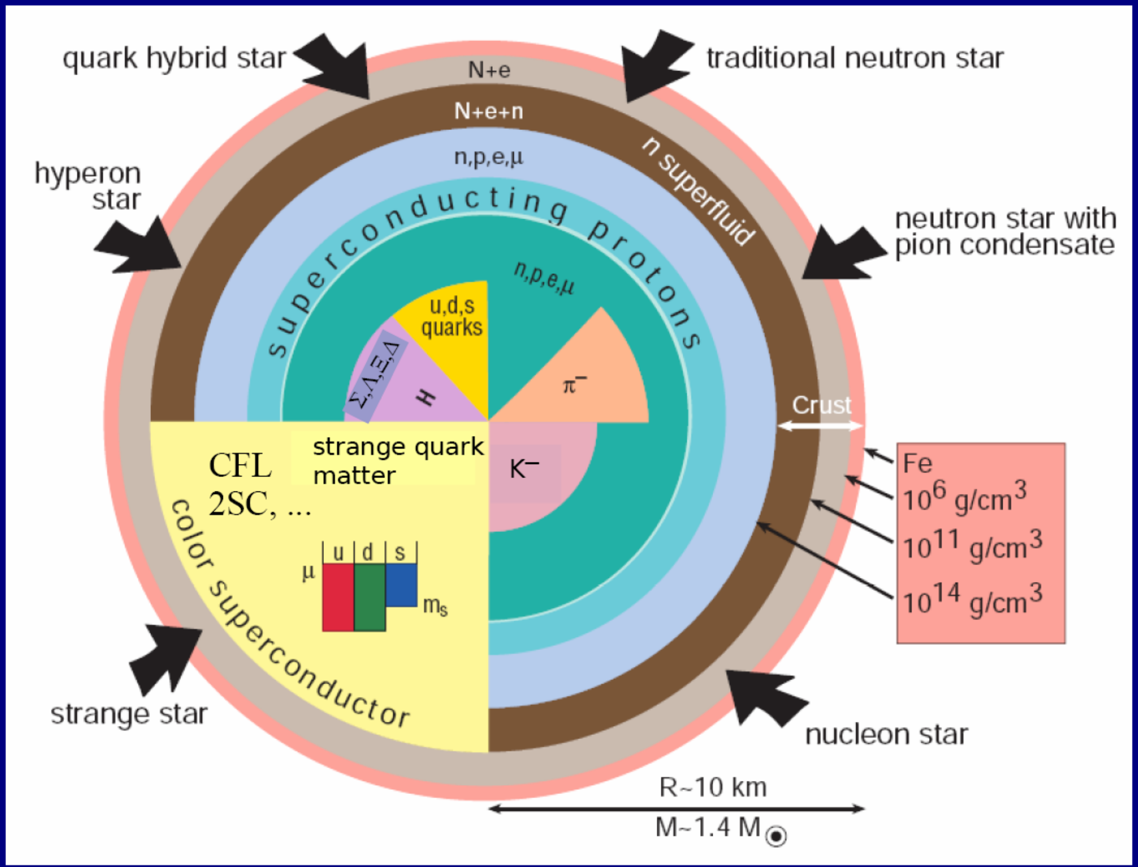
An extreme challenge

Neutron star modelling involves the very extremes of physics:



F. Weber, Prog. Part. Nucl. Phys. 54 (2005) 193-288

- **rapid (differential) rotation**
- **general relativity**
- **superfluidity**
- **strong magnetic fields**
- **crust-core interface Ekman/Alfven layer**
- **exotic nuclear physics strange quarks, hyperons**

Can GW, x-ray, γ-ray observations constrain the theoretical models?

Rotating Relativistic Stars

- Spacetime

$$ds^2 = -e^{\nu(r,\theta)}dt^2 + e^{\mu(r,\theta)}(dr^2 + r^2 d\theta^2) + e^{\psi(r,\theta)} r^2 \sin^2\theta (d\varphi - \omega(r,\theta)dt)^2$$

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) - 2\omega(r) r^2 \sin^2\theta dt d\varphi$$

- Energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

- Energy-momentum conservation + Einstein equations

$$\nabla_\mu T^{\mu\nu} = 0$$

$$R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

- EoS ***p=p(ρ,s,…)***

Slow rotation is not too bad!
P=1.5ms, R=10km, M=1.4$M_\odot$

$$\varepsilon = \Omega / \Omega_{\mathrm{Kepler}} \approx 0.3$$

Stellar Perturbation Theory

Take variations of Einstein's equations and the energy-momentum conservation

$$\delta \begin{pmatrix} \nabla_\mu T^{\mu\nu} = 0 \\ R_{\mu\nu} = k\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \end{pmatrix}$$

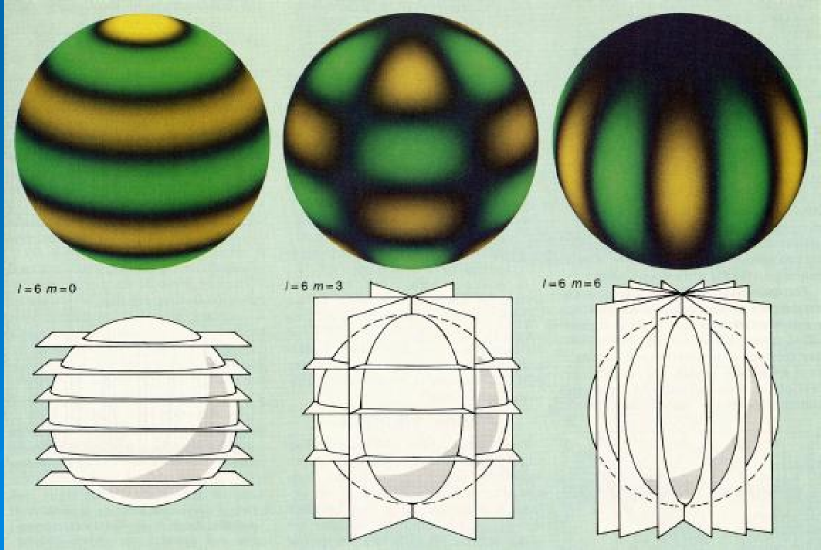
Assume small variation in pressure, density, fluid velocities and in the metric.

$$\delta P \sim \delta P(t,r) \cdot Y_m^l(\theta,\varphi)$$

$$\delta\rho \sim \delta\rho(t,r) \cdot Y_m^l(\theta,\varphi)$$

$$h_{\mu\nu} \sim h_{\mu\nu}(t,r) \cdot Y_m^l(\theta,\varphi)$$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}(t,r,\theta,\varphi)$$



Hydrodynamical Evolution

General-Relativistic Hydrodynamics:

$\nabla_a T^{ab} = 0$	Energy and momentum conservation
$\nabla_a (\rho u^a) = 0$	Baryon number conservation

1st-order hyperbolic form:

$$\partial_t \vec{U} + \partial_i \vec{F}^i = \vec{S}$$

$\vec{U}$	State vector
$\vec{F}^i$	Fluxes
$\vec{S}$	Sources

ρ	Rest-mass density
e	Specific int. energy
v^i	3-velocity

Primitive variables

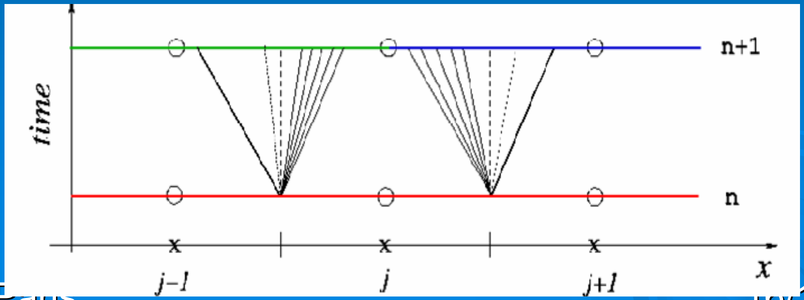
$$D = \sqrt{\gamma} W \rho$$
$$\tau = \sqrt{\gamma}(\rho h W^2 - p - W\rho)$$
$$S_i = \sqrt{\gamma}\rho h W^2 v_i$$

Conserved variables

$$W = \alpha u^t \qquad h = 1 + e + \frac{p}{\rho}$$

HRSC methods:

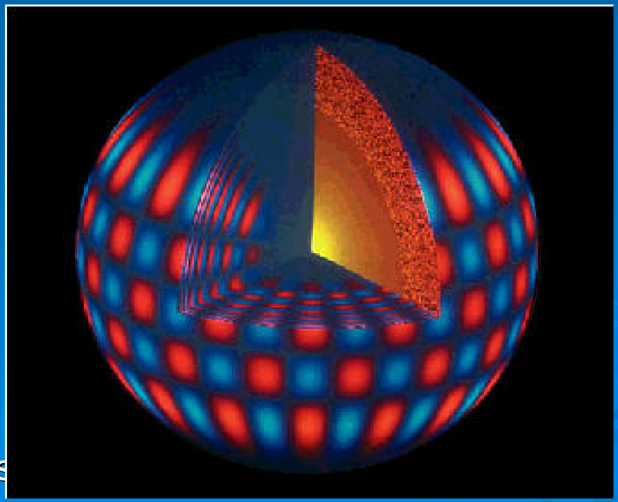
Solution of local Riemann problem in each cell:

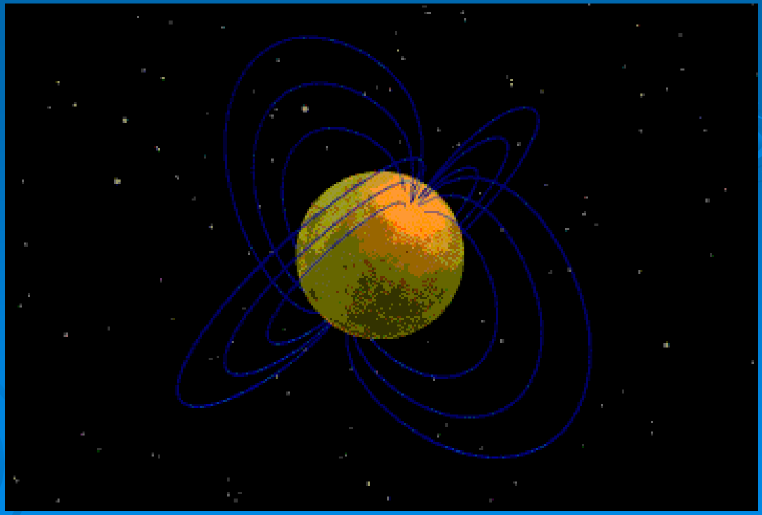


- Shock capturing without artificial viscosity
- High-order, oscillation-free reconstructions

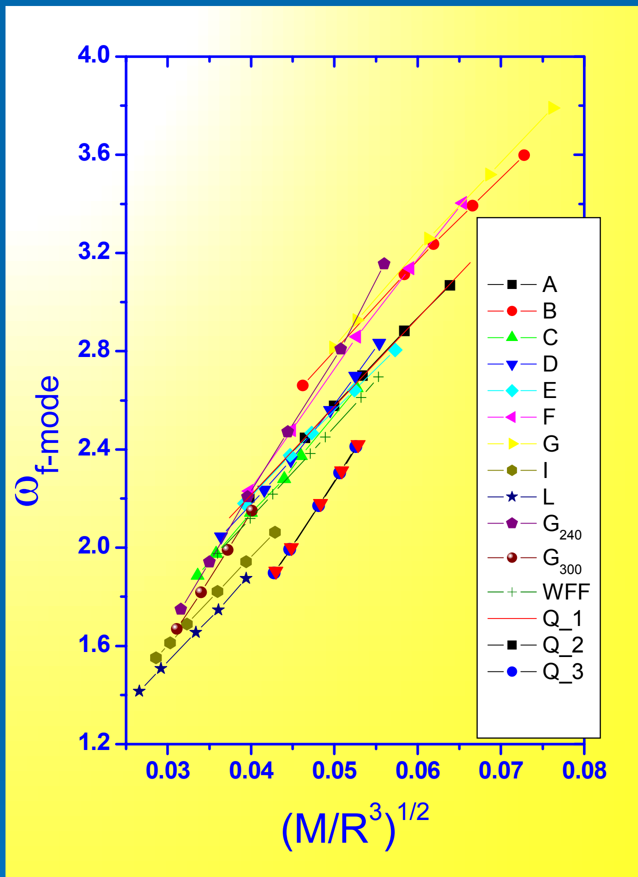
Neutron Star “ringing

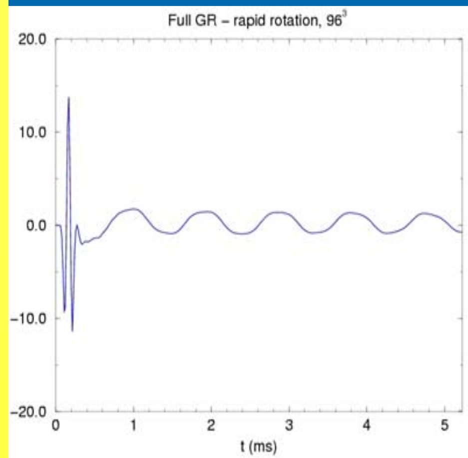
- **p-modes:** main restoring force is the pressure **(f-mode)** (can become unstable) $f^2 \sim M/R^3$ *(>1.5 kHz)*
- **Inertial modes (r-modes)** main restoring force is the Coriolis force (can become unstable) $f \sim \Omega$
- **Torsional modes** (t-modes) $f^2 \sim u_s/R$ *(>20 Hz)* shear deformations, divergence-free, with no-radial components. Restoring force, the weak Coulomb force of the crystal ions.
- **w-modes:** pure space-time modes (only in GR) (can become unstable) $f \sim 1/R$ *(>5kHz)*

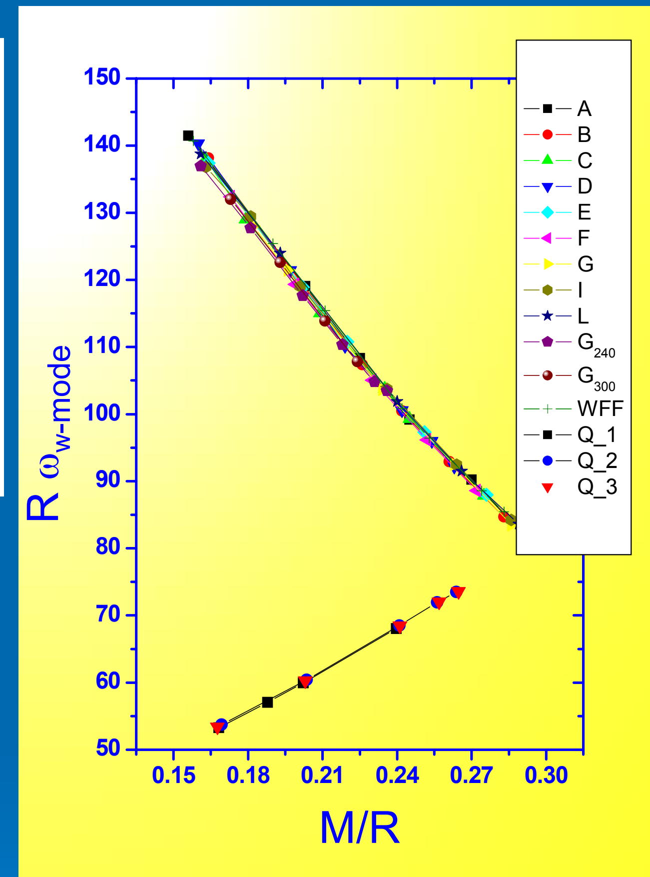




Grav. Wave Asteroseismology







$$\omega_f (kHz) \approx 0.78 + 1.637 \left(\frac{M_{1.4}}{R_{10}{}^3} \right)^{1/2} + \delta_f m \frac{\Omega}{\Omega_K}$$

$$\omega_w (kHz) \approx \frac{1}{R_{10}} \left[20.92 - 9.14 \frac{M_{1.4}}{R_{10}} \right] + \delta_w m \frac{\Omega}{\Omega_K}$$

 Andersson + KK (96,98,01)

Stability of Rotating Stars

Non-Axisymmetric Perturbations

A general criterion is:

$$\beta = \frac{T}{W}$$

T : rot. kinetic energy
W : grav. binding energy

Dynamical Instabilities

- Driven by hydrodynamical forces (bar-mode instability)
- Develop at a time scale of about one rotation period

$$\beta \geq 0.27$$

Secular Instabilities

- **Driven by dissipative forces (*viscosity, gravitational radiation*)**
- **Develop at a time scale of several rotation periods.**
- **Viscosity driven instability causes a Maclaurin spheroid to evolve into a non-axisymmetric Jacobi ellipsoid.**
- **Gravitational radiation driven instability causes a Maclaurin spheroid to evolve into a stationary but non-axisymmetric Dedekind ellipsoid.**

  Chandrasekhar-Friedman-Schutz (CFS)

$$\beta \geq 0.14$$

GR and/or differential rotation suggest considerably lower β for the onset of the instabilities

Instability window (r-mode)

- For the r-mode ($\ell=2$) we get:

$$\tau_{BV} \approx 2.4\times10^{10}\left(\frac{1.4M_\odot}{M}\right)\left(\frac{R}{10\text{km}}\right)^5\left(\frac{10^9K}{T}\right)^6\left(\frac{P}{1\text{ms}}\right)^2 \text{sec}$$

$$\tau_{SV} \approx 1.2\times10^{8}\left(\frac{1.4M_\odot}{M}\right)^{5/4}\left(\frac{R}{10\text{km}}\right)^{23/4}\left(\frac{T}{10^9K}\right)^2 \text{sec}$$

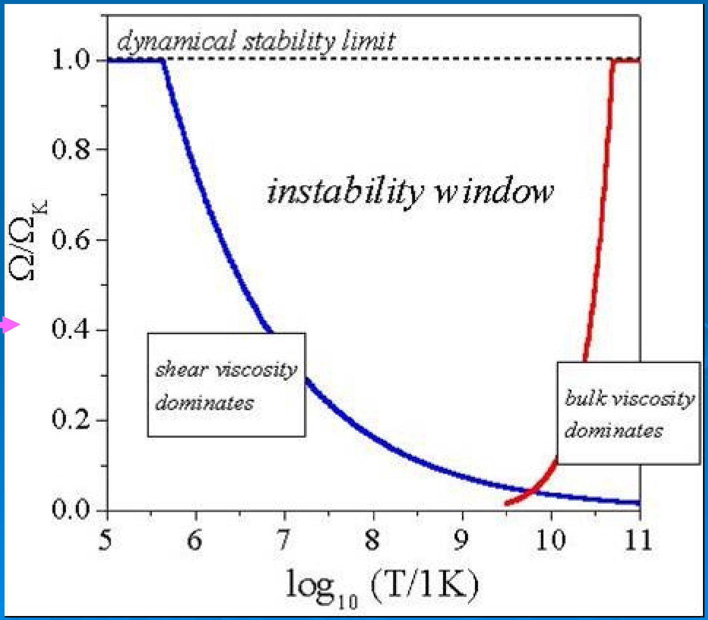
$$\tau_{GW} \approx -22\left(\frac{1.4M_\odot}{M}\right)\left(\frac{R}{10\text{km}}\right)^{-4}\left(\frac{P}{1\text{ms}}\right)^6 \text{sec}$$

The instability will grow if

$$\tau_{\text{visc}}(\Omega,T) \geq \tau_{\text{inst}}(\Omega)$$

The $l=m=2$ r-mode grows on a timescale 20-50secs

- Instability window
- Many astrophysical applications both on newly born and old NS



f-mode

- f-mode is the fundamental pressure mode of the star
- It corresponds to polar perturbations
- Frequency for uniform density stars

$$\omega^2 = \frac{2l(l-1)}{2l+1}\frac{GM}{R^3}$$

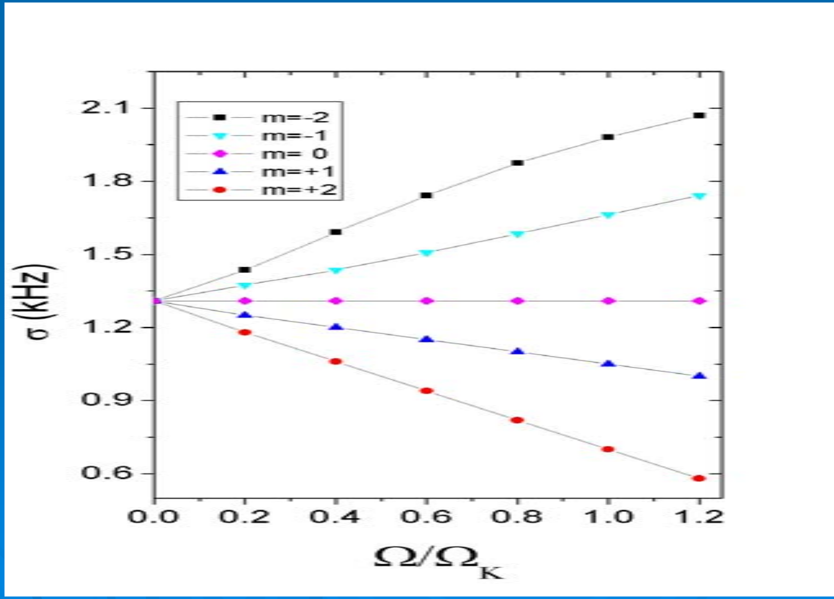
growth time(if unstable)

$$t_{GW} \approx f(l)R\left(\frac{R}{M}\right)^{l+1} \sim 0.07\left(\frac{1.4M_\odot}{M}\right)^3\left(\frac{R}{10km}\right)^4 \text{sec}$$

- For ℓ=2 is ~1.2-4kHz

- Rotation breaks the symmetry: the various **-ℓ≤m≤ℓ** decouple
- There is coupling between the polar and axial modes
- The frequency shifts:

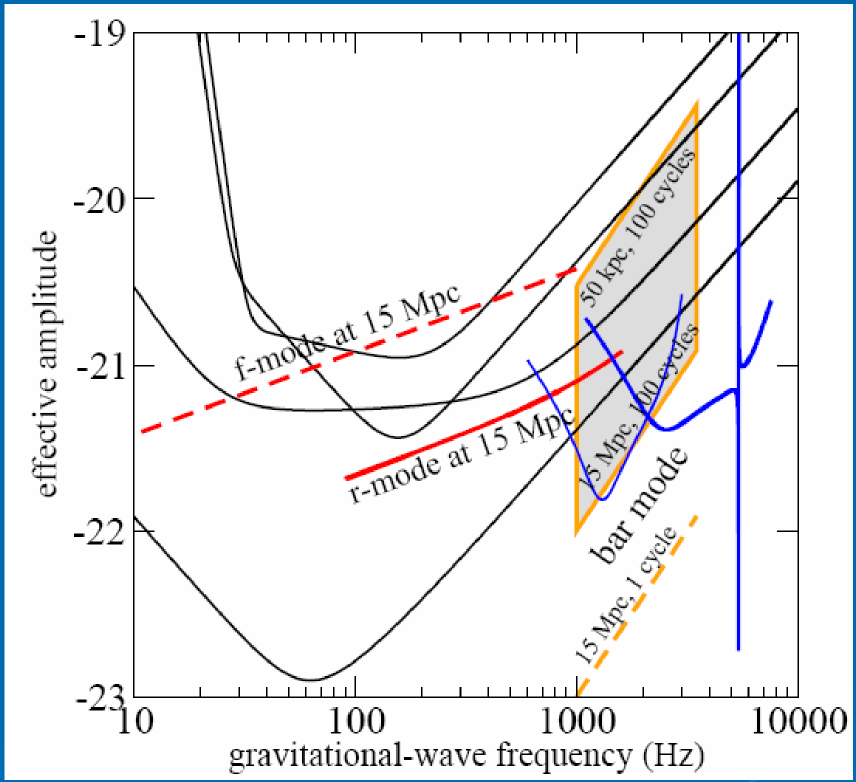
$$\omega_{\text{inert}}(\Omega) = \omega(\Omega=0) + \kappa m\Omega$$



Stavridis & KK 2005

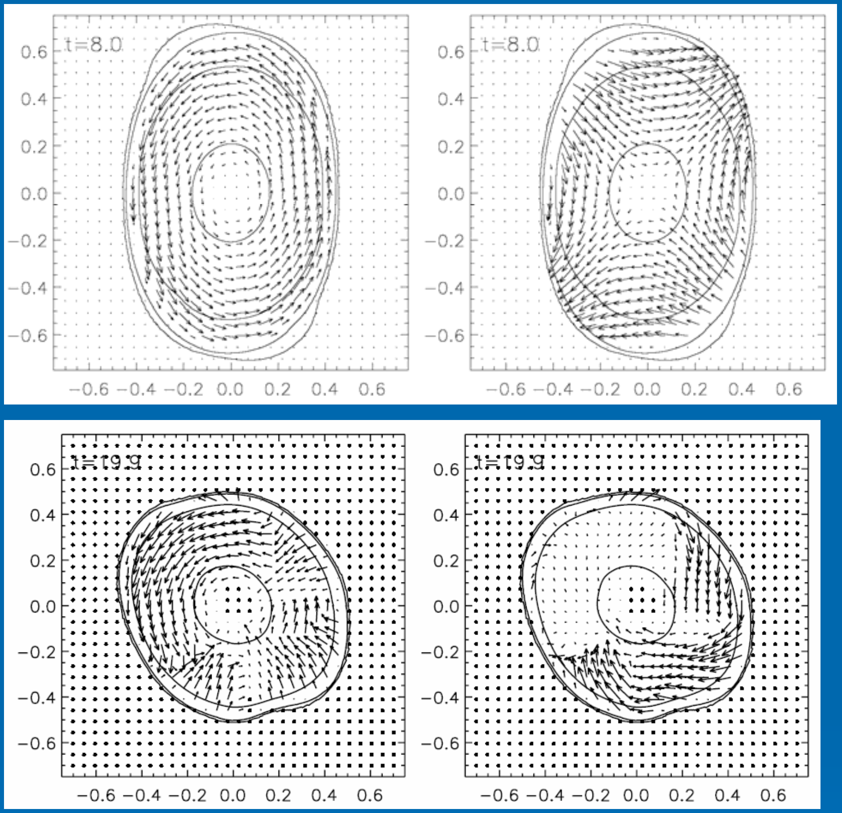
f-mode (astrophysics)

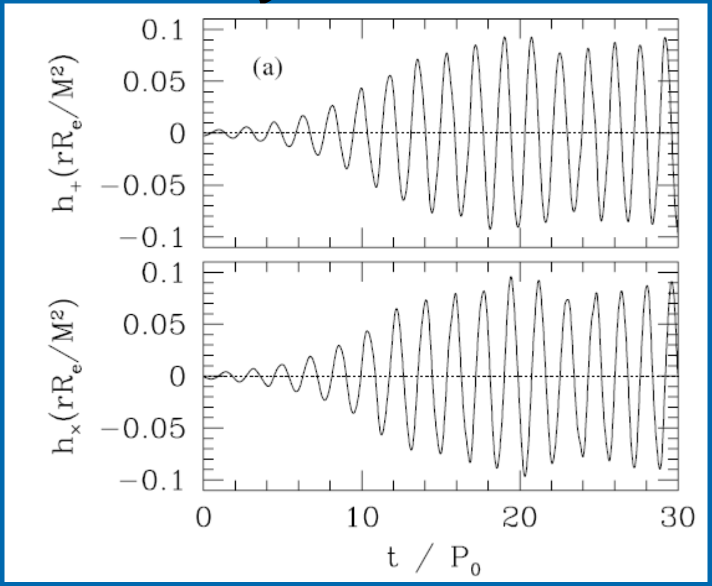
- In GR the ***m=2*** mode becomes unstable for ***$\Omega > 0.85\Omega_{Kepler}$*** or ***$\beta > 0.06$-$0.08$***
- Detectable from as far as 15Mpc (LIGO-I), 100Mpc (LIGO-II) (*depending on the saturation amplitude*).
- Differential rotation affects the onset of the instability
- Non-linear calculations by Shibata & Karino (2004) suggest that:
  - Up to 10% of energy and angular momentum will be dissipated by GWs.
  - Amplitude (at ~500Hz):



$$h_{\rm eff} \sim 5\times10^{-22}\left(\frac{R_e}{20km}\right)^{1/4}\left(\frac{M}{1.4M\odot}\right)^{3/4}\left(\frac{100\text{Mpc}}{r}\right)$$

f-mode Instability





Lai & Shapiro, 1995

Ou, Lindblom & Tohline, 2004

Shibata & Karino, 2004

In the *best-case scenario*, the GWs are easily detectable out to 140 Mpc !

Major uncertainties:

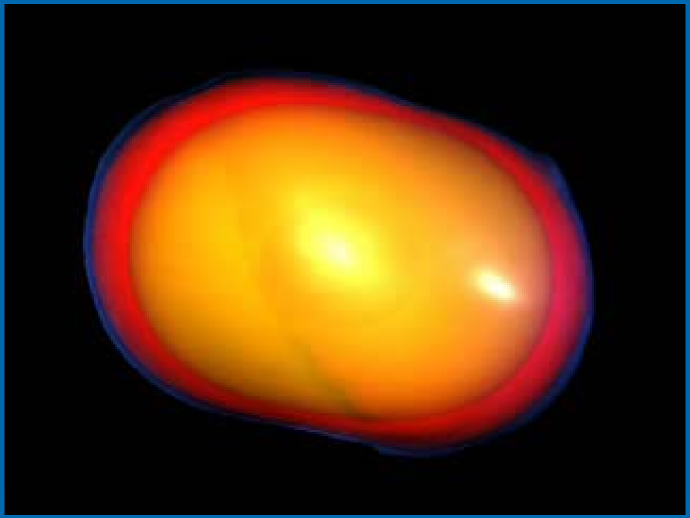
1. **Relativistic growth times**
2. **Nonlinear saturation**
3. **Initial rotation rates of protoneutron stars – event rate**
4. **Effect of magnetic fields**

r-modes

- A non-rotating star has only trivial axial modes. Rotation provides a restoring force (Coriolis) and leads in the appearance of the inertial modes. The *l*=*m*=2 inertial mode is called r-mode.
- In a frame rotating with the star, the r-modes have frequency

$$\omega_{\rm rot} = \frac{2m}{l(l+1)}\Omega$$

- GW amplitude depends on α (the saturation amplitude).
- Mode coupling might not allow the growth of instability to high amplitudes (Schenk etal)
- The existense of *crust*, hyperons in the core, magnetic fields, affects the efficiency of the instability.
- For newly born neutron stars might be quite weak ; unless we have the creation of a strange star
- Old accreting neutron (or strange) stars, probably the best source! (400-600Hz)



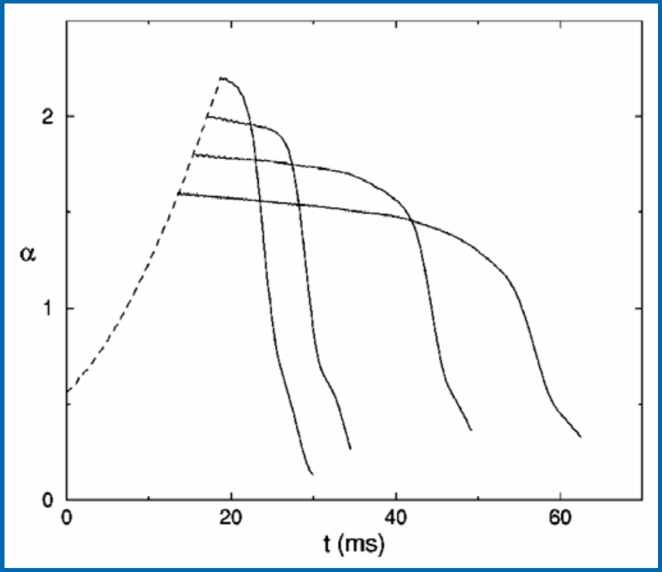
Lindblom-Vallisneri-Tohline

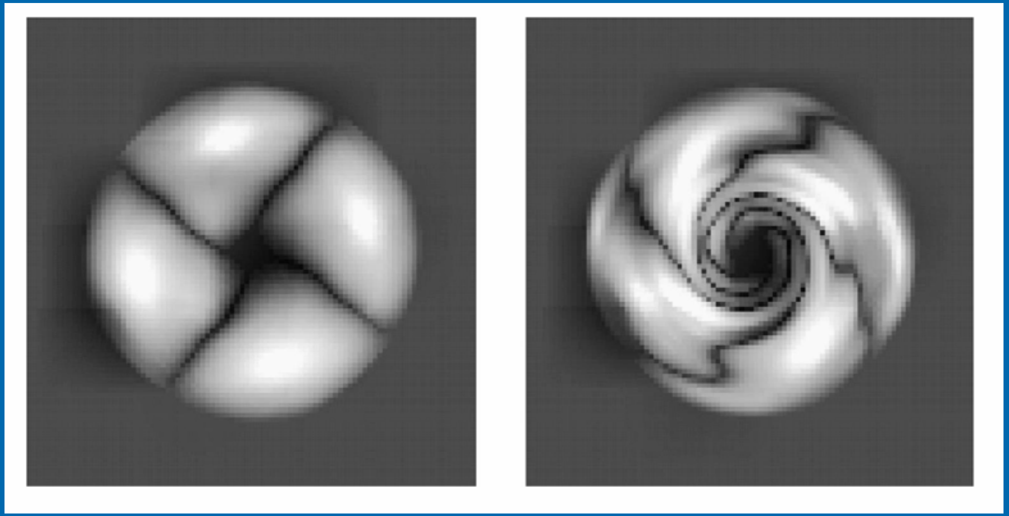
$$h(t) \approx 10^{-20}\,\alpha \left(\frac{\Omega}{1\ \text{kHz}}\right)\left(\frac{10\text{Kpc}}{d}\right)$$

$$\alpha \simeq 10^{-3} - 10^{-4}$$

Saturation of Nonlinear R-Modes

Long-term nonlinear evolution of r-mode grown to O(1), using accelerated gravitational-radiation-reaction force.





Gressman, Lin, Suen, Stergioulas & Friedman (2002)

- When r-mode exceeds its saturation amplitude, it ultimately breaks down into a vortex-like motion.
- More detailed analysis by Lin & Suen (2004) showed that this break-down is due to nonlinear 3-mode coupling of the r-mode to two other inertial modes. Saturation amplitude of $O(10^{-2})$.
- However, resolution not sufficient to resolve inertial modes with very high mode number.

Saturation of Nonlinear R-Modes

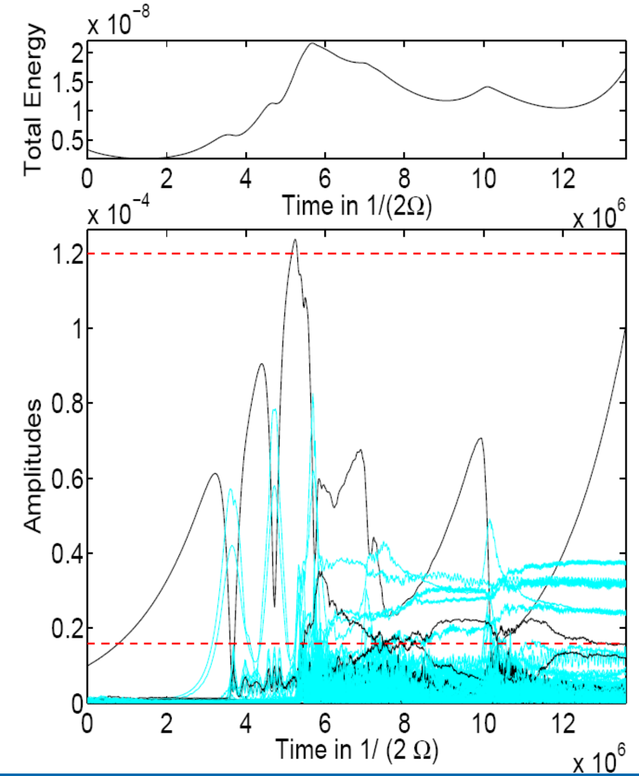
Morsink (2002)
Schenk, Arras, Flanagan, Teukolsky, Wasserman (2002)
Arras, Flanagan, Morsink, Schenk, Teukolsky, Wasserman (2003
Brink, Teukolsky, Wasserman (2004a, 2004b)

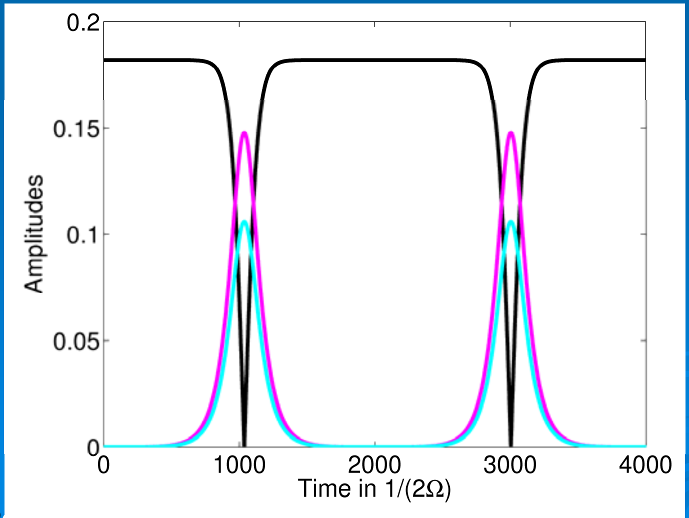
Second-order perturbative evolutions (Neutonian).

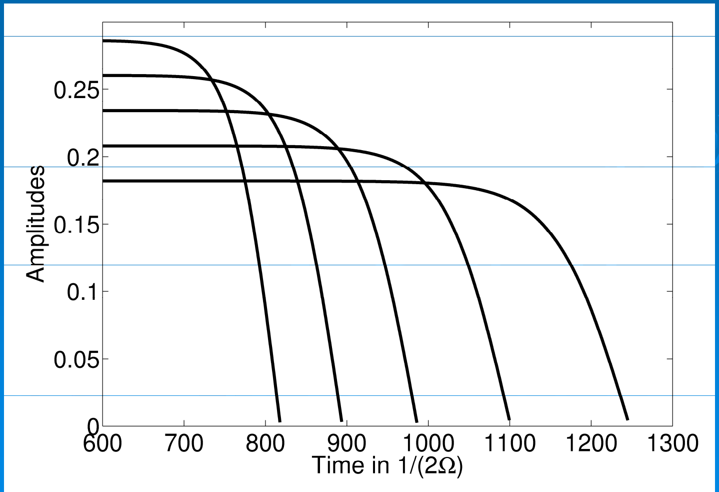
Several *3-mode couplings* of r-mode to other *high-order inertial modes*.

Saturation amplitude may be of $O(10^{-3}$-$10^{-4})$, *still fine* for *GWs from LMXBs*.

(see Andersson, Jones, KK & Stergioulas, 2001)

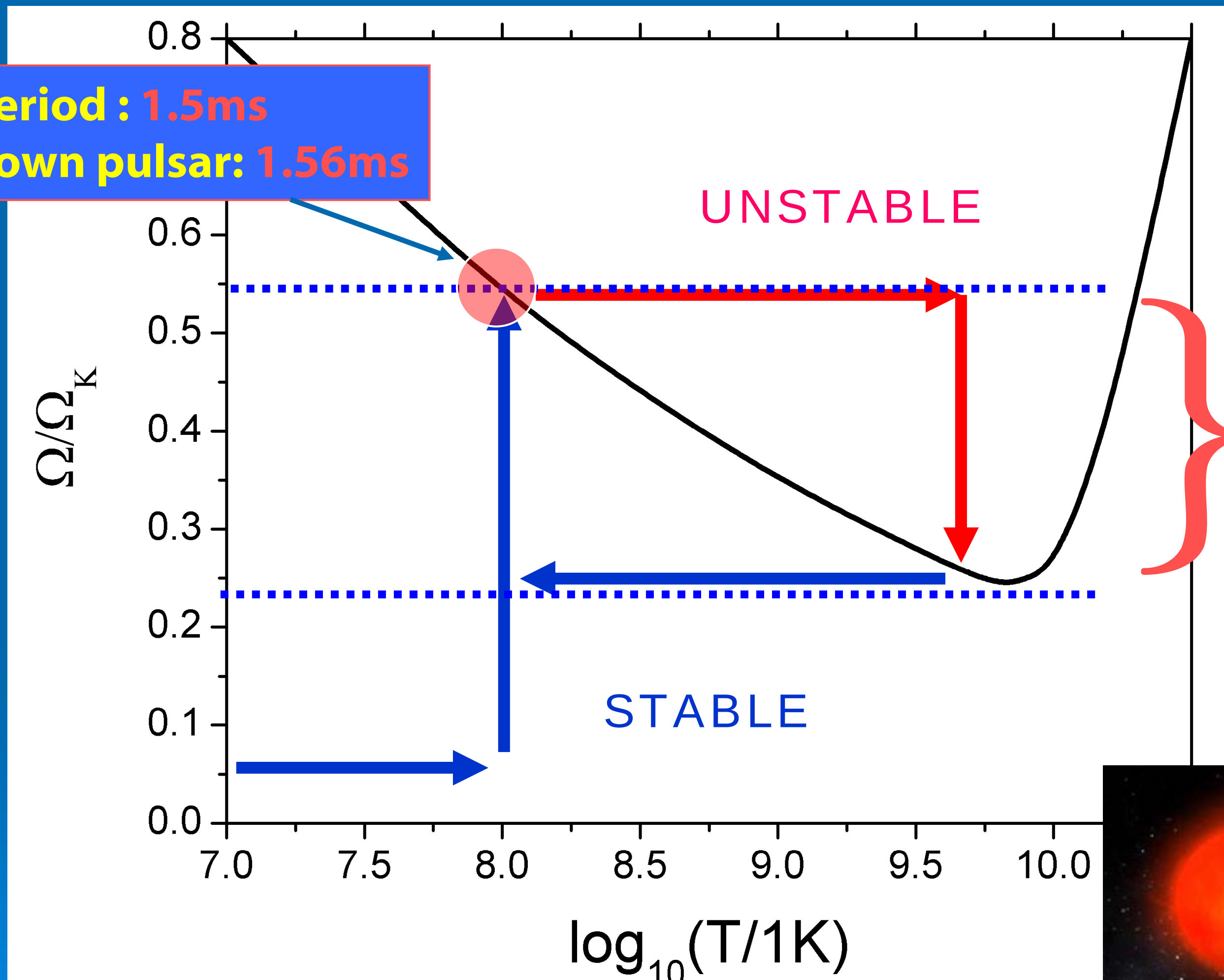




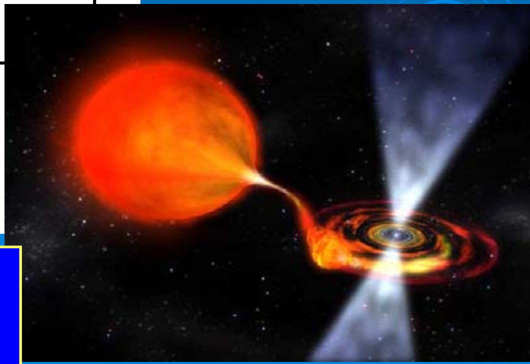


LMXBs & r-modes

Limiting Period : 1.5ms
Fastest known pulsar: 1.56ms



Period clustering of ms pulsars

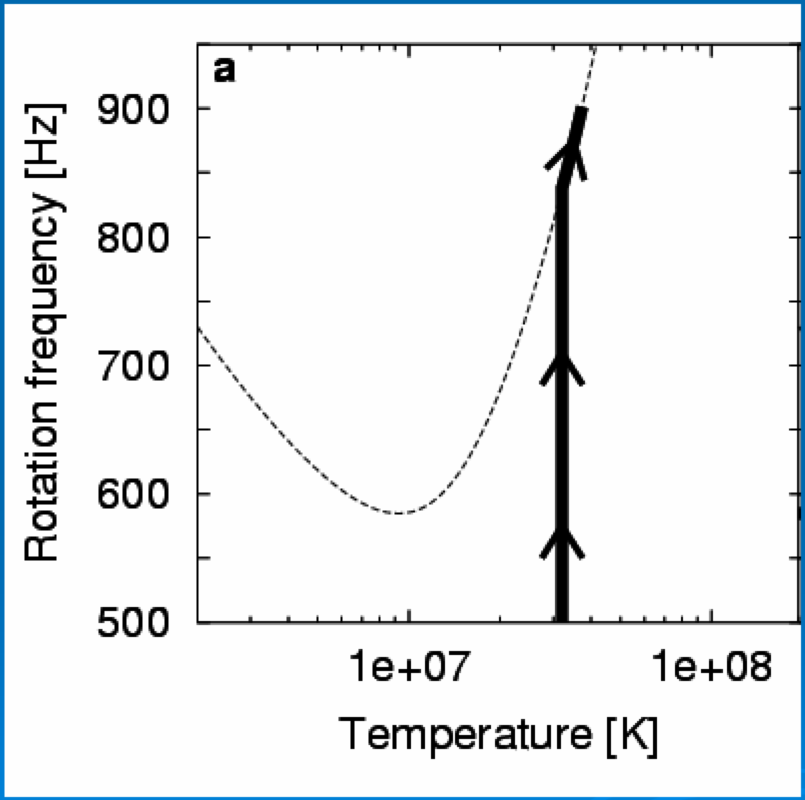


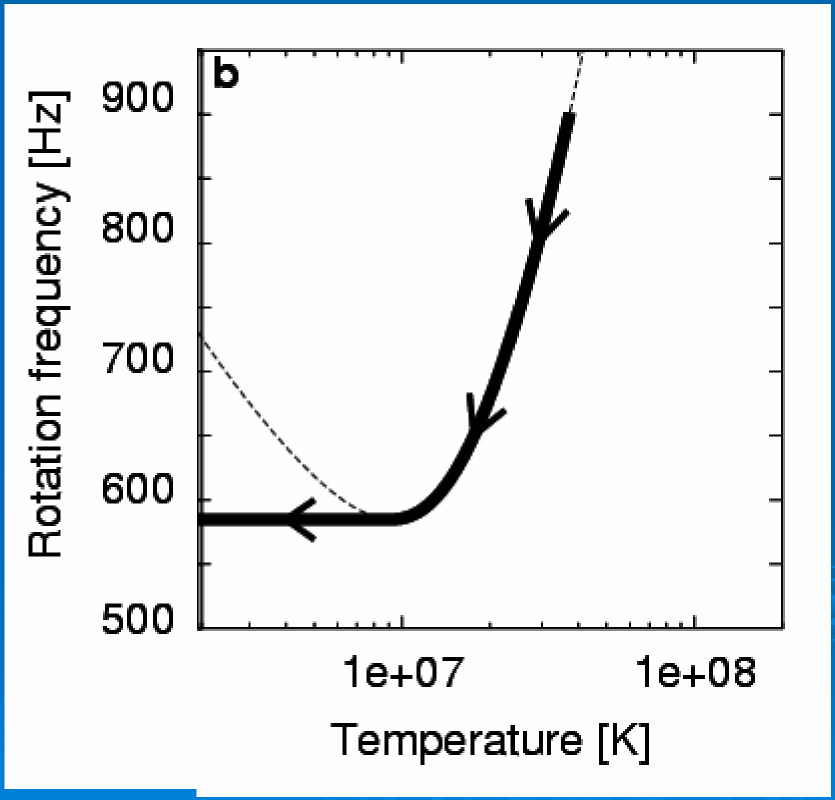
Andersson, KK, Stergioulas 1999, Levin 2000
Andersson, Jones, KK, Stergioulas 2000, Heyl 2002

Non-standard evolution: LMXBs

- Andersson, KK, Jones (2001) : Strange stars
- Wagoner (2002) : Hyperons
- Reisenegger & Bonačić (2003) : Hyperons

Neutron stars in LMXBs may evolve to an **equilibrium state**:



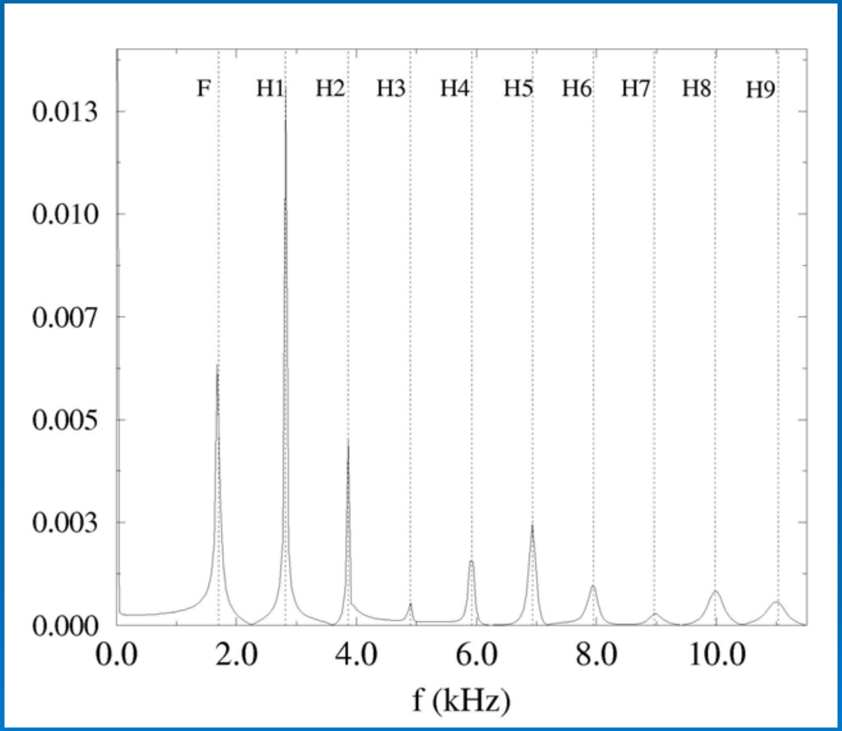


During accretion

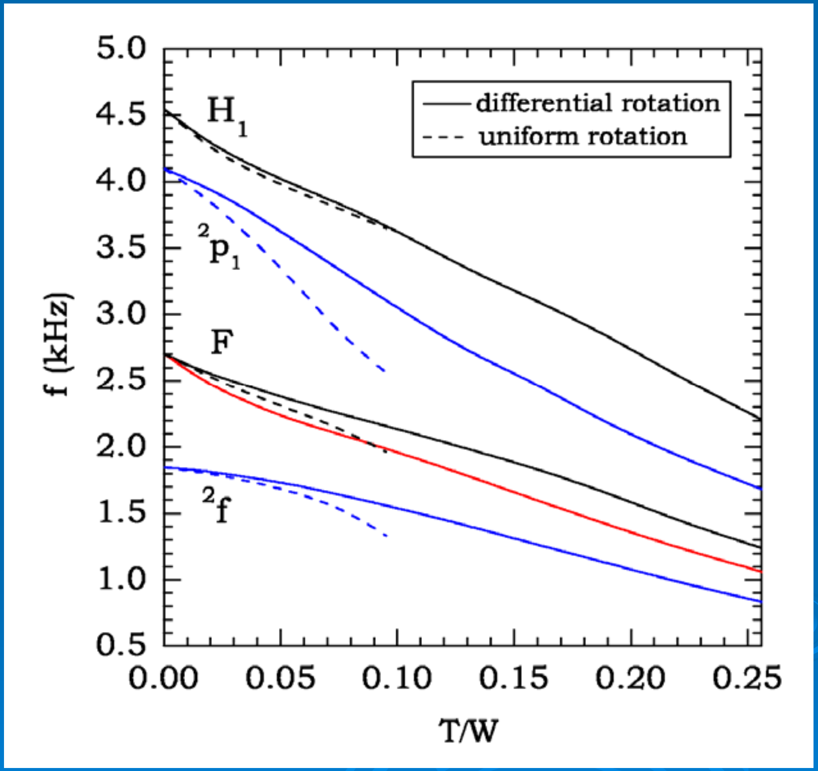
After accretion

Axisymmetric Modes
in Cowling Approximation

Axisymmetric modes for uniform or differential rotation with *fixed* spacetime evolution (Cowling approximation) :



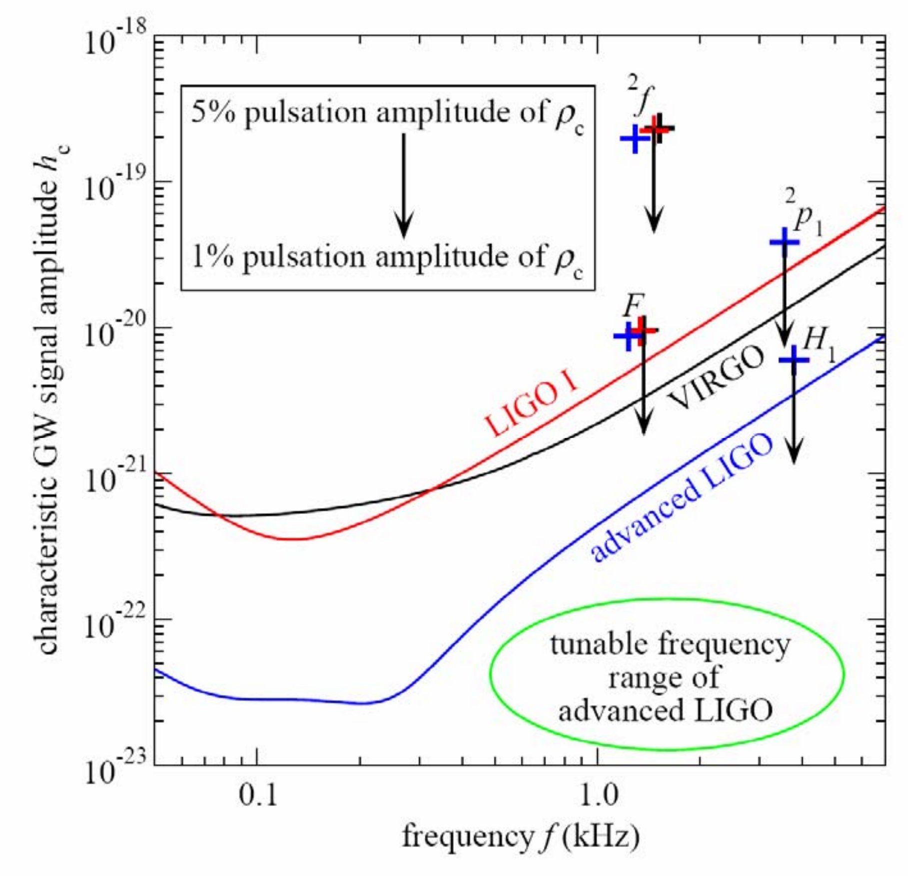
Font, Stergioulas, KK(2002)



Stergioulas, Apostolatos, Font (2004)

Gravitational Wave Emission

Characteristic signal amplitude for slowest rotating model ($T/|W|$~2%) at 10kpc (individually excited modes with 20ms integration time).



For GW-Asteroseismology, need advanced detectors with better sensitivity at several kHz.

Dimmelmeier, Stergioulas & Font (2005)

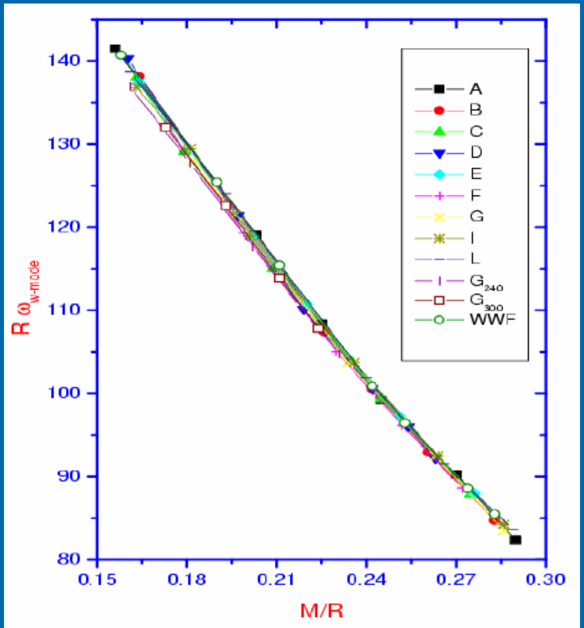
w-modes

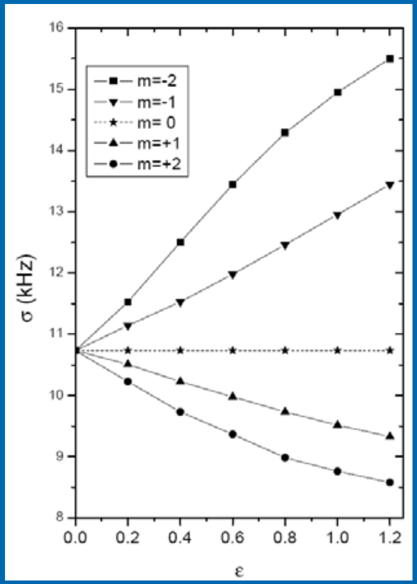
Very high frequency (short lived) modes 6-12kHz
(KK & Schutz 1986,1992, Leins etal 1993)

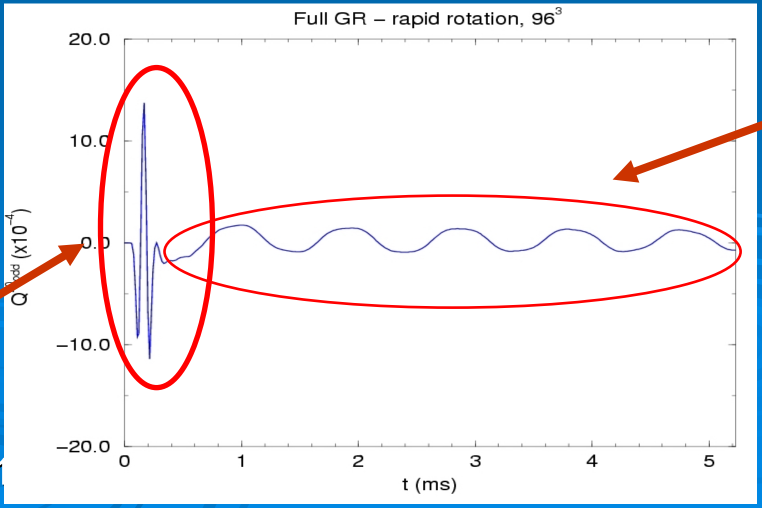
The frequency of the w-modes show a very accurate *scaling behavior* for a large sample of realistic EOSs. **(KK, Apostolatos & Andersson 2001)**

W-modes of rotating stars in slow rotation and in Inverse Cowling Approximation (ICA) have been calculated
(Stavridis & KK 2005)

W-modes of ultra compact stars R<3M become CFS unstable for small rotational rates ***Ω > 0.20 Ω_{Kepler}*** (ergoregion instability)
(KK, Ruoff & Andersson 2004)







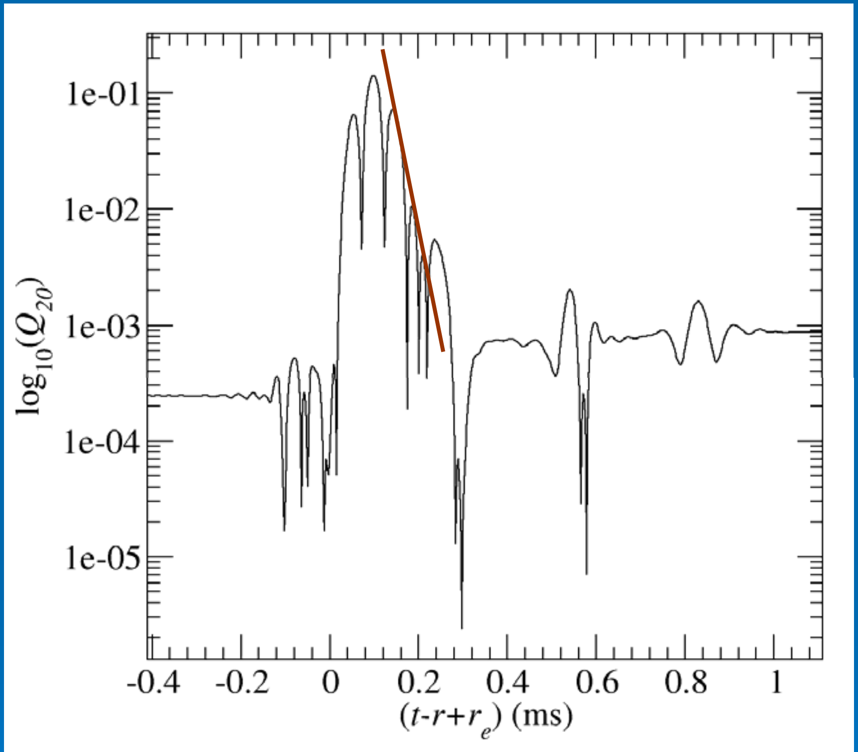
F-mode

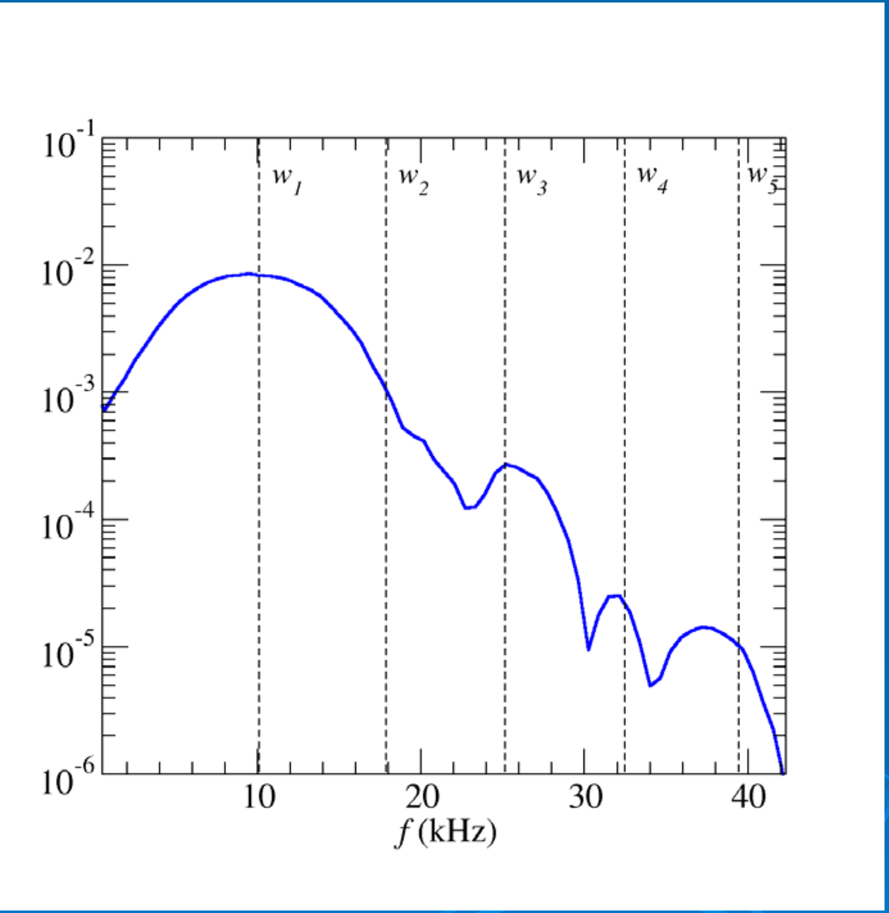
W-mode

W-Modes: Comparison with Linear Frequencies

Stergoulas, KK, Hawke 2006

We excite l=2 w-modes for a 1.4 M_{sun} non-rotating star and extract Q_{20}.





Good agreement also in damping time:

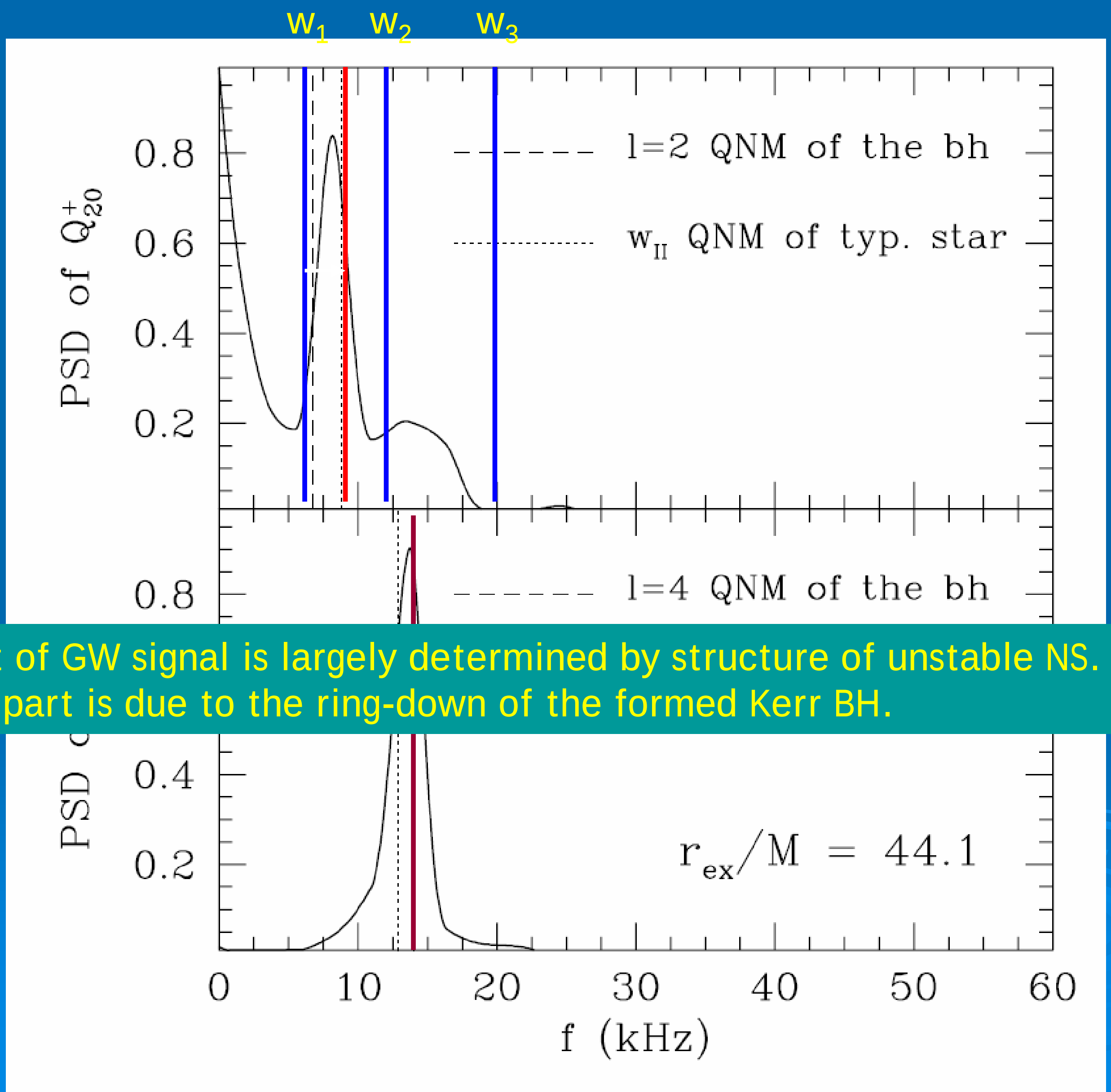
τ= 0.0235 ms

vs. 0.0226 ms in linear theory.

The frequencies agree very well with those of known linear modes.

W-Modes of Rotating Stars

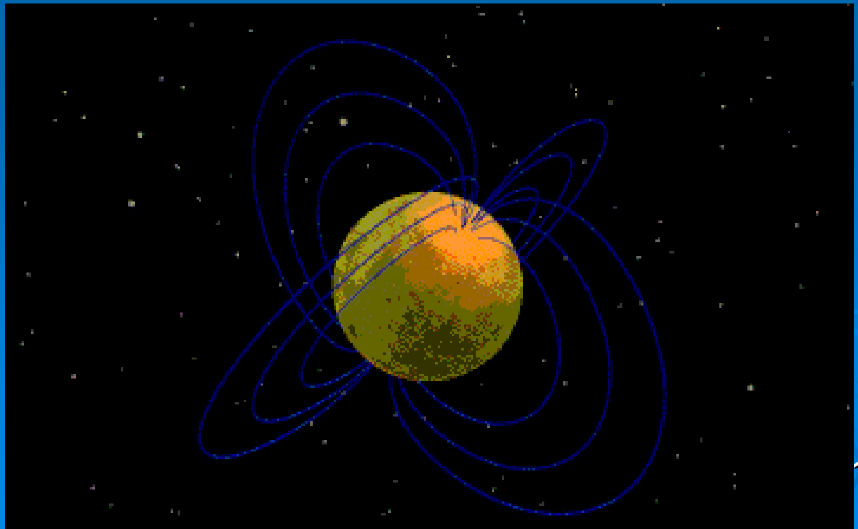
Comparison to GW signal during Kerr BH formation via collapse of unstable neutron star (Baiotti, Hawke, Rezzolla & Schnetter, PRL, 2005)



Initial part of GW signal is largely determined by structure of unstable NS.
Only later part is due to the ring-down of the formed Kerr BH.

Soft Gamma Repeaters & Stellar Oscillations

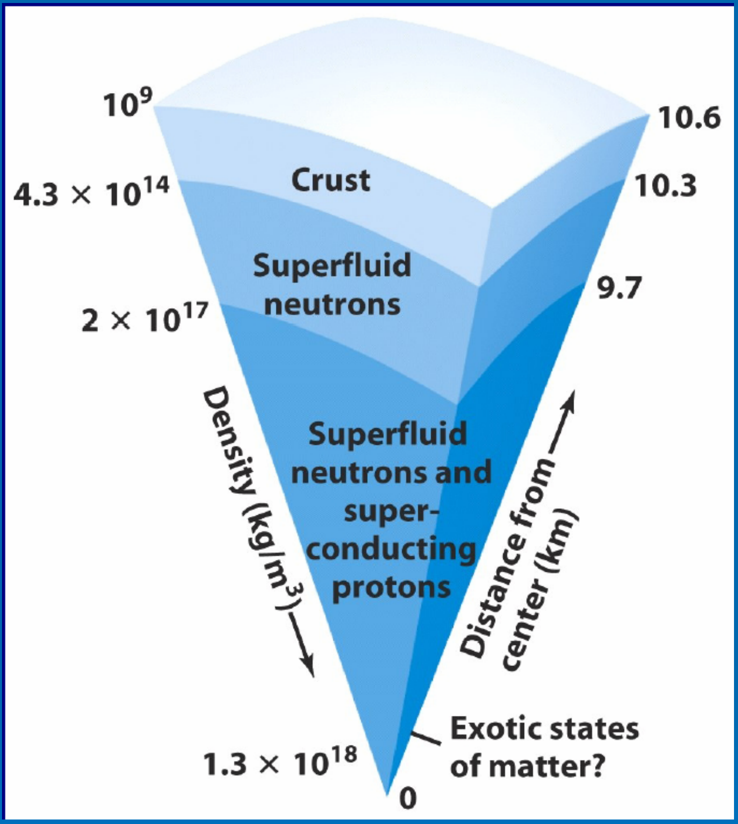
- The Soft gamma Repeaters (SGRs) are objects exhibiting recurrent bouts of γ-ray flare activity. They thought to be **magnetars** i.e. neutron stars with strong magnetic field $>10^{14}$ G.
- **SGRs exhibit giant flares (10^{44}-10^{46} ergs/s)**
- The catastrophic magnetic instability that powers the giant flares is thought to be associated with large-scale fracturing of NS crust.
- **Global seismic (23 mag) vibrations are excited.**
- **Pulsations are visible in the tail and reveal the neutron star period.**
- **3** such flares have been detected from satellite high-energy detectors.

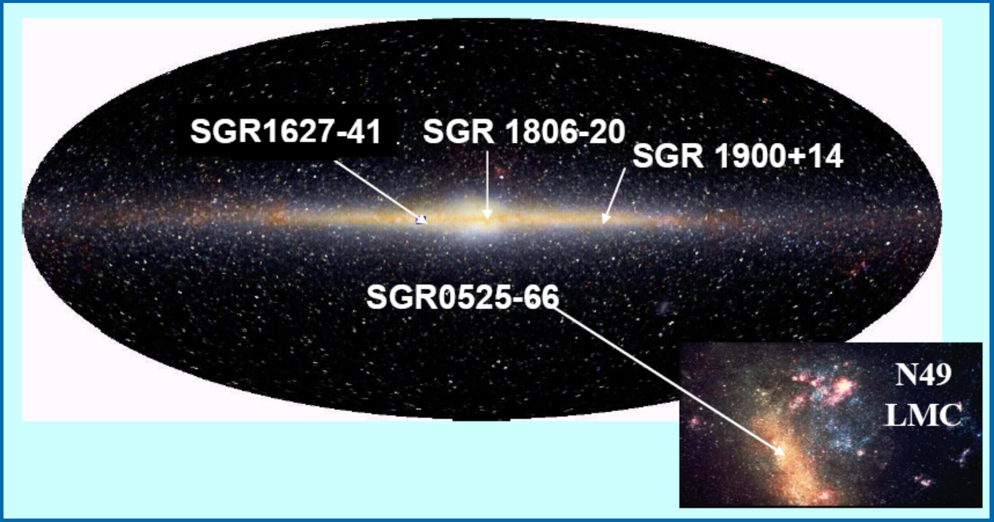


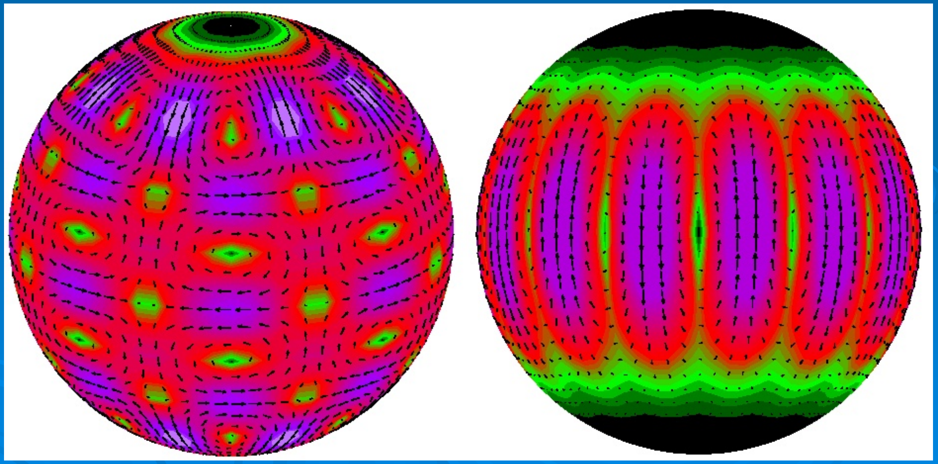
6 more potential candidates are known (anomalous X-ray pulsars)

The Dec 2004 flare from SGR 1806-20 was the most energetic ever recorded (Rossi X-ray Timing Explorer, RXTE).

Starquake reveals hidden structure of a neutron star







•Surface patterns for different torsional modes that may have been excited by the hyperflare.

Strohmayer & Watts 2006

Torsional modes

…the first NS oscillations **ever** observed

- **1979** (March 5): (SGR 0526-66) Barat et. al. (1983)
  - **T=23ms (43Hz) $_2t_0$ or $_3t_0$ –mode!**
  - B ~10^{14}-5x10^{14}G, E ~ 10^{44} ergs
- **1998** (August 27): (SGR 1900+14) Strohmayer & Watts (2005)
  - **$_2t_0$ = 28Hz, $_4t_0$ =53.5Hz, $_7t_0$ =84Hz & $_{13}t_0$ =155.1Hz**
- **2004** (Dec 27): (SGR 1806-20) Israel et. al. (`05) Strohmayer & Watts (`06)
  - **$_7t_0$ =92.5Hz, $_2t_0$ =30.4 Hz, $_2t_1$ =625.5 Hz, $_?t_?$=26Hz and $_?t_?$=18Hz**
  - B~10^{15}G, E~ 10^{46} ergs

New data: 150, 1840, 720 ?, 2384 ? Hz

$$P = \frac{2\pi}{\sqrt{\ell(\ell+1)}}\left(\frac{R}{u_s}\right)$$

$$P({}_l t_0) = P({}_2 t_0)\left(\frac{6}{l(l+1)}\right)^{1/2}\left[1+\left(\frac{B}{B_\mu}\right)^2\right]^{-1/2}$$

Torsional Modes and GWs

The torsional modes of the NS crust are not (?) likely to be significant source of GWs. For an oscillation with l=2 the gravitational wave strain is estimated to be:

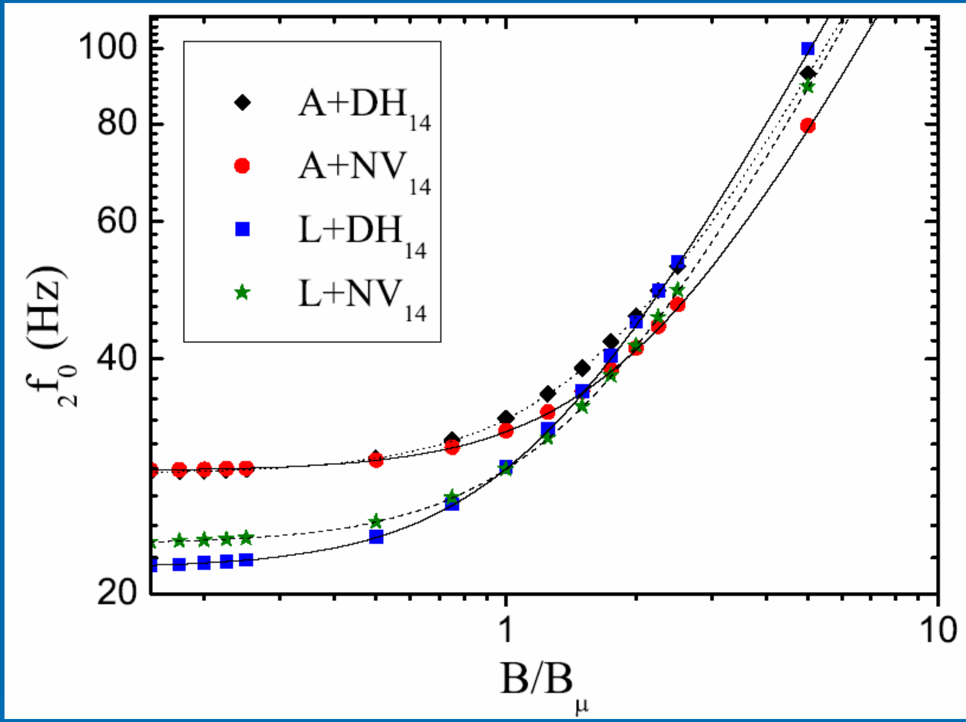
$$h \sim 10^{-25} - 10^{-28} \times \left(\frac{10kpc}{r}\right)\left(\frac{\beta}{10^{-3}}\right)$$

The strong magnetic field might/could couple these surface modes with core oscillations which could emit significantly stronger GWs.

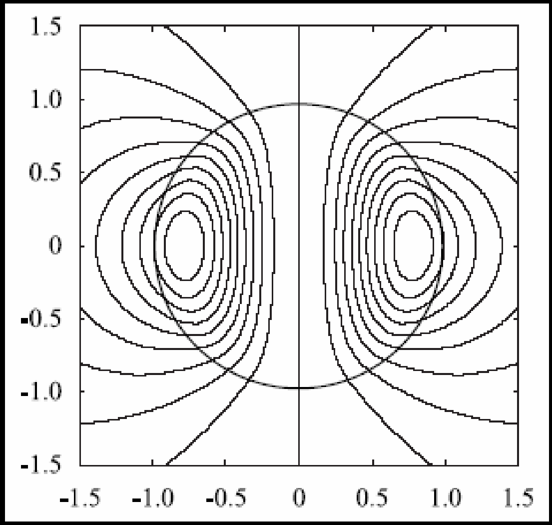
- **The Dec 2004 flare from SGR 1806-20 was the most energetic ever recorded.**
- **LIGO was in operation at that specific period (with a sensitivity far bellow the designed one), the analysis of the data did not show any sign of incoming GWs.**

Some recent results…

Sotani, KK, Stergioulas astro-ph/0608626



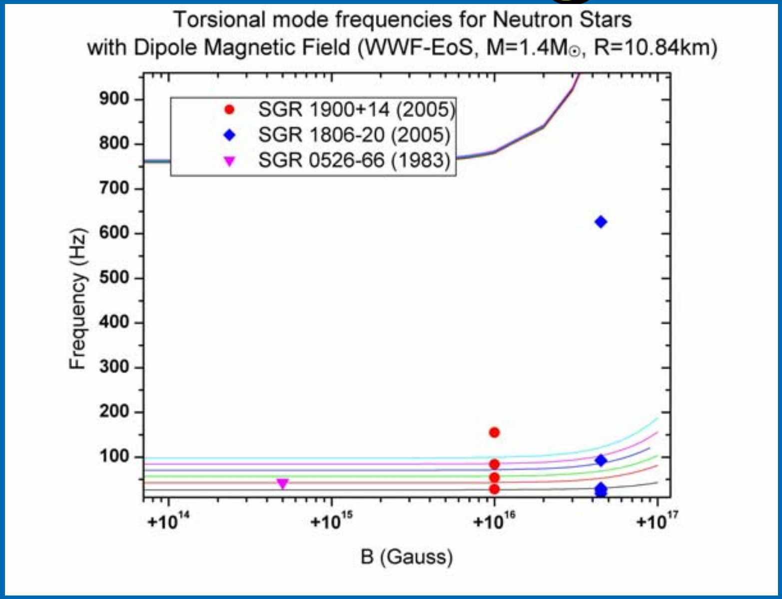
- Dipole Magnetic Field



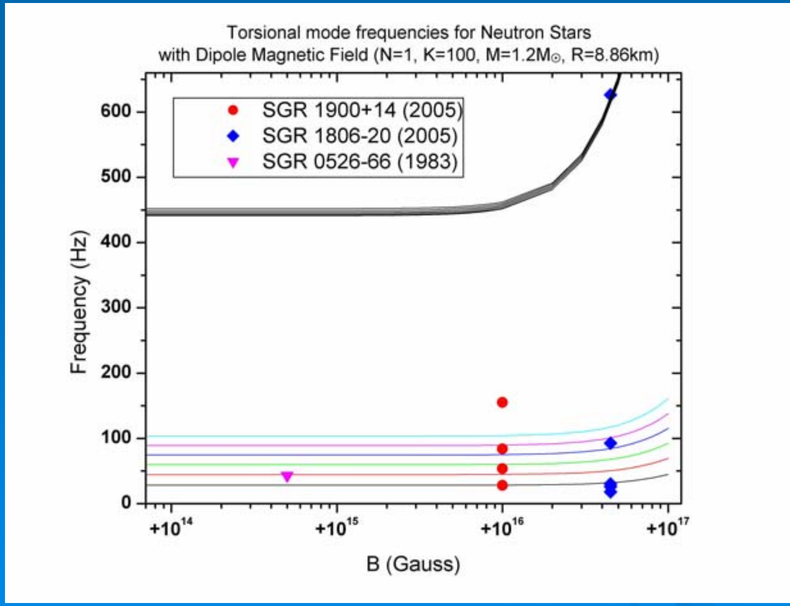
- 4 EoS for the core
- 2 for the crust

- **The GR periods 15-30% off the Newtonian ones**
- **The main difference is due to the redshift factor $e^{2\Phi}$**

…higher harmonics?



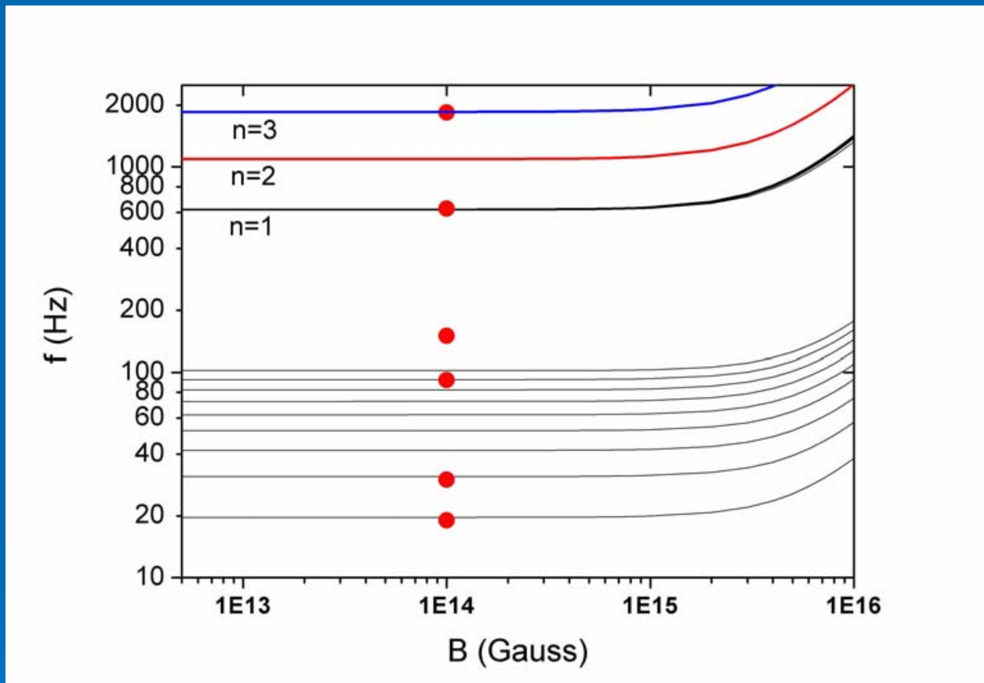
- Higher overtones are also present e.g. **626.5** **and** **1840 Hz!**
- They are independent from angular index ℓ
- Unique information about the thickness of the crust.



$$\frac{\Delta r}{R} \approx \frac{n\pi}{\sqrt{\ell(\ell+1)}} \frac{{}_{\ell}f_0}{{}_{\ell}f_n} \left(\frac{R}{u_s} \right)$$

The “model”…

We “identified” a specific NS model which fits quite well the observed frequencies for SGR 1806-20 !



The “**correct**” model !

Core EoS **L**

Crust EoS **NV**

$M=1.9\text{-}2.0M_{\odot}$, $R=14.0$ km

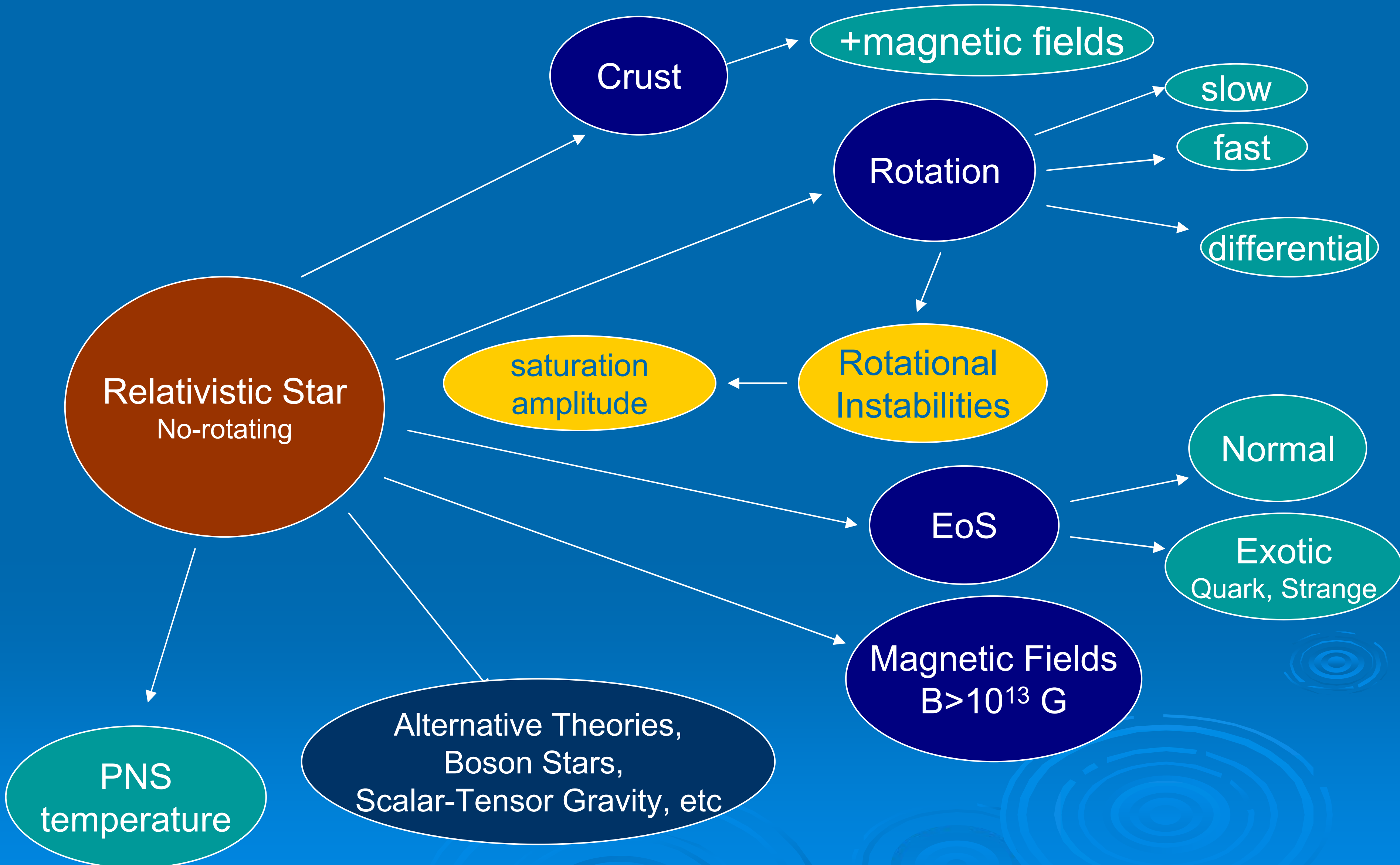
crust thickness = 1.05 km

NOTICE: results are model dependent, a different magnetic field geometry might alter considerably the picture !

Where we are now :

- For a first time we observed NS oscillations (t-modes)
- f-modes: (unstable) good source of GWs
- r-modes: (unstable) good source only from LMXB
- w-modes: can be seen in core collapse to BH
- Non-linear evolutions “overtook” perturbation methods
- Complicated cases e.g. *dynamics of hot newly born fast (differential) rotating NS with complicated magnetic fields* are still to be studied.

…still long way to go!



…needs synergism of perturbation theory and nonlinear numerical GR