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Perfect fluid and magnetic field conservation laws in the theory of black hole accretion rings

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1. Introduction

The purpose of this article is to give a systematic introduction to the many conservation laws of relativistic perfect fluid and magnetic field theory that have been utilized on various occasions by numerous authors (including Abramowicz, Bardeen, Novikov & Thorne, Blandford & Znajek to mention but a few) in discussions of accretion onto black holes. The class of laws to be derived here includes relativistic generalizations of Bernoulli's theorem, the vorticity conservation theorem, and von Zeipel's theorem, as well as many other less well-known classical results such as Moffat's helicity conservation theorem whose relativistic generalization has not been published before.

It will be shown that these laws can all be derived with surprisingly little mathematical effort (in some cases less than is required for their classical analogues!) without any need to use the machinery of Riemannian geometry. In order to understand the discussion that follows, the reader will need only a vague familiarity with the concept of curvilinear coordinates and the description of curved space–time geometry in terms of a Lorentzian metric tensor with variable components g_{ab}, but it will *not* be necessary to have any prior knowledge of more technical Riemannian concepts such as covariant differentiation using an affine connection, or the construction of the Ricci curvature tensor that appears in Einstein's equations. Our discussion will in fact make no mention of Einstein equations, since the results are of such general validity as to be applicable in quite general background space–times *regardless* of any field equations (whether those of Einstein or of any alternative theory) that may or may not be satisfied.

The only necessary mathematical concept that may be unfamiliar to some readers is that of the *Lie derivative* of a tensor or other quantity with respect to the flow generated by some tangent vector field $\vec{\xi}$ say. Lie differentiation may be defined very simply in terms of appropriate *comoving coordinates* defined with respect to the flow, i.e. in terms of a system in which x^1, x^2, x^3 say are chosen in such a way as to be constant along the flow lines, while the zero coordinate, τ say, is a parameter (which may or may not measure the proper time) varying along the flow lines in such a way as to ensure that the generator $\vec{\xi}$ has components ξ^a given by the quadruplet of values (1, 0, 0, 0). In terms of such a coordinate system *Lie differentiation with respect to* $\vec{\xi}$ can be interpreted as simply meaning *partial differentiation* (with respect to τ) along the flow.

We shall be applying the concept of Lie differentiation to two different kinds of flow field. First we shall of course be concerned with the flow of the *physical fluid* under consideration, in which case $\vec{\xi}$ will be proportional to the fundamental unit flow vector $\vec{u}$, which is defined as satisfying the normalization condition,

$$u^a u_a = -1 \tag{1.1}$$

(we restrict ourselves to units in which the speed of light c is unity). However we shall also be concerned with cases in which $\vec{\xi}$ is the generator of a space–time *symmetry*. In particular, when we are concerned with a *stationary* situation it will be possible to choose coordinates in such a way that all physical tensor components are *independent* of an ignorable time coordinate t say. In terms of such a system the *generator* $\vec{k}$ of the symmetry may be *defined* as having components k^a given by the quadruplet (1, 0, 0, 0) or equivalently by

$$k^a(\partial/\partial x^a) = \partial/\partial t. \tag{1.2}$$

Thus if we choose to identify $\vec{\xi}$ with $\vec{k}$ then we may take the Lie differentiation parameter τ to be the same as t. Similarly when we are concerned with an axisymmetric situation in which there is an ignorable angle coordinate ϕ say, then we shall have a corresponding symmetry generator $\vec{h}$ say, defined by

$$h^a(\partial/\partial x^a) = \partial/\partial\phi, \tag{1.3}$$

and Lie differentiation with respect to $\vec{h}$ could be interpreted as meaning partial differentiation with respect to ϕ. In either of these examples the condition of (stationary or axial) *symmetry* is equivalent to the condition that the corresponding *Lie derivatives* (as defined with respect to $\vec{k}$ or $\vec{h}$) of the physical tensor fields under consideration *be zero*.

The mathematical formula for the Lie derivative of an arbitrary tensor field may be rather complicated and depends not only on the order but on the covariant or contravariant character of the tensor. However, differential and integral conservation laws are usually expressible in terms of differential *forms*, i.e. *covariant* tensors that are *antisymmetric* under interchange of any pair of indices (this definition includes all covariant *vectors* as well as scalar fields). For these the Lie derivative with respect to a vector field $\vec{\xi}$ (which we shall denote by the prefix $\vec{\xi}\pounds$) can be expressed by a very simple formula due to Cartan which will be the basis of all that follows. For a differential form $\boldsymbol{\omega}$ of arbitrary order (0, 1, 2, . . .) the Cartan formula may be written as

$$\vec{\xi}\pounds\boldsymbol{\omega} \equiv \vec{\xi}\cdot(\nabla\wedge\boldsymbol{\omega}) + \nabla\wedge(\vec{\xi}\cdot\boldsymbol{\omega}), \tag{1.4}$$

where a raised point denotes scalar contraction of adjacent indices (e.g. for a 2-form with components F_{ab}, $\vec{\xi}\cdot\boldsymbol{F}$ has components $\xi^b F_{ba}$) and where a wedge denotes the exterior product defined as a *sum* over combinations of the indices, with the *sign* chosen according as an even or odd permutation of ordering is involved. (This definition is fairly standard and is used in particular by Misner, Wheeler & Thorne (1973). Nevertheless the reader should be warned that a minority of authors, including Hawking & Ellis (1973), use a less convenient normalization condition in which the exterior product is defined in terms of an *average*, rather than a sum, over combinations, with the regrettable consequence that many simple formulae such as (1.4) require dimension and order-dependent adjustment factors.) Thus in particular for a scalar ϕ, a 1-form $\boldsymbol{\pi}$ or a 2-form $\boldsymbol{F}$, the exterior derivative appearing in (1.4) would have components given by

$$\left.\begin{aligned} (\nabla\phi)_a &\equiv \nabla_a\phi, \\ (\nabla\wedge\boldsymbol{\pi})_{ab} &= \nabla_a\pi_b - \nabla_b\pi_a, \\ (\nabla\wedge\boldsymbol{F})_{abc} &= \nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab}. \end{aligned}\right\} \tag{1.5}$$

Within these formulae the nabla symbol, ∇, may be interpreted as denoting simple *partial* differentiation (so that in terms of coordinates x^a we may interpret ∇_a as being equivalent to $\partial/\partial x_a$), since *antisymmetrized* partial derivatives of components of a *covariant* tensor will themselves transform as components of a well-behaved tensor field, (which is in fact the *same* as would be obtained if ∇_a were interpreted as denoting a more sophisticated *covariant* differentiation operation – in terms of some affine connection – of the kind whose use we have eschewed). This special property of exterior derivatives of differential forms (i.e. the property that $\nabla \wedge \boldsymbol{\omega}$ is independent of whether ∇ is interpreted in terms of covariant or partial differentiation) was first systematically exploited by Cartan, whose disciples customarily replace the combination $\nabla \wedge$ by the shorthand symbol d. I shall refrain from doing this here, since I wish to reserve the symbol d for its more traditional use to denote small displacements, as for example in the expression

$$\mathrm{d}\phi = \mathrm{d}\vec{x} \cdot \nabla\phi = \mathrm{d}x^a(\partial\phi/\partial x^a),$$

where $\mathrm{d}\phi$ does *not* mean the exterior derivative of ϕ (which would be $\nabla \wedge \phi$, or more briefly $\nabla\phi$, since we are dealing with a zero form) but the differential change in ϕ associated with the infinitessimal displacement $\mathrm{d}\vec{x}$ with components $\mathrm{d}x^a$.

In order to exploit the Cartan formula (1.4) we shall make frequent use of another important property of exterior differentiation – traditionally associated with the name of Poincaré – which is that for a p-form $\boldsymbol{F}$ to be (at least locally) the exterior derivative of some $(p-1)$-form $\boldsymbol{A}$ say, it is both sufficient and necessary that the exterior derivative of $\boldsymbol{F}$ be zero, i.e.

$$\nabla \wedge \boldsymbol{F} = 0 \Leftrightarrow \exists\, \boldsymbol{A} : \boldsymbol{F} = \nabla \wedge \boldsymbol{A}. \tag{1.6}$$

(The best-known application of this is to the case where $\boldsymbol{F}$ is a Maxwellian 2-form and $\boldsymbol{A}$ is the electromagnetic 4-potential.)

Much of the importance of differential forms stems from the fact that they play a fundamental role in the construction of surface integrals. A general p-surface S say may be thought of as being decomposed into a mesh of approximate parallelopipeds whose sides may be represented by infinitesimal displacement vectors $\mathrm{d}\vec{x}_{(i)}$ $(i = 1, \ldots, p)$. An integral over such a surface is definable in terms of a p-form $\boldsymbol{\Omega}$ in the form

$$\int \boldsymbol{\Omega}\, \mathrm{d}\vec{S} = \frac{1}{p!} \operatorname{Lt} \sum \boldsymbol{\Omega}_{ab\ldots x}\, \mathrm{d}S^{ab\ldots x}, \tag{1.7}$$

where for each element of the mesh, $\mathrm{d}\vec{S}$ is the p-vector defined by

$$\mathrm{d}\vec{S} = \mathrm{d}\vec{x}_{(1)} \wedge \mathrm{d}\vec{x}_{(2)} \wedge \ldots \wedge \mathrm{d}\vec{x}_{(p)}, \tag{1.8}$$

and where *ht* $\sum$ denotes the sum over all the elements in the limit when the mesh is made infinitesimally small. In the particular case when $S = \partial\Sigma$, i.e. when S is the *boundary* of some higher $(p+1)$ dimensional

surface or volume Σ, then the surface integral will obey the well-known Stokes law:

$$\oint_{S=\partial\Sigma} \boldsymbol{\Omega}\, \mathrm{d}\vec{S} = \int_{\Sigma} (\nabla \wedge \boldsymbol{\Omega})\, \mathrm{d}\vec{S}. \tag{1.9}$$

(For mathematical derivations of basic formulae such as (1.4), (1.6) or (1.9) the reader is referred to Choquet-Bruhat (1968) or Flanders (1963).)

2. The concept of flux conservation

We are now ready to consider the general properties of *integral conservation laws* of the kinds that are important throughout physics. It can be seen directly from (1.9) that we shall obtain a conservation law of the strongest form whenever the p-form $\boldsymbol{\Omega}$ is closed in the sense that its exterior derivative vanishes, i.e.

$$\nabla \wedge \boldsymbol{\Omega} = 0, \tag{2.1}$$

since in this case the integral of $\boldsymbol{\Omega}$ will have the same magnitude for any pair of surfaces Σ_1, and Σ_2 which together enclose a higher-dimensional surface or volume. A more interesting kind of conservation law is defined with respect to a flow generated by a vector field $\vec{\xi}$ say. Let us consider a family of surfaces that is *comoving* with flow in the sense that each member is characterized by a variable value of the flow parameter τ, and otherwise specified by *fixed* values of the other coordinates, in a comoving coordinate system of the kind described at the outset of the previous section, i.e. a system such that

$$\xi^a \partial/\partial x^a \equiv \partial/\partial\tau. \tag{2.2}$$

It is evident from our definition of Lie differentiation that the rate of change with respect to τ of the integral of a field $\boldsymbol{\Omega}$ over a member of such a family will be given by

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Sigma(\tau)} \boldsymbol{\Omega}\, \mathrm{d}\vec{\Sigma} = \int_{\Sigma} (\vec{\xi} \pounds \boldsymbol{\Omega})\, \mathrm{d}\vec{\Sigma}, \tag{2.3}$$

which is equivalent by the theorems of Stokes (1.9) and Cartan (1.4) to

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\Sigma(\tau)} \boldsymbol{\Omega}\, \mathrm{d}\vec{\Sigma} = \int_{\Sigma} \vec{\xi} \cdot (\nabla \wedge \boldsymbol{\Omega})\, \mathrm{d}\vec{\Sigma} + \oint_{S=\partial\Sigma} \vec{\xi} \cdot \boldsymbol{\Omega}\, \mathrm{d}\vec{S}. \tag{2.4}$$

The field $\boldsymbol{\Omega}$ is therefore said to be *conserved under transport by* $\vec{\xi}$ if

$$\vec{\xi}\pounds\boldsymbol{\Omega}=0, \tag{2.5}$$

since in this case $\int\boldsymbol{\Omega}\,\mathrm{d}\vec{\Sigma}$ will be independent of τ for any comoving family of surfaces. A particularly important special case is that in which the *flux of* $\boldsymbol{\Omega}$ *is conserved* in the sense that $\int\boldsymbol{\Omega}\,\mathrm{d}\vec{\Sigma}$ is the same for *any* surface Σ cutting across a given flux tube, i.e. intersecting a given family of flow lines. The condition for flux conservation in this sense is that the Lie derivative should be zero with respect to *any* vector field parallel to the flow, regardless of its normalization, i.e.

$$(\alpha\vec{\xi})\pounds\boldsymbol{\Omega}=0 \tag{2.6}$$

for all possible choices of the arbitrary scale factor α. Using the easily verified general formula

$$\nabla\wedge(\alpha\boldsymbol{\omega})=\alpha\nabla\wedge\boldsymbol{\omega}+(\nabla\alpha)\wedge\boldsymbol{\omega} \tag{2.7}$$

for any differential form $\boldsymbol{\omega}$, one can easily deduce from the Cartan formula (1.9) that

$$(\alpha\vec{\xi})\pounds\boldsymbol{\Omega}=\alpha(\vec{\xi}\pounds\boldsymbol{\Omega})+(\nabla\alpha)\wedge(\vec{\xi}\cdot\boldsymbol{\Omega}) \tag{2.8}$$

and hence that (2.6) can be satisfied for general α if and only if *both* the conditions

$$\vec{\xi}\cdot(\nabla\wedge\boldsymbol{\Omega})=0, \tag{2.9}$$

$$\vec{\xi}\cdot\boldsymbol{\Omega}=0, \tag{2.10}$$

are satisfied. In the particular case when $\boldsymbol{\Omega}$ is closed (as in the Maxwellian example) then (2.10) alone will be a sufficient condition for flux conservation. For a given field $\boldsymbol{\Omega}$ the set of vectors $\vec{\xi}$ (if any exist) simultaneously satisfying (2.10) and (2.9) may be conveniently termed *flux vectors* since the flux of $\boldsymbol{\Omega}$ is conserved along any tube generated by lines parallel to such vectors. The flux vectors necessarily have the remarkable property that if at each point they span a tangent subspace of more than one dimension, then these flux tangent subspaces mesh together in such a way as to form a well-behaved family of *flux surfaces* (which would be called magnetic field surfaces in the Maxwellian case) which form a foliation in the sense that there is one surface through each point. With a little geometric intuition one can see that to prove that the flux tangent subspaces are indeed integrable to form such a foliation it suffices to show that the direction of the subspaces, as measured in terms of any coordinate system comoving with one of the flux vector fields, is invariant

under changes of the corresponding flow parameter τ. In coordinate-independent terms this is clearly equivalent to the requirement that the Lie derivative of any one of the flux vector fields $\vec{\eta}$ say with respect to any other one, $\vec{\xi}$ say, should always lie in the flux tangent subspace and thus itself be a flux vector field. The Lie derivative of one vector field with respect to another is known as their *commutator* since Lie differentiation with respect to the commutator gives the net result of commuting the order of successive lie differentiations, i.e.

$$(\vec{\xi}\pounds\vec{\eta})\pounds\equiv(\vec{\xi}\pounds)(\vec{\eta}\pounds)-(\vec{\eta}\pounds)(\vec{\xi}\pounds), \tag{2.11}$$

when acting on any arbitrary fields (see e.g. Yano 1965). Since $\vec{\eta}$ is contravariant the commutator $\vec{\xi}\pounds\vec{\eta}$ cannot be obtained from the Cartan formula (1.4) but is given instead by the even more widely known formula

$$\vec{\xi}\pounds\vec{\eta}\equiv(\vec{\xi}\cdot\nabla)\vec{\eta}-(\vec{\eta}\cdot\nabla)\vec{\xi}\equiv-\vec{\eta}\pounds\vec{\xi}. \tag{2.12}$$

This expression shares with exterior differentiation the property that it makes *no difference* whether ∇ is interpreted in terms of covariant derivatives or of the simple partial derivatives to whose use we are restricting ourselves here. It follows from (2.12) that for any scalar field α,

$$(\alpha\vec{\xi})\pounds\vec{\eta}=\alpha(\vec{\xi}\pounds\vec{\eta})-(\vec{\eta}\cdot\nabla\alpha)\vec{\eta}. \tag{2.13}$$

Now if $\vec{\xi}$ and $\vec{\eta}$ both satisfy (2.5), it is evident from the commutator formula (2.10) that the same will be true of $\vec{\xi}\pounds\vec{\eta}$. Furthermore, if $\vec{\xi}$ satisfies (2.6) for abitrary α then one can similarly deduce that $(\alpha\vec{\xi})\pounds\vec{\eta}$ will satisfy (2.5) and hence, with the aid of (2.13), that $\vec{\xi}\pounds\vec{\eta}$ will satisfy (2.6) for arbitrary α. This completes the demonstration that the commutator of the two flux tangent vector fields is itself a flux tangent vector field and hence that the flux tangent spaces are integrable to form a well-behaved foliation by flux surfaces.

The foregoing results apply to a differential form $\boldsymbol{\Omega}$ of arbitrary order in a space of arbitrary dimension. (The most commonly occurring applications in which the dimension is higher than four are those involving *phase spaces*). However the most familiar application is where $\boldsymbol{\Omega}$ is the Maxwellian 2-form $\boldsymbol{F}$ in ordinary four-dimensional space–time. Since a Maxwell field $\boldsymbol{F}$ always satisfies the closure condition (2.1), the single condition $\vec{\xi}\cdot\boldsymbol{F}=0$ will be both necessary and sufficient for $\vec{\xi}$ to be a flux tangent vector. In other words the flux tangent vectors are the *zero-eigenvalue* characteristic vectors of the component matrix of $\boldsymbol{F}$. Now the component matrix will have a zero eigenvalue if and only if it is *degenerate* in the sense that its determinant is zero, which is equivalent to the condition

$$(*F^{ab})F_{ab} = 0, \tag{2.14}$$

where $*F$ is the *bivector* (i.e. the antisymmetric contravariant tensor that is adjoint to F in the sense that

$$*F^{ab} = \tfrac{1}{2}\vec{\varepsilon}^{abcd}F_{cd}, \tag{2.15}$$

where $\vec{\varepsilon}$ is the fundamental four-dimensional *alternating tensor*, whose normalization is defined in terms of the metric by the condition $\vec{\varepsilon}^{abcd}\varepsilon_{abcd} = -4!$. Since F is antisymmetric its matrix rank must in any case be even, and hence if F is non-zero but has zero determinant its *matrix rank will be two*. In terms of the rest frame defined by some unit timelike vector $\vec{u}$ we can decompose F into electric and magnetic parts in the form

$$F = u \wedge E + *(\vec{u} \wedge \vec{B}), \tag{2.16}$$

where E and $\vec{B}$ are respectively a covariant and a contravariant vector defined by

$- \vec{u} \cdot F$

$$E = \vec{u} \cdot F, \qquad \vec{B} = (*F) \cdot u, \tag{2.17}$$

and satisfying

$$\vec{u} \cdot E = 0 = \vec{B} \cdot u. \tag{2.18}$$

In terms of these the scalar invariants of the field are expressed by

$$\tfrac{1}{2}F^{ab}F_{ab} = \vec{B} \cdot B - \vec{E} \cdot E, \tag{2.19}$$

and

$$\tfrac{1}{2}(*F^{ab})F_{ab} = \vec{B} \cdot E \tag{2.20}$$

(whose transformations from contravariant to covariant form are performed by using the metric). When the degeneracy condition is satisfied, i.e. when the second invariant (2.20) vanishes, the field will be said to be purely magnetic or purely electric according to whether the first invariant (2.19) is positive or negative. (If the first invariant is also zero, as in the case of plane waves, the field is said to be null.) It is only in these cases that *non-zero flux vectors* can exist, and when they do exist our foregoing work shows that they will form well-behaved *two-dimensional flux surfaces*, which will clearly be *timelike* in the purely magnetic case. Purely magnetic fields play an important role in idealized models for astrophysical purposes. The most familiar example is the *perfect magneto-hydrodynamic fluid*, for which the flow vector $\vec{u}$ itself can be substituted in place of $\vec{\xi}$ in (2.8) i.e.

$$\vec{u} \cdot F = 0,$$

so that

$$F = *(\vec{u} \wedge \vec{B}).$$ (E = 0 ds le ref. du fluide

Another example whose relevance to black hole models of pulsars has been emphasized by Blandford & Znajek (1977) is that of a *force-free* electromagnetic field, in which the vector $\vec{\xi}$ in (2.8) may be identified with the electromagnetic current vector $\vec{j}$.

3. Canonical formulation of perfect fluid mechanics

We now come to the main point of this paper which is to show how the concepts described in the previous section provide a very powerful and efficient approach to the derivation of conservation laws in perfect fluid mechanics. The results to be derived here will in fact apply to any motion in which the flow lines obey a *Lagrangian variation principle* in the sense that (for any particular flow configuration) it is possible to construct a Lagrangian function $L(x, \vec{v})$, space–time coordinates x^a, and 4-velocity components v^a where the velocity

$$v^a = dx^a/d\tau \tag{3.1}$$

(which is not necessarily a unit vector) is defined in terms of a (not necessarily proper) time parameter τ, in such a way that the equations of the flow lines can be expressed in the standard variational form

$$d\pi_a/d\tau = \partial L(x, \vec{v})/\partial x^a, \tag{3.2}$$

where the covector π, which (subject to suitable normalization conditions) plays the role of an *effective momentum* per idealized particle of the fluid, is given by

$$\pi_a = \partial L(x, \vec{v})/\partial v^a. \tag{3.3}$$

Now although (3.2) is the standard form of Lagrange's equation for point particle mechanics it is not the most convenient starting point for a fluid treatment. It has the disadvantage that neither side transforms as a well-behaved covariant vector under space–time coordinate transformations since on the one hand the operation $d/d\tau$ denotes ordinary (not covariant) differentiation, so that the left hand side has the non-covariant form

$$d\pi_a/d\tau = v^b \frac{\partial \pi_a}{\partial x^b}, \tag{3.4}$$

while on the other hand $\partial L(x, \vec{v})/\partial x^a$ does not represent a space–time gradient since the components v^a are supposed to be held constant during the partial differentiation, despite the fact that they are *variables* over space–time. It can be seen from (3.3) that the true gradient components $\nabla_a L$ of the Lagrangian are related to the right hand side of (3.2) by

$$\nabla_a L = \partial L(x, \vec{v})/\partial x^a + \pi_b \partial v^b/\partial x^a. \tag{3.5}$$

This shows that the Lagrangian equation (3.2) may be converted to a more satisfactory form, in which each side is a respectable covector, by adding $\pi_b \partial v^b/\partial x^a$ to both sides, and with the aid of Cartan's formula (1.4) one can easily see that the result takes the remarkably elegant form

$$\overline{\underline{\vec{v}\pounds\boldsymbol{\pi} = \nabla L.}} \tag{3.6}$$

One of the main purposes of this paper is to emphasize that in the context of fluid theory it is most convenient to take (3.6), rather than the traditional single particle form (3.2), as the basic Lagrangian variation equation.

One of the most immediately obvious consequences of (3.6) is a generalized Kelvin–Helmholtz theorem to the effect that the circulation,

$$\mathscr{C}(c) = \oint_c \boldsymbol{\pi} \cdot \mathrm{d}\vec{x}, \tag{3.7}$$

around any closed circuit c, is *conserved* as the surface is dragged along by the flow lines at a uniform rate as measured in terms of the Lagrangian time parameter τ. For *any* (not necessarily closed) curve s the equation (2.3) leads to

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_s \boldsymbol{\pi} \cdot \mathrm{d}\vec{x} = \int_s \nabla L \cdot \mathrm{d}\vec{x} = \Delta L, \tag{3.8}$$

where ΔL is to be interpreted as the difference (if any) between the values of L at the end points (if any) of the curve s; this leads immediately to the required result

$$\mathrm{d}\mathscr{C}/\mathrm{d}\tau = 0 \tag{3.9}$$

in the closed circuit case.

We can convert the fundamental variational equation (3.6) into a form that is even more useful for the derivation of conservation laws by introducing the generalized *vorticity* 2-form $\boldsymbol{w}$ defined by

$$\boldsymbol{w} = \nabla \wedge \boldsymbol{\pi}, \tag{3.10}$$

and the *Hamiltonian scalar* field H defined by

$$H = \vec{v} \cdot \boldsymbol{\pi} - L. \tag{3.11}$$

In terms of these (3.6) may be written in what we shall refer to as the fundamental *canonical* form, namely

$$\overline{\underline{\vec{v} \cdot \boldsymbol{w} = -\nabla H,}} \tag{3.12}$$

whose equivalence with (3.6) is immediately evident from the Cartan formula (1.4).

Provided the defining equation (3.3) for the momentum can be solved to give the Lagrangian velocity components v^a as functions of x^a and π_a, the expression (3.11) can be used to define not just a scalar field but a genuine Hamiltonian *function* $H(x, \boldsymbol{\pi})$, in terms of which the velocity will be given by the Hamilton equation

$$v^a = \partial H/\partial \pi^a, \tag{3.13}$$

and under these circumstances (3.6) will be equivalent to the second Hamilton equation,

$$\mathrm{d}\pi_a/\mathrm{d}\tau = -\partial H/\partial x^a, \tag{3.14}$$

whose familiar single-particle form shares with (3.2) the disadvantage (as compared with (3.12)) that the sides are not separately covariant. However it is worth emphasizing that even in the degenerate special case (which will be shown to be relevant to the discussion of *purely magnetic* fields) when (3.3) is not soluble, so that a proper Hamiltonian *function* (of x^a and π_a) does not exist, there is still no impediment to the definition of a Hamiltonian *scalar field* H as given by (3.11), so that the *canonical equation of motion* (3.12) will *always* be valid.

The circulation theorem that we have already derived is intimately related to conservation of vorticity. If we define the *vorticity flux* $\mathscr{W}$ across a 2-surface S (which we may suppose to be bounded by the circuit c introduced above) as

$$\mathscr{W}(S) = \int_S \boldsymbol{w}\ \mathrm{d}\vec{S}, \tag{3.15}$$

then we can immediately deduce that it will be unchanged as the surface is dragged along the flow lines at a uniform rate in terms of the Lagrangian parameter (as defined by 3.1) simply by applying Stokes theorem (1.9) which gives

$$\mathscr{W}(S) = \mathscr{C}(\partial S), \tag{3.16}$$

and hence by (3.9)

$$\mathrm{d}\mathcal{W}/\mathrm{d}\tau = 0. \tag{3.17}$$

This general flux-conservation theorem could also have been deduced directly, by using (2.3), from the local *vorticity-conservation* theorem

$$\vec{v}\pounds w = 0, \tag{3.18}$$

which is obtainable from the Cartan formula (1.4) by substituting the canonical equation of motion (3.12) and using the *closure condition*

$$\nabla \wedge w = 0, \tag{3.19}$$

which results from the construction of w as an exterior derivative.

An immediate consequence of (3.18) is the possibility of *potential flow*, as characterized by the requirement that the momentum 1-form be the *gradient* of some (at least locally defined) scalar *action* function S, i.e.

$$\pi = \nabla S, \tag{3.20}$$

which is equivalent to the *zero-vorticity* requirement,

$$w = 0, \tag{3.21}$$

since it is evident from (3.18) that if (3.21) is satisfied as a constraint on an *initial hypersurface* then it will remain satisfied *throughout* the subsequent flow.

It is directly evident from the canonical equation of motion (3.12) (simply by contracting with $\vec{v}$ and using the antisymmetry of w) that the Hamiltonian scalar itself is *always* constant along the flow lines, i.e.

$$\vec{v} \cdot \nabla H = 0. \tag{3.22}$$

An important special case, which we shall refer to as that of a *uniformly* canonical system, occurs when the stronger condition,

$$\nabla H = 0, \tag{3.23}$$

is satisfied. By (3.2) it is clear that for H to satisfy this uniformity condition throughout the fluid it is sufficient to impose a uniform value on H as a *constraint* on an initial hypersurface. (We shall see later that this uniformity condition holds *automatically* for a perfect fluid that is isentropic (in the sense that the entropy per particle is uniform) and hence in particular for a perfect fluid at *zero temperature*.)

We are now ready to consider the question of when the fluid flow lines will be *flux-conservation* lines for the vorticity flux in the strong sense discussed in § 2, i.e. the question of when (3.18) can be strengthened to the form,

$$(\alpha\vec{v})\pounds w = 0, \tag{3.24}$$

for an arbitrary variable velocity renormalization factor α. In view of the *closure* property (3.19) it follows from the results of § 2 that (3.24) will hold if and only if w satisfies the *uniformly canonical equation of motion*

$$\vec{v} \cdot w = 0, \tag{3.25}$$

to which (3.12) reduces if and only if the uniformity condition (3.23) is satisfied. If the uniformity requirement is satisfied then the vorticity form w will have the same properties as those described in § 2, for the Maxwell form F in the purely magnetic case. The vorticity form will have rank *two* (unless it vanishes altogether), which incidentally is the necessary and sufficient condition for its exterior product with itself to vanish, i.e.

$$w \wedge w = 0, \tag{3.26}$$

and so the space–time will be foliated by a well-defined congruence of *vorticity 2-surfaces* whose tangent vectors are all zero-eigenvalue characteristic vectors of w. Under these circumstances the flux $\mathcal{W}(S)$ will be invariant not merely when S is translated uniformly with respect to τ along the flow lines, but for any change whatsoever, provided the new position of the surface S is such that it still intersects the same set of vorticity 2-surfaces as before.

Having discussed circulation round one-dimensional circuits and vorticity flux across 2-surfaces we now come to consider the *helicity* integral over 3-surfaces, that is the natural relativistic generalization of the Newtonian helicity volume integral whose conservation under appropriate circumstances was demonstrated by Moffat (1969). For any hypersurface Σ we define the helicity integral $\mathcal{H}$ to be

$$\mathcal{H}(\Sigma) = \int_\Sigma \pi \wedge w \,\mathrm{d}\vec{\Sigma}. \tag{3.27}$$

Now the closure of w implies that the exterior derivative of the *helicity* 3-form, $\pi \wedge w$, will be given by

$$\nabla \wedge (\pi \wedge w) = w \wedge w \tag{3.28}$$

(which shows by (3.26) that the helicity form is itself closed in the *uniformly* canonical case). Using this in conjunction with the purely algebraic identity,

$$\vec{v} \cdot (w \wedge w) = 2(\vec{v} \cdot w) \wedge w, \tag{3.29}$$

one can show that *whether or not* the uniformity condition (3.23) is satisfied, the Cartan equation (1.4) taken in conjunction with the canonical equation of motion (3.12) leads to

$$\vec{v}\pounds(\boldsymbol{\pi} \wedge \boldsymbol{w}) = \nabla \wedge (L\boldsymbol{w}). \tag{3.30}$$

This implies quite generally by (2.4) that if Σ is translated along the flow lines uniformly with respect to the Lagrangian time coordinate τ, then the helicity integral varies according to

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\tau} = \oint_{S=\partial\Sigma} L\boldsymbol{w}\ \mathrm{d}\vec{S}. \tag{3.31}$$

Since we have

$$L\boldsymbol{w} = \boldsymbol{\pi} \wedge \nabla L + \nabla \wedge (L\boldsymbol{\pi}), \tag{3.32}$$

the theorem of Stokes allows us to rewrite (3.30) in the equivalent alternative form

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}\tau} = \oint_{S=\partial\Sigma} \boldsymbol{\pi} \wedge \nabla L\ \mathrm{d}\vec{S}. \tag{3.33}$$

It can be seen from (3.31) that in particular the helicity will be conserved for any uniformly comoving volume surrounded by a domain of zero vorticity (i.e. potential) flow. The condition that the volume should be comoving uniformly (i.e. at a constant rate as measured with respect to the Lagrangian time coordinate τ) may of course be relaxed when the system itself is *uniformly canonical*, owing to the previously remarked fact that (by (3.26) and (3.28)) the helicity 3-form will then satisfy the closure condition

$$\nabla \wedge (\boldsymbol{\pi} \wedge \boldsymbol{w}) = 0. \tag{3.34}$$

We conclude our remarks on helicity by pointing out that any 3-form may be thought of as the adjoint of a *current vector*, and in particular we may consider the helicity form $\boldsymbol{\pi} \wedge \boldsymbol{w}$ as the *adjoint* of a *helicity current vector* $\vec{\eta}$ say, i.e.

$$\boldsymbol{\pi} \wedge \boldsymbol{w} = *\vec{\eta}, \tag{3.35}$$

where

$$(*\eta)_{abc} = \varepsilon_{abcd}\eta^d. \tag{3.36}$$

Conversely we may express the helicity current vector explicitly by

$$\vec{\eta} = (*\boldsymbol{w}) \cdot \boldsymbol{\pi}.$$

(The helicity $\mathscr{H}(\Sigma)$ is thus interpretable as the flux of $\vec{\eta}$ across the hypersurface Σ.) The closure condition (3.34) that is satisfied in the uniformly canonical case is precisely equivalent to the condition that $\vec{\eta}$ have zero divergence (i.e. $\nabla \cdot \vec{\eta} = 0$, in the language of covariant differentiation).

This completes our discussion of conservation laws that are valid independently of any special (e.g. stationary) symmetry of space–time. Before going on, in § 5, to discuss the more specialized conservation laws that hold when such space–time symmetries are present, we shall digress in the next section to describe concrete examples of idealized systems to which our analysis applies, in order to provide the reader with an insight into the physical interpretation of the results.

4. Examples of canonical systems

Before listing particular examples of canonical systems we shall recapitulate the definition of the *kinematic* rotation vector $\vec{\omega}$ of a fluid, since it will be of interest to examine its relationship to the canonically defined *dynamic* vorticity form $\boldsymbol{w}$. For a fluid with unit flow 4-vector $\vec{u}$, the rotation vector $\vec{\omega}$ is defined by

$$\omega^a = \tfrac{1}{2}\varepsilon^{abcd}u_b\nabla_c u_d \tag{4.1}$$

(where, as in all our numbered equations, the symbol ∇_c can be interpreted as indicating either covariant or simple partial differentiation, without affecting the result). This vector is, by construction, orthogonal to the flow (i.e. $\omega^a u_a = 0$) and it can be interpreted as measuring the *local angular velocity* of the fluid relative to an inertial frame at the point under consideration. The kinematic rotation vector also has the property that (by the theorem of Frobenius; see Choquet-Bruhat, Bleick-Dillard & DeWitt (1977) or Flanders (1963)) its *vanishing* is a necessary and sufficient condition for the local existence of a well-behaved family of hypersurfaces everywhere *orthogonal* to the flow lines.

We now provide a list of examples, labelled (a) to (d) in progressively increasing order of complexity. Except for the first example (which is degenerate in the sense that a proper Hamiltonian function does not even exist) the four-dimensional systems here differ formally from their corresponding three-dimensional Newtonian analogues in that the physically meaningful solutions of the system are characterized by a *restraint* fixing the initial (and hence the subsequent) values of the Hamiltonian scalar, thereby removing what would otherwise be a redundant degree of freedom.

(a) An example of a canonical system that is *degenerate* (in the sense that there is a Lagrangian but no proper Hamiltonian function) as well as satisfying the uniformity condition (that the Hamiltonian scalar field be zero) is provided by the case of a *purely magnetic field* (as defined in § 2), which may be characterized by a Lagrangian function of the form

$$L(x, \vec{v}) = \vec{v} \cdot \boldsymbol{A}, \tag{4.2}$$

where $\boldsymbol{A}$ is the electromagnetic 4-potential (treated as a function of space–time position only) and where $\vec{v}$ may be variously interpreted as the electric charge current vector in the *force-free* case or as the ordinary fluid flow velocity in the perfect *magnetohydrodynamic* case.

For this particular Lagrangian the effective momentum convector is given simply by

$$\boldsymbol{\pi} = \boldsymbol{A}, \tag{4.3}$$

so that the canonical vorticity 2-form w will simply be the electromagnetic field itself, i.e.

$$w = \boldsymbol{F}; \tag{4.4}$$

while the corresponding Hamiltonian scalar field will simply vanish, i.e.

$$H = 0. \tag{4.5}$$

This implies that the equation of motion has the *uniformly* canonical form (3.25), which will be equivalent to the *purely magnetic* field condition (2.21) provided $\vec{v}$ is timelike and so parallel to a unit vector $\vec{u}$ that can be used for making the decomposition (2.17). We saw that in the uniformly canonical case the helicity, which in this case is given by

$$\vec{\eta} = (*\boldsymbol{F}) \cdot \boldsymbol{A}, \tag{4.6}$$

will necessarily be strictly conserved (meaning that $\vec{\eta}$ is divergence free). This particular application of our general helicity-conservation law was originally found by Woltjer in 1958.

In this degenerate case the vorticity has no direct relation to the rotation vector, but is related to the magnetic field vector by

$$*w = \vec{u} \wedge \vec{B}. \tag{4.7}$$

(b) Our next-simplest example is that of *geodesic motion* which is satisfied by free particles and hence also by a *pressure-free* perfect fluid ('dust'). It

is well known that geodesics are derivable from a Lagrangian of the form

$$L(x, \vec{v}) = \tfrac{1}{2} g_{ab} v^a v^b, \tag{4.8}$$

where g_{ab} are the covariant metric components, so that the corresponding Hamiltonian function is given in terms of the contravariant (inverse) metric components g^{ab} by

$$H(x, \boldsymbol{\pi}) = \tfrac{1}{2} g^{ab} \pi_a \pi_b. \tag{4.9}$$

If we impose the constraint

$$H = -\tfrac{1}{2} m^2, \tag{4.10}$$

where m is a constant representing the *mass* in the free particle case, or the *mass per particle* in the pressure-free fluid case, then we see that the canonical velocity $\vec{v}$ and the canonical momentum $\boldsymbol{\pi}$ will be given in terms of the *unit* 4-velocity $\vec{u}$ by

$$v^a = m u^a, \qquad \pi_a = m u_a. \tag{4.11}$$

(In the zero-rest-mass limit, $\vec{u}$ cannot be defined but $\vec{v}$ and $\boldsymbol{\pi}$ remain finite.) In this particular case not only the Hamiltonian but also the *Lagrangian* scalar field L is constant, since (4.9) clearly implies

$$L = -\tfrac{1}{2} m^2 \tag{4.12}$$

as well. Under such circumstances it is evident from (3.32) that no special boundary conditions are needed to ensure that helicity is conserved over a hypersurface that is transported uniformly with respect to canonical time τ (which in this example is proportional to ordinary *proper* time) along the geodesic flow lines.

In this case the vorticity form will be given in terms of the rotation vector by

$$*w = 2m\vec{u} \wedge \vec{\omega}, \tag{4.13}$$

and hence the conserved helicity will simply be represented by the divergence-free spacelike vector

$$\vec{\eta} = 4m^2 \vec{\omega}. \tag{4.14}$$

(c) We now come to the first non-trivial example of a uniformly canonical fluid system, namely a *single-constituent perfect fluid*, with finite internal pressure and with allowance for the possibility that there is a net electric charge per particle, e say. This example includes both the preceding examples (a) and (b) as limiting special cases.

A perfect fluid is defined by the condition that its energy momentum tensor T has components given by

$$T^a{}_b = (\rho + P)u^a u_b + P g^a{}_b, \qquad (4.15)$$

where ρ is the energy density in the rest frame defined by the unit flow vector $\vec{u}$ and where P is the pressure. A *single-constituent* perfect fluid is characterized by the further requirement that both P and ρ are functions of the rest-frame number density n of a single type of conserved particle. By the first law of thermodynamics the pressure and energy-density functions cannot be independent but are related by

$$P = n\mu - \rho, \qquad (4.16)$$

where the *enthalpy per particle*, μ, is given by

$$\mu = \mathrm{d}\rho/\mathrm{d}n. \qquad (4.17)$$

It is well known (see e.g. Lichnerowicz 1967) that the flow lines of such a fluid have a variational form, since they have the same form as would *free-particle* trajectories in the conformally modified space–time constructed by setting $\mu^2 g_{ab}$ in place of g_{ab}. This means that in the electrically neutral subcase the system could be expressed in canonical form simply by replacing the Lagrangian $\frac{1}{2}g_{ab}v^a v^b$ of example (b) by $\frac{1}{2}\mu^2 g_{ab}v^a v^b$, an approach which would retain the technical advantage of having a uniform *Lagrangian* as well as Hamiltonian scalar field. What I shall actually present however is a slightly modified Lagrangian that has been chosen to satisfy instead the requirement that the canonical time parameter is the same as the ordinary proper time. This is achieved by taking

$$L(x, \vec{v}) = \tfrac{1}{2}\mu g_{ab}v^a v^b + eA_a v^a - \tfrac{1}{2}\mu, \qquad (4.18)$$

where e is the net charge per particle if any, subject to the very simple restraint,

$$H = 0, \qquad (4.19)$$

imposed on the corresponding Hamiltonian function which is

$$H(x, \boldsymbol{\pi}) = \frac{1}{2\mu} g^{ab}(\pi_a - eA_a)(\pi_b - eA_b) + \tfrac{1}{2}\mu. \qquad (4.20)$$

It can be seen that the canonical velocity vector $\vec{v}$ and the momentum covector $\boldsymbol{\pi}$ appearing in this system will be given in terms of the *unit* flow vector $\vec{u}$ of the fluid by

$$v^a = u^a, \qquad \pi_a = \mu u_a + eA_a, \qquad (4.21)$$

which shows that μ plays the role of a variable *effective mass* per particle. The restraint (4.19) immediately ensures that the system is *uniformly* canonical, but we now have a *variable* Lagrangian scalar given by

$$L = e\boldsymbol{A} \cdot \vec{u} - \mu. \qquad (4.22)$$

(We leave it as an easy exercise for readers familiar with covariant differentiation to show for this system that the uniformly canonical equation of motion (3.25) is equivalent, when taken in conjunction with the particle conservation law $\nabla \cdot (n\vec{u}) = 0$, to the standard covariant energy momentum conservation law $\nabla \cdot T = en\boldsymbol{E}$ with $\boldsymbol{E}$ as in (2.17).)

With the aid of the uniformly canonical equations of motion one can show that the conserved vorticity form is related to the magnetic field and rotation vectors (as defined by (2.17) and (4.1)) by

$$*\boldsymbol{w} = \vec{u} \wedge (2\mu\vec{\omega} + e\vec{B}), \qquad (4.23)$$

which means that the vorticity flux 2-surfaces through each point have independent tangent vectors given by $\vec{u}$ and by the linear combination $2\mu\vec{\omega} + e\vec{B}$. The conserved helicity vector is now rather complicated, being given by

$$\vec{\eta} = (\mu - e\boldsymbol{A} \cdot \vec{u})(2\mu\vec{\omega} + e\vec{B}) + (2e\mu\boldsymbol{A} \cdot \vec{\omega} + e^2 \boldsymbol{A} \cdot \vec{B})\vec{u}. \qquad (4.24)$$

It can be seen that these quantities reduce to the corresponding values as given in example (a) in the limit $\mu = 0$, $e = 1$ and to those in example (b) in the limit $e = 0$, $\mu = m$.

(d) We now come to an example which unlike the preceding ones is *non-uniformly* canonical. This is the case of a non-conducting *two-constituent perfect fluid*. Such a fluid is characterized by an energy momentum tensor of the same form (4.15) as in the preceding example, but the energy ρ is now supposed to be a function of two *independently conserved* number densities, n and s say, which to be definite we shall interpret as representing conserved physical particles (e.g. baryons or under less extreme conditions ordinary molecules) and entropy respectively. (The non-conducting condition may then be interpreted as meaning that the flow is adiabatic.)

In such a case the energy density function $\rho(n, s)$ will determine a pair of potentials μ^N and Θ (which may respectively be interpreted as the relativistic chemical potential of the particles and the thermodynamic temperature) by the variation rule,

$$\mathrm{d}\rho = \mu^N \,\mathrm{d}n + \Theta \,\mathrm{d}s, \qquad (4.25)$$

and by the first law of thermodynamics the enthalpy per particle will be given in terms of these by

$$\mu = \frac{\rho + P}{n} = \mu^N + \frac{\Theta s}{n}. \tag{4.26}$$

Our previous example (c) can be thought of as representing the *isentropic* special case in which the ratio s/n is uniform throughout (this ratio being in any case constant along the separate flow lines). The physical relevance of the isentropic case is most plausible in the zero-temperature limit.

The possibility of expressing the equations of motion for the non-isentropic case in variational form is less well known than in the simpler isentropic case (indeed I know of no published reference) but it can be achieved by taking as Lagrangian the function

$$L(x, \vec{v}) = \tfrac{1}{2}\Theta\mu g_{ab}v^a v^b + \tfrac{1}{2}\left(\frac{s}{n} - \frac{\mu^N}{\Theta}\right), \tag{4.27}$$

subject to the restraint

$$H = -s/n, \tag{4.28}$$

imposed on the corresponding Hamiltonian, namely

$$H(x, \pi) = \frac{1}{2\Theta}\left(\frac{g^{ab}\pi_a\pi_b}{\mu} + \mu\right) - \frac{s}{n}. \tag{4.29}$$

For simplicity I have allowed here only for the *electrically neutral* case but the charged possibility can easily be allowed for in the usual way by adding a term $eA_a v^a$ to the Lagrangian, which corresponds to replacing π_a by $\pi_a - eA_a$ in the Hamiltonian, while the restraint is unchanged. In terms of the unit flow vector $\vec{u}$, the canonical velocity and momentum derived from (4.27) and (4.29) will then be given by

$$v^a = (1/\Theta)u^a, \qquad \pi_a = \mu u_a, \tag{4.30}$$

so that as before the enthalpy per particle plays the role of an effective mass. (The appearance of a velocity vector with normalization inversely proportional to the temperature is already familiar in statistical mechanics where $1/\Theta$ turns up as the Lagrange multiplier associated with the energy.) The resulting Lagrangian scalar is given by

$$L = \frac{\mu^N}{\Theta}. \tag{4.31}$$

The total momentum per particle can be decomposed naturally in the form

$$\pi = \pi^N + (s/n)\pi^s, \tag{4.32}$$

where the momenta directly associated with the physical particle and energy contributions are defined by

$$\pi^N{}_a = \mu^N u_a, \qquad \pi^s{}_a = \Theta u_a, \tag{4.33}$$

In terms of these the canonical vorticity tensor (which will still be conserved, but no longer of rank two as in the uniformly canonical preceding examples, so that there will *not* be any family of vorticity flux 2-surfaces) will be given by

$$w = \pi^s \wedge \nabla(s/n) + 2\mu * (\vec{u} \wedge \vec{\omega}). \tag{4.34}$$

(This is rather analogous to the general form of the electromagnetic field, with $\mu\vec{\omega}$ playing the role of the magnetic field and $\Theta\nabla(s/n)$ playing the role of the electric field.) The helicity vector still retains the comparatively simple form,

$$\vec{\eta} = 2\mu^2\vec{\omega}, \tag{4.35}$$

as in the isentropic case, but it is no longer divergence free.

5. Application to stationary and/or axisymmetric systems

Having seen that the canonical equations derived in § 3 are applicable to a useful class of non-dissipative systems we now examine some of the more specialized consequences that can be drawn when there are continuous space–time symmetries present.

To start with, one knows from the more familiar theory of single-particle mechanics that the scalar contraction of the momentum with a symmetry generator will always be a constant of the motion and hence in the fluid context, a constant along the flow lines. Thus, e.g. for a *stationary* symmetry generator $\vec{k}$ or an *axial* symmetry generator $\vec{h}$ respectively, we shall have

$$\vec{v} \cdot \nabla\mathscr{E} = 0 \quad \text{or} \quad \vec{v} \cdot \nabla\mathscr{J} = 0, \tag{5.1}$$

where the effective energy per particle, $\mathscr{E}$, and the effective angular momentum per particle, $\mathscr{J}$, are given by

$$\mathscr{E} = -\vec{k} \cdot \pi \quad \text{or} \quad \mathscr{J} = \vec{h} \cdot \pi. \tag{5.2}$$

(The constancy of $\mathscr{E}$ is the relativistic generalization of the classical *Bernoulli* theorem.) Such a conservation law can be deduced for an arbitrary symmetry generator $\vec{l}$ directly from the contraction,

$$\vec{l} \cdot \boldsymbol{w} \cdot \vec{v} = \vec{l} \cdot \nabla H, \tag{5.3}$$

of the canonical equation of motion (3.12) by using the Cartan formula (1.4) to express the relevant symmetry requirements in the form

$$\vec{l}\pounds\boldsymbol{\pi} \equiv \vec{l} \cdot \boldsymbol{w} + \nabla(\vec{l} \cdot \boldsymbol{\pi}) = 0, \tag{5.4}$$

$$\vec{l} \cdot \pounds H = \vec{l} \cdot \nabla H, \tag{5.5}$$

which leads immediately to the required result

$$\vec{v} \cdot \nabla(\vec{l} \cdot \boldsymbol{\pi}) = 0. \tag{5.6}$$

(It is to be noted that there is no direct need to assume that the metric itself is invariant, i.e. that $\vec{l}$ is a 'Killing' vector, although it might be difficult for (5.4) and (5.5) to be satisfied otherwise.)

In the *uniformly* canonical case the conclusions can be considerably strengthened since there will then be a two-parameter family of vectors $\vec{\xi}$ tangent to the vorticity flux 2-surfaces, i.e. satisfying

$$\vec{\xi} \cdot \boldsymbol{w} = 0. \tag{5.7}$$

Contracting any one of these with (5.4) leads to

$$\vec{\xi} \cdot \nabla(\vec{l} \cdot \boldsymbol{\pi}) = 0, \tag{5.8}$$

which shows that the relevant scalar $\vec{l} \cdot \boldsymbol{\pi}$ must be constant not only along each flow line but over each vorticity flux 2-surface. In the special case where the *vorticity is zero* (potential flow) one can make the even stronger deduction that the generalized Bernoulli scalar $\vec{l} \cdot \boldsymbol{\pi}$ must be constant over the whole of space–time, since setting $\boldsymbol{w}$ to zero in (5.4) obviously gives

$$\nabla(\vec{l} \cdot \boldsymbol{\pi}) = 0. \tag{5.9}$$

We can also reach the same conclusion for motion that is *rigid* in the sense that the flow itself is parallel to a symmetry generator, i.e. when

$$\vec{v} = \beta \vec{l}, \tag{5.10}$$

for some scalar function β, since in this case the uniformly canonical equation of motion (3.25) can be used to eliminate the term $\vec{l} \cdot \boldsymbol{w}$ from (5.4). A classic example of such a rigid motion – to which this argument could be applied – is the case of a corotating binary star system, wherein

the constant $\vec{l} \cdot \boldsymbol{\pi}$ could be interpreted as meaning the *energy in the corotating frame*, which is known as the 'Jacobi integral' in the context of non-relativistic single-particle mechanics.

Let us now consider the more detailed conclusions that may be drawn when we have (just) *two* linearly independent symmetry generators $\vec{k}$ and $\vec{h}$ which must *commute* (since otherwise by (2.10) there would be a third independent one) and hence generate a family of invariant 2-surfaces. In terms of a system in which the symmetries are made manifest by corresponding ignorable coordinates $t = x^4$ and $\phi = x^3$ (so that $k^a = \delta^a_4$, $h^a = \delta^a_3$) the energy and angular momentum scalars will have the forms

$$\mathscr{E} = -\pi_4, \qquad \mathscr{J} = \pi_3. \tag{5.11}$$

Let us first consider what may be deduced when the flow is *circular* in the sense that the velocity 4-vector is confined to the invariant 2-surfaces defined by the symmetries, i.e.

$$\phi\vec{u} = \vec{k} + \Omega\vec{h}, \tag{5.12}$$

where Ω is the local angular velocity scalar, and ϕ is an 'average redshift' factor. By substituting $\vec{k}$ and $\vec{h}$ in place of $\vec{l}$ in (5.4) we can see that the symmetry requirements lead directly to

$$(\phi\vec{u})\pounds\boldsymbol{\pi} = \mathscr{J}\nabla\Omega. \tag{5.13}$$

However, the canonical equation of motion (3.12) gives

$$(\phi\vec{u})\pounds\boldsymbol{\pi} = -\zeta\nabla H - \nabla\Phi, \tag{5.14}$$

where ζ is defined by

$$\phi\vec{u} = \zeta\vec{v}, \tag{5.15}$$

(its reciprocal ζ^{-1} is the rate of change of coordinate time t with respect to canonical time τ) and Φ is the *zero angular momentum injection energy per particle* (cf. Bardeen 1973; Novikov & Thorne 1973) which is defined by

$$\Phi = -\zeta\vec{v} \cdot \boldsymbol{\pi} = -\phi\vec{u} \cdot \boldsymbol{\pi} = \mathscr{E} - \Omega\mathscr{J}. \tag{5.16}$$

Combining (5.13) and (5.14) gives the useful relation

$$\nabla\Phi + \zeta\nabla H + \mathscr{J}\nabla\Omega = 0, \tag{5.17}$$

which gives strong conclusions in many particular cases, such as:

(i) *If Ω is uniform* – i.e. if the rotation is *rigid* – then the hypersurfaces of constant H and constant Φ must coincide with each other and hence also (except in the uniformly canonical case where ∇H and $\nabla\Phi$ will both be zero) with the hypersurfaces of constant $\zeta(= -\mathrm{d}\Phi/\mathrm{d}H)$.

(ii) If $\mathcal{J}$ is uniform – which we have seen to be *necessary* in the zero-vorticity case – then the hypersurfaces of constant H and $\mathcal{E}$ must coincide with each other and hence also (except in the uniformly canonical case where ∇H and $\nabla\mathcal{E}$ will both be zero) with the hypersurfaces of constant $\zeta(=-\mathrm{d}\mathcal{E}/\mathrm{d}H)$.

(iii) If ζ is uniform – which in example (d) (wherein $\zeta=\phi$) corresponds to effective *thermal equilibrium* – then the hypersurfaces of constant Ω and of constant ζL (which is equal to $-\phi\mu^N$ in example (d)) must coincide with each other, and hence also (except in the rigid case where $\nabla\Omega$ and $\nabla(\zeta L)$ will both be zero) with the hypersurfaces of constant $\mathcal{J}(=\mathrm{d}(\zeta L)/\mathrm{d}\Omega)$.

(iv) In the uniformly canonical (isentropic) case where H is uniform, the hypersurfaces of constant Φ and Ω must coincide with each other and hence also (except in the rigid case where $\nabla\Phi$ and $\nabla\Omega$ are both zero) must coincide with those of $\mathcal{J}$ $(=\mathrm{d}\Phi/\mathrm{d}\Omega)$ and thus also (by 5.16) with those of $\mathcal{E}$.

This fourth application is the generalized von Zeipel theorem of Bardeen (1973) and Abramowicz (1974). (The hypersurfaces on which $\Omega, \mathcal{J}, \mathcal{E}, \Phi$ are all constant are exactly cylindrical in the Newtonian limit.)

Let us now consider what may be deduced if the flow is *not* circular (as for example when there is a finite velocity of accretion onto a central black hole). Since this means that the flow lines – on which H, $\mathcal{E}$ and $\mathcal{J}$ are constant – do *not* lie in the invariant 2-surfaces, we see at once that the hypersurfaces of constant H, $\mathcal{E}$ and $\mathcal{J}$ must *all* coincide with each other (since each such hypersurface will have $\vec{k}$, $\vec{h}$ and $\vec{v}$ as linearly independent tangent vector fields).

When vorticity flux 2-surfaces exist, (i.e. in the *uniformly* canonical case where H is constant everywhere) they must lie in the hypersurfaces of constant $\mathcal{E}$ and $\mathcal{J}$ by (5.8), and hence they must have a common tangent vector, $\vec{\xi}$ say, with the invariant 2-surfaces at each point (provided $\mathcal{E}$ and $\mathcal{J}$ are not actually uniform throughout). We may take this vector $\vec{\xi}$ to be normalized in such a way as to satisfy

$$\vec{\xi}=\vec{k}+\Omega\vec{h}, \tag{5.18}$$

where the quantity Ω represents an effective *angular velocity of rotation* of the flux surfaces. Since it is postulated to satisfy (5.7) as well as (5.18) we can apply the same arguments as in the derivation of (5.13) and (5.14) so as to obtain

$$\vec{\xi}\pounds\pi=\mathcal{J}\nabla\Omega=-\nabla\Phi, \tag{5.19}$$

where

$$\Phi=-\vec{\xi}\cdot\pi=\mathcal{E}-\Omega\mathcal{J}. \tag{5.20}$$

We can thus draw the same conclusions as in example (iv) of the circular flow case: *either* the system of flux 2-surfaces rotates rigidly with uniform Ω, in which case Φ is uniform throughout, *or else* the common hypersurfaces of $\mathcal{E}$ and $\mathcal{J}$, which are known to contain the flux 2-surfaces, must coincide with the hypersurfaces of constant Ω (and thus also of constant Φ). Either way we can conclude that *each individual flux 2-surface rotates rigidly* in the sense that Ω is constant on it. In the particular case of a purely magnetic field (our example (a)) this result is already known as *Ferraro's theorem* (whose relativistic form has been utilized in the context of black hole accretion problems by Blandford & Znajek) but our present derivation shows that (like the more specialized von Zeipel theorem to which it reduces in the circular flow limit) this result is applicable to all *uniformly* canonical systems, including our examples (b) and (c), although not example (d). It is worth drawing attention to this view of the point (recently emphasized by Mestel (1977) in the specific context of pulsar magnetospheres, although it would be equally relevant to Blandford–Znajek type black hole accretion models of quasars) that electromagnetic fields can easily accelerate particles sufficiently for inertial effects to prevent them from corotating with the magnetic field lines of a force-free magnetosphere even in the domain where radiative dissipation is still negligible. What I hope to have made clear here is that in such a regime the particles can *still* be thought of as confined to rigid flux 2-surfaces but that one must make appropriate allowance for inertia in defining the flux 2-surfaces, which are to be considered *not* as being generated just by $\vec{u}$ and $\vec{B}$ as in the force-free case, but by $\vec{u}$ and $e\vec{B}+2\mu\vec{\omega}$ (in which μ can be approximated by the particle mass m when pressure is unimportant).

Let us finish by mentioning some of the detailed conclusions that can be drawn in the more specialized case of *circular flow* (which may often be considered as a good first approximation in the study of black hole accretion discs or rings when radial inflow velocity is sufficiently small), restricting ourselves to the uniformly canonical case described by our example (iv). This has been discussed in detail by Abramowicz *et al.* (1978) who have shown that it is convenient to work with the *specific angular momentum*, which we shall denote by α, as defined by

$$\alpha=\mathcal{J}/\mathcal{E} \tag{5.21}$$

(this is the local analogue of the ratio $a=J/M$ as defined for a star or black hole taken as a whole). Abramowicz *et al.* describe how it is possible to construct an explicit solution for any assumed distribution of α as a function of Ω, by first finding α and Ω in terms of any known (e.g. Schwarzschild or Kerr) background metric as explicit functions of the non-ignorable coordinates, $x^1 = r$ and $x^2 = \theta$ say, using the fact that in terms of a manifestly symmetric system of the kind already described (with $x^3 = \phi$, $x^4 = t$) one will have

$$\alpha = -u_3/u_4, \qquad \Omega = u^3/u^4. \tag{5.22}$$

This equation enables one to find the explicit form of the generalized von Zeipel cylinders, and hence to calculate $\mathscr{E}$, $\mathscr{J}$ and Φ as functions of r and θ to within an overall scale factor of integration. Since explicit knowledge of Ω also enables one to evaluate the redshift factor, which is given by

$$\phi = u^4, \tag{5.23}$$

one can obtain the enthalpy per particle from the expression

$$\mu = \Phi/\phi + e\bar{u}\cdot\mathbf{A}, \tag{5.24}$$

which is derivable directly from (5.16). In the electrically neutral case considered by Abramowicz *et al.* the possibility of rescaling Φ by a constant-of-integration factor merely gives rise to a corresponding rescaling of μ which means that the constant μ surfaces can be plotted independently of the adjustable factor (as well as independently of the equation of state, which has played no part in our discussion). A particularly simple case worked out analytically by Abramowicz *et al.* is the one where the function relation between α and Ω is determined simply by taking α to be *constant* which can be seen to arise automatically for *potential* (i.e. zero-vorticity) flow. Surfaces of constant μ (and hence also of constant P and ρ) are sketched for flow that is irrotational except near the axis in fig. 1. The boundary of the matter ring must lie on one such surface representing the locus $P = 0$. The maximum value of μ occurs at the position of a stable circular orbit. The ring can be filled up to any contour one chooses by adding more matter, until it starts to overflow at a *cusp* which occurs at the position of an *unstable circular orbit.*

For a realistic model the simple constant α approximation may be good enough for study of the immediate neighbourhood of the cusp but farther out one would expect a transition to a Keplevian angular momentum function of the form $\alpha \sim (GM)^{2/3}\Omega^{-1/3}$, so that the shape of the accretion rings would be closer to that indicated in fig. 2. The particular form of the angular momentum function does not affect the fact that the

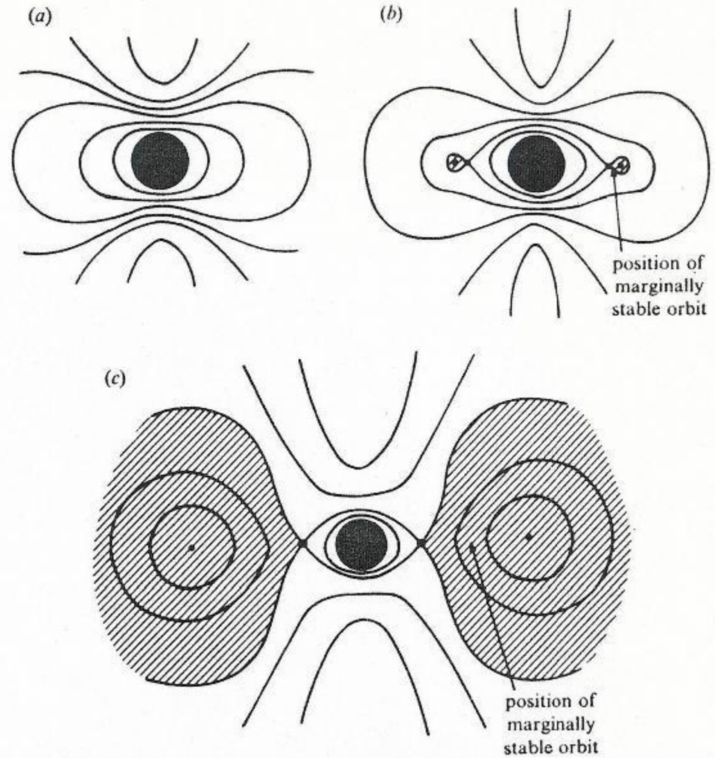

Fig. 1. The qualitative character of constant μ contours is sketched on polar section through a stationary black hole for distributions of α tending to successively larger uniform values outside the neighbourhood of the axis, where α must tend to zero to avoid a singularity of Ω. The blacked in region represents holes, and the shaded region represents the maximum region occupied by matter rings when filled to overflow point.

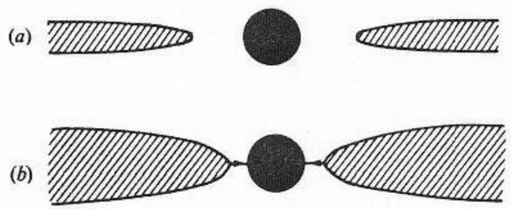

Fig. 2. The qualitative character of region occupied by matter is sketched on polar section through a stationary black hole for approximately Keplerian angular momentum distribution. The blacked in region represents hole and the shaded region represents region occupied by matter for (*a*) unfilled and (*b*) overflowing ring.

cusp will occur at the location at an unstable circular orbit. As excess matter trickles out through the cusp (in a manner precisely analogous to Roche lobe overflow in the more familiar binary star problem) it will take with it the energy of the unstable circular orbit. The energy release will therefore be most efficient in the limiting case of an extremely thin disc for which the cusp occurs very close to the marginally stable circular orbit. For high accretion rates, particularly when the Eddington limit is approached, one would expect radiation pressure to provide significant thickening of the inner part of the disc. This would cause the cusp to move inwards to a less tightly bound unstable circular orbit and as a result would *reduce* the efficiency of the energy release. Such an effect might by itself be sufficient to prevent the Eddington limit from being exceeded.

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Only authors explicitly named in the text are indexed; thus, co-authors are not necessarily included, nor are authors noted in figures and tables.
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