Generalized total angular momentum operator for the Dirac equation in curved space-time

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It is found that an operator of the form $i\gamma_5\gamma^\mu[f_\mu^{\ \nu}\nabla_\nu-(1/6)\gamma^\nu\gamma^\rho f_{\mu\nu;\rho}]$ commutes with the Dirac operator $\gamma^\mu\nabla_\nu$ whenever $f_{\mu\nu}$ is an antisymmetric tensor satisfying the Penrose-Floyd equation $f_{\mu(\nu;\rho)}=0$. Such a tensor exists notably in the Kerr solutions and in the flat-space limit wherein the operator can be interpreted as the square root of the ordinary total squared angular momentum Casimir operator of the rotation group.

Recent progress in the detailed analysis of the properties of the Kerr black-hole solutions has been to a large extent based on the key discovery by Teukolsky[1] of the possibility of analytically solving the Maxwell (spin 1) and perturbed Einstein (spin 2) field equations therein. Teukolsky's method proceeds by *first* obtaining a *decoupled* equation for *one* particular component of the field, and *subsequently separating the variables* in the equation thus obtained. The latter step is a direct generalization of the separability procedure that had previously been found to be successful in the simpler case of the ordinary scalar (D'Alembertian or Klein-Gordon) wave equation.[2] The Teukolsky procedure depended on specifying the field components in terms of a certain very particular null tetrad that was introduced by Kinnersley[3] in a general study of vacuum Einstein solutions with a type-D conformal curvature tensor. The procedure of successive decoupling and separation was almost immediately extended to the Weyl equation for zero-mass and spin $\frac{1}{2}$ by Teukolsky[4] and Unruh[5] independently. The latter work is noteworthy for demonstrating that the method works equally well using, instead of the Kinnersley tetrad, the more fundamentally symmetric canonical tetrad that had been brought to light in the original classical-particle-orbit separability studies by one of us.[6] However, progress on extending the procedure to the full Dirac equation with nonzero mass was held up until a new breakthrough was achieved by Chandrasekhar,[7] who succeeded in devising an ingenious method of separation of variables *without* prior decoupling of the equations.

Examination of Chandrasekhar's procedure (which has been generalized by Page[8] and Toop[9] to allow for electric charge as well as mass) shows that it can be interpreted as consisting of two essential steps, of which the first involves replacing the original wave equation

$$H\psi=0\,, \tag{1}$$

where H is the Dirac operator (which Chandrasekhar expressed in terms of the Weyl representation using the Kinnersley tetrad) by a modified but equivalent wave equation

$$W\psi'=0\,, \tag{2}$$

where W is obtained from H by the combined result of an appropriate four-spinor basis transformation

$$\psi\to\psi'=S^{-1}\psi \tag{3}$$

combined with the effect of a premultiplication by an appropriate separation factor Υ which, like S, is a (variable) nonsingular 4×4 matrix, so that one has

$$W=\Upsilon S^{-1}HS\,. \tag{4}$$

If Υ is adjusted so as to include certain further sign modifications (beyond those explicitly introduced by Chandrasekhar) then it turns out *not only* that the resulting operator splits up directly as a sum of the form

$$W=W_\gamma+W_\theta \tag{5}$$

where (apart from derivatives with respect to the ignorable coordinates φ and t) W_γ depends only on γ and $\partial/\partial\gamma$, while W_θ depends only on θ and $\partial/\partial\theta$, but *furthermore* the matrix coefficients are arranged in just such a way that the two operators *commute*, i.e.,

$$[W_\gamma,W_\theta]=0\,. \tag{6}$$

Chandrasekhar's second step did not make any explicit reference to the operators W_γ and W_θ as such, but implicitly depended on their very specialized form, which made it possible to factorize the components of ψ' into pairs of single-variable (γ or θ) interdependent functions that obey a system of ordinary differential equations involving a *separation constant* λ which we have found to be interpretable as being in fact an *eigenvalue* of W_γ or equivalently (after adjustment of sign) of W_θ. [It transpires at this stage that as in the zero-mass case the use of Kinnersley's tetrad at the outset was unnecessary. Indeed, apart from a complex two-spinor conformal factor, Chandrasekhar's transformation S can be interpreted in terms of the Geroch-Held-Penrose (GHP) transformation needed to get back to the canonical symmetric tetrad[2,5,6] whose advantages have recently been emphasized, in another context, by Znajek.[10]]

Since they commute with each other, it is obvious that the separate operators W_γ and W_θ also commute with the total transformed wave operator W. More remarkably it turns out that they can be used to construct a *new* operator that commutes with the *original* wave operator H. The construction depends on the fact that the premultiplication factor Υ decomposes in the form

$$\Upsilon = \Upsilon_\gamma + \Upsilon_\theta \,, \tag{7}$$

where the coefficients of the matrices Υ_γ and Υ_θ are single-variable functions, respectively, of γ and θ, which are arranged in such a way as to satisfy the commutation relations

$$[\Upsilon_\gamma, \Upsilon_\theta] = 0 \,, \tag{8}$$

$$[\Upsilon_\gamma, W_\theta] = 0 = [\Upsilon_\theta, W_\gamma] \,. \tag{9}$$

It can easily be checked that the preceding relations (4) to (9) automatically imply that we can construct a new operator $S\Upsilon^{-1}(\Upsilon_\gamma W_\theta - \Upsilon_\theta W_\gamma)S^{-1}$ that will commute with the original wave operator, i.e.,

$$[H, S\Upsilon^{-1}(\Upsilon_\gamma W_\theta - \Upsilon_\theta W_\gamma)S^{-1}] = 0 \,. \tag{10}$$

The occurrence of an operator such as the one thus brought to light—as characterized by the property that its *commutation* with the relevant wave operator underlies the *separability* of the corresponding wave equation, is not a feature unique to the Dirac equation. A precisely analogous operator had emerged previously in the simpler context of the separation of the Klein-Gordon wave equation[2] for which the necessary separating factor Υ was just the square root of the metric determinant in the canonical coordinate system, while the transformation S was the trivial unit multiplication. In the scalar case the occurrence of an operator commuting with the wave operator has recently been shown[11] to be a feature that arises automatically from the existence of an appropriate *Killing tensor* field in a space-time under consideration. It has also been shown[11] that for any appropriately self-adjoint wave equation, such as the Klein-Gordon and Dirac equations, the occurrence of an operator commuting with the wave operator implies the existence of a corresponding *conserved current* distribution associated with each solution of the equation. Familiar simple cases are the operators of energy or momentum, and the corresponding conserved currents, that arise from ordinary *Killing vector* fields, i.e., the generators of ordinary space-time symmetries. We shall now show that the occurrence of the more mysterious kind of operator exemplified by the one underlying the Chandrasekhar separability of the Dirac equation can be considered to arise in a corresponding way from the presence of an appropriate *Killing spinor* field on the space-time under consideration.

The concept of a *Killing spinor* has its origin in the work of Penrose and Walker,[12] who demonstrated the existence in any type-D vacuum space-time of a second-order symmetric two-spinor satisfying the "twistor equation"

$$\nabla_{\dot{A}(B}\kappa_{CD)} = 0 \,, \tag{11}$$

where the capital two-spinor indices run from 1 to 2, and where we introduce the use of round and square brackets, respectively, to denote symmetrization or antisymmetrization over the indices within. Any solution of (11) represents what may appropriately be described as a *conformal* Killing spinor, since (by double contraction with the parallel propagated tangent two-spinor) it evidently determines a constant of the motion along any null geodesic. It has recently been shown[13] that in *all* the Kinnersley type-D vacuum solutions the corresponding constants can also be derived from *separability* of the *null-geodesic* Hamilton-Jacobi equation. In order to qualify as a *Killing spinor* in the strong (not merely conformal) sense, it is appropriate to demand that the symmetric two-spinor should satisfy not just (11) but also the further condition that the contraction $\nabla_{\dot{A}C}\kappa^C{}_B$ be skew Hermitian, i.e.,

$$\nabla_{\dot{A}C}\kappa^C{}_B + \bar{\nabla}_{B\dot{C}}\bar{\kappa}^{\dot{C}}{}_{\dot{A}} = 0 \tag{12}$$

(where the bars denote complex conjugation). It may be remarked that unlike (11) Eq. (12) is not invariant under duality rotations by a constant phase factor. The additional restriction (12) can *not* be satisfied in the full set of Kinnersley type-D solutions but only (with an adjustment of phase by a factor i relative to the original Penrose-Walker convention) in the previously discovered subset[2] of solutions characterized by separability of the Hamilton-Jacobi equation for *massive* (not just null) particle orbits. The appropriateness of imposing such a subsidiary condition did not become apparent until the more recent work of Penrose and Floyd[14] who made the key discovery that in the latter subset of solutions (including that of Kerr) the antisymmetric tensor defined by the canonical correspondence

$$f_{\mu\nu} \leftrightarrow \bar{\epsilon}_{\dot{A}\dot{B}}\kappa_{AB} + \epsilon_{AB}\bar{\kappa}_{\dot{A}\dot{B}} \tag{13}$$

(again with a factor i relative to the original Penrose-Walker convention for κ_{AB}) will satisfy the strikingly simple equation

$$f_{\mu(\nu;\rho)} = 0\,, \tag{14}$$

where a semicolon precedes an index of covariant differentiation. In view of the algebraic antisymmetry property

$$f_{(\mu\nu)} = 0\,, \tag{15}$$

Eq. (14) is equivalent to the total antisymmetry condition

$$f_{\mu\nu;\rho} = f_{[\mu\nu;\rho]}\,. \tag{16}$$

It can be checked that in order for (14) to hold it is necessary and sufficient that the symmetric two-spinor specified by (13) should satisfy both the original Penrose-Walker condition (11) *and* the subsidiary condition (12).

Equations (14) and (15), whose solutions may appropriately be described as *Killing spin two-forms*, can be considered[15,16] as belonging to a class of systems extensively studied by Yano.[17] It is easy to verify[14] that the *square* of a Killing spin two-form will give a symmetric tensor,

$$a_{\mu\nu} = f_{\mu\rho}f_\nu{}^\rho\,, \tag{17}$$

$$a_{[\mu\nu]} = 0\,, \tag{18}$$

which will automatically satisfy the Stackel-Killing equation

$$a_{(\mu\nu;\rho)} = 0\,. \tag{19}$$

It may also be remarked that a Stackel-Killing tensor obtained in this way will necessarily have the Segre type [(11)(11)] property on which recent work of Hauser and Malhiot[18] and Dietz[19] is based. It can be shown furthermore that such a tensor will always satisfy the restriction

$$a^\rho{}_{[\mu}R_{\nu]\rho} = 0\,, \tag{20}$$

where $R_{\mu\nu}$ are the Ricci tensor components, which implies[11] that $a_{\mu\nu}$ will necessarily be a Killing tensor in the *strong* sense, meaning that the operator $\nabla_\mu a^{\mu\nu}\nabla_\nu$ will necessarily commute with the D'Alembertian wave operator on scalar fields, i.e.,

$$[\nabla_\rho\nabla^\rho, \nabla_\mu a^{\mu\nu}\nabla_\nu] = 0\,, \tag{21}$$

where ∇_μ denotes the ordinary covariant differentiation operator. The condition (20) follows directly from a corresponding condition

$$f^\rho{}_{(\mu}R_{\nu)\rho} = 0 \tag{22}$$

(note that symmetrization rather than antisymmetrization is involved this time) which is itself obtained by contracting the integrability condition

$$R_{\mu\nu[\sigma}{}^\tau f_{\rho]\tau} + R_{\sigma\rho[\mu}{}^\tau f_{\nu]\tau} = 0 \tag{23}$$

on the Riemann tensor for any solution of (14). It is to be remarked that in addition to this last condition (23), any solution of (14) must satisfy the further integrability condition

$$f_{\mu\nu;\rho;\sigma} = \tfrac{3}{2} f_{\tau[\nu}R_{\mu\rho]\sigma}{}^\tau \tag{24}$$

(using the sign conventions of Misner *et al.*[20] for the metric and curvature tensors) which is analogous to the well-known

$$k_{\mu\nu;\rho} = k_\tau R_{\mu\nu\rho}{}^\tau \tag{25}$$

satisfied by any solution of the ordinary Killing equation

$$k_{(\mu;\nu)} = 0\,. \tag{26}$$

The foregoing conclusions may be extended to take into account the presence of an electromagnetic field. It has been shown by Hughston *et al.*[21] that when the principal null vectors of the electromagnetic field coincide with those of the type-*D* Weyl tensor the symmetric Maxwell two-spinor with components φ_{AB} will be related to the Killing spinor by

$$\kappa^C{}_{(A}\varphi_{B)C} = 0\,. \tag{27}$$

Translated into tensor language this gives

$$f^\rho{}_{[\mu}F_{\nu]\rho} = 0\,, \tag{28}$$

which expresses the condition that the Killing spin two-form be a linear combination of the Maxwell field and its dual. It evidently implies that the Killing tensor will satisfy the corresponding requirement

$$a^\rho{}_{(\mu}F_{\nu)\rho} = 0\,, \tag{29}$$

which is well known[22] to be sufficient to ensure that the scalar $a_{\mu\nu}u^\mu u^\nu$ be conserved not just along geodesics but also along the charged-particle orbits given by

$$\left(u^\rho\nabla_\rho\delta^\mu_\nu - \frac{e}{\mu}F^\mu{}_\nu\right)u^\nu = 0\,, \tag{30}$$

where u^μ are components of the unit tangent vector. It can in fact be seen that when conditions (14) and (28) are satisfied, the vector

$$L^\mu = f^\mu{}_\nu u^\nu \tag{31}$$

will satisfy a formally identical equation of motion

$$\left(u^\rho \nabla_\rho \delta^\mu_\nu - \frac{e}{\mu} F^\mu{}_\nu\right) L^\nu = 0 \tag{32}$$

and in particular, as pointed out by Penrose and Floyd,[14] the vector L^μ will be *parallel-propagated* in the geodesic case, i.e., when the charge/mass ratio e/μ is zero. (This result is a generalization of the ordinary law of conservation of angular momentum in flat space-time: In the particular case where χ^μ are ordinary *Minkowski coordinates* and k^μ is the covariantly constant generator of *time translations*, then the combination $f_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} k^\rho \chi^\sigma$ can easily be seen to be a Killing spin two-form, and in this case the L^μ will just be the components of the ordinary angular momentum vector.) The condition (32) evidently implies conservation along the orbits of the squared *scalar*

$$L_\mu L^\mu = a_{\mu\nu} u^\mu u^\nu \tag{33}$$

(which in the flat-space case is just the total squared angular momentum).

Let us consider the conservation laws for four-spinor fields ψ satisfying the Dirac equation

$$(\gamma^\mu D_\mu - m)\psi = 0\,, \tag{34}$$

where the γ matrices are defined in accordance with the conventions of Streater and Wightman,[23] and where

$$D_\mu = \nabla_\mu - ieA_\mu \tag{35}$$

is the operator of gauge-covariant partial differentiation. In accordance with the principles described in the previously cited work,[11] whose actual applications concerned only the scalar wave equations, the analogs of the ordinary constants of the classical particle motion will be the appropriately self-adjoint operators that commute with the Dirac wave operator $\gamma^\rho D_\rho$. When there is a manifest space-time symmetry with Killing vector generator k^μ, it is easy to check that an appropriate operator is given by

$$K = i(k^\mu \nabla_\mu - \tfrac{1}{4}\gamma^\mu \gamma^\nu k_{\mu;\nu})\,, \tag{36}$$

which will satisfy the commutation relation

$$[\gamma^\rho D_\rho, K] = 0 \tag{37}$$

for arbitrary values of the electric charge e provided the electromagnetic four-potential itself satisfies the condition

$$k^\rho A_{\rho;\mu} + k^\rho{}_{;\mu} A_\rho = 0 \tag{38}$$

of invariance under the action generated by k^μ. [The operator K as defined by (36) agrees—apart from the factor i introduced for the sake of self-adjointness—with the operator referred to by Kosmann[24] and Unruh[25] as the covariant Lie derivative: It coincides with the ordinary Lie derivative on the spinor components treated as scalars *provided* the spinor gauge is chosen so that the scalar components of the γ matrices are themselves invariant under the action generated by k^μ.] If (37) is replaced by the condition of *pure magneticity*, i.e., vanishing of the electric part $F_{\mu\nu} k^\nu$ of the field, then one may obtain a *gauge-invariant* constant of the motion by replacing ∇_μ by D_μ in the definition (36) of K.

The main purpose of this work is to point out the existence of a corresponding gauge-invariant operator

$$L = i\gamma_5 \gamma^\mu (f_\mu{}^\nu D_\nu - \tfrac{1}{6}\gamma^\nu \gamma^\rho f_{\mu\nu;\rho}) \tag{39}$$

which satisfies the commutation relation

$$[\gamma^\rho D_\rho, L] = 0\,, \tag{40}$$

whenever the Killing two-form Eq. (14) and the corresponding electromagnetic field condition (28) are satisfied. As K, this operator satisfies the self-adjointness condition that permits the construction[11] of a corresponding conserved current. The spinor operator L can be considered as a square root of the scalar operator $D_\mu a^{\mu\nu} D_\nu$ [which satisfies a corresponding commutation law when conditions (19), (20), (29), and the source-free Maxwell equations are satisfied] in the same loose sense in which the Dirac operator $\gamma^\mu D_\mu$ may be considered as a square root of the scalar wave operator $D^\mu D_\mu$. One can check that the further commutation relation

$$[K, L] = 0 \tag{41}$$

will be satisfied whenever the invariance condition

$$k^\mu f_{\mu\nu;\rho} + 2k^\rho{}_{;[\nu} f_{\mu]\rho} = 0 \tag{42}$$

holds.

In the Coulombian flat-space limit the operator L may be interpreted rather precisely as a square root of the three-dimensional rotation-group Casimir operator that is given by the sum of the squares of the ordinary angular momentum operators with respect to mutually orthogonal axes, and it may be remarked that within this rather restricted context such an operator was originally discovered by Dirac himself.[26] In the more general context of the Kerr-Newman solutions we have confirmed by an explicit calculation that the Chandrasekhar separation constants are indeed interpretable as eigenvalues of this operator L.

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