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Gravitational and acoustic waves in an elastic medium

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Relativistic equations governing perturbations of an elastic medium under the influence of gravitational waves are derived firstly in a gauge-independent way in terms of relative strains, and secondly in terms of gauge-dependent displacements. The derivations are based on the exact nonlinear theory of elasticity in conjunction with Einstein's theory of gravity, and hence are applicable to the solid crusts and cores of neutron stars. It is shown that in the approximately Minkowskian weak-field limit the equations reduce to those derived by previous workers for application in terrestrial contexts such as the detection of gravitational waves by a Weber bar.

I. INTRODUCTION

The purpose of the present work is to describe the application of the exact nonlinear theory of an elastic medium to the description of perturbations such as would be generated by an influx of gravitational waves according to Einstein's general theory of relativity.

The equations describing such perturbations in the approximately Minkowskian *weak-field* limit are now fairly well known owing to the work of Rayner,[1] Dyson,[2] Papapetrou,[3] and others. (For a recent account with many references see Maughin.[4])

The *full nonlinear* theory of relativistic elasticity, suitable for application in contexts such as that of a neutron-star crust or core where gravitational curvature is *large*, has been developed by several authors, but no comprehensive and authoritative review of the relevant literature is yet available. The absence of any adequate historical review is attributable to the fact that the subject has been developed with very diverse physical motivations, using different languages and disparate notation systems, by isolated groups or individuals sometimes laboring under serious but not always superficially apparent misconceptions. We shall not attempt to remedy this situation here, but simply remark that to the best of our present knowledge the earliest completely correct treatment of the nonlinear fully relativistic elasticity theory is that of Souriau[5] (cf. Souriau[6,7]). The present work will be entirely based on the mathematically equivalent but more fully developed approach that we introduced ourselves (Carter and Quintana[8]) at a time when we (and virtually all other workers in the field) were unaware of Souriau's work. This particular line of approach has already been applied effectively to specific problems arising in an astrophysical context, (see Carter,[9,10,11] Friedman and Schutz,[12] Carter and Quintana,[13,14] and Quintana[15]). For a brief introductory account, see Ehlers.[16]

The plan of the present work is as follows: In Sec. II we derive an *exact* equation relating the second derivatives of the *relative strain* tensor to the Weyl tensor of the gravitational field. In Sec. III we apply linearized perturbation theory in a background space that may be strongly curved, and derive a system of equations of motion for an infinitesimal gauge-dependent *relative displacement* vector, and for the gravitational perturbation of the metric itself. In Sec. IV we show how the well-known equations describing the approximately Minkowskian weak-field limit may be obtained directly from the formulas either of Sec. II or of Sec. III. We have included a list of minor copying errors in our original paper (Carter and Quintana[8]) as an Appendix.

II. STRAIN VARIATIONS IN A GRAVITATIONAL FIELD

The most direct way to obtain an equation of motion for strain variations in a gravitational field described by Einstein's theory is to start from the generalized Raychadhuri equation (see, e.g., Ellis[17]) obtained by applying the Ricci identity to the matter flow velocity whose components (in terms of coordinates x^a) are given by

$$u^a = \frac{dx^a}{d\tau}, \tag{2.1}$$

where the proper-time differential $d\tau$ is defined in terms of the Lorentzian metric tensor by

$$ds^2 = g_{ab}dx^a dx^b = -c^2 d\tau^2, \tag{2.2}$$

so that the velocity vector $\vec{u}$ satisfies the normalization condition

$$u^a u_a = -c^2 . \tag{2.3}$$

Following Ellis[17] we introduce a flow gradient tensor with components v_{ab} defined by

$$u_{a;b} = v_{ab} - \frac{1}{c^2}\dot{u}_a u_b , \tag{2.4}$$

where a semicolon denotes covariant differentiation and the acceleration vector is defined by

$$\dot{u}_a = u_{a;b}u^b . \tag{2.5}$$

This tensor is *orthogonal* to the flow in the sense that

$$u^b v_{ba} = 0 = v_{ab}u^b . \tag{2.6}$$

We shall use the symbol $\vec{\xi}\mathfrak{L}$ to denote the operation of Lie differentiation with respect to any vector field $\vec{\xi}$, and we shall use the abbreviation $\mathfrak{L}_\tau$ for differentiation with respect to the matter flow field $\vec{u}$, i.e., we set

$$\mathfrak{L}_\tau \equiv \vec{u}\mathfrak{L} . \tag{2.7}$$

For any *covariant* tensor that is *orthogonal* to the flow [in the sense defined by (2.6)] the Lie derivative with respect to the flow will be given by the formula

$$\mathfrak{L}_\tau f_{ab} \equiv \gamma^c_{\ a}\gamma^d_{\ b} f_{cd;e}u^e + f_{cb}v^c_{\ a} + f_{ac}v^c_{\ b} , \tag{2.8}$$

where the *orthogonal projector* is defined by

$$\gamma^a_{\ b} = g^a_{\ b} + \frac{1}{c^2}u^a u_b . \tag{2.9}$$

In particular for the *Cauchy strain tensor*, which has covariant components γ_{ab} obtained by lowering the indices of the orthogonal projector itself (in the standard way by contraction with the metric), we shall have

$$\mathfrak{L}_\tau \gamma_{ab} = 2\theta_{ab} , \tag{2.10}$$

where θ_{ab} is the symmetric part of v_{ab}, i.e.,

$$v_{ab} = \theta_{ab} + \omega_{ab} , \tag{2.11}$$

with

$$\theta_{[ab]} = 0, \quad \omega_{(ab)} = 0 \tag{2.12}$$

using round and square brackets on indices to denote symmetrization and antisymmetrization, respectively. Using the Ricci identity

$$u_{a;[b;c]} = \tfrac{1}{2}u_d R^d_{\ abc} \tag{2.13}$$

(where $R^d_{\ abc}$ are Riemann tensor components) we can go on to express the Lie derivative of the flow gradient tensor in the form

$$\mathfrak{L}_\tau v_{ab} = \gamma^c_{\ a}\gamma^d_{\ b}\dot{u}_{c;d} + v^c_{\ a}v_{cb} + \frac{1}{c^2}\dot{u}_a\dot{u}_b - u^c u^d R_{acbd} . \tag{2.14}$$

(A better known, but for the purposes of elasticity theory less useful, form of this identity is given by Ellis[17] in terms of the covariant derivative instead of the Lie derivative of v_{ab}.) The antisymmetric part of this identity gives

$$\mathfrak{L}_\tau \omega_{ab} = \gamma^c_{\ a}\gamma^d_{\ b}\dot{u}_{[c;d]} \tag{2.15}$$

and the symmetric remainder gives a second-order strain variation equation of the form

$$\tfrac{1}{2}\mathfrak{L}_\tau\mathfrak{L}_\tau\gamma_{ab} = \gamma^c_{\ a}\gamma^d_{\ b}\dot{u}_{(c;d)} + v^c_{\ a}v_{cb} + \frac{1}{c^2}\dot{u}_a\dot{u}_b - u^c u^d R_{acbd} . \tag{2.16}$$

For purposes of physical interpretation it is convenient to express this in terms of the Weyl conformal tensor given by

$$C^{ab}_{\ \ cd} = R^{ab}_{\ \ cd} - 2g^{[a}_{\ [c}(R^{b]}_{\ d]} - \tfrac{1}{6}R g^{b]}_{\ d]}) , \tag{2.17}$$

where the Ricci tensor and scalar are defined by

$$R_{ab} = R^c_{\ acb}, \quad R = R^c_{\ c} . \tag{2.18}$$

It will also be useful to introduce a *relative* strain tensor with components defined by

$$e_{ab} = \tfrac{1}{2}(\gamma_{ab} - \kappa_{ab}) , \tag{2.19}$$

where κ_{ab} are components of some as yet unspecified *strain reference tensor* satisfying

$$\mathfrak{L}_\tau \kappa_{ab} = 0 \tag{2.20}$$

so that by (2.10) the expansion tensor with components θ_{ab} may be interpreted as measuring the *relative strain rate*, i.e.,

$$\theta_{ab} = \mathfrak{L}_\tau e_{ab} . \tag{2.21}$$

(When the physical circumstances do not suggest any other more useful choice, we are always free to set κ_{ab} equal to zero for the sake of definiteness.) In terms of these quantities the strain variation equation (2.16) may be written as

$$\mathfrak{L}_\tau\mathfrak{L}_\tau e_{ab} = \gamma^c_{\ a}\gamma^d_{\ b}\dot{u}_{(c;d)} - u^c u^d C_{acbd} - \tfrac{1}{2}\gamma_{ab}(R_{cd}u^c u^d + \tfrac{1}{3}Rc^2) + \tfrac{1}{2}\gamma^c_{\ a}\gamma^d_{\ b}R_{cd}c^2 + v^c_{\ a}v_{cb} + \frac{1}{c^2}\dot{u}_a\dot{u}_b . \tag{2.22}$$

So far we have been dealing purely with kinematics, i.e., properties of the motion that would be valid for any medium—elastic or otherwise—in terms of any metric theory of gravity. We now introduce the condition that the medium has an elastic energy-momentum tensor of the form

$$T^{ab} = p^{ab} + \rho u^a u^b , \tag{2.23}$$

where p^{ab} are components of the *pressure tensor*, which is symmetric and orthogonal, i.e.,

$$p^{[ab]} = 0, \quad p^{ab}u_b = 0 \tag{2.24}$$

and where ρ is the local *mass density*, which may if one wishes be expressed in terms of an invariant conserved particle (e.g., baryon) number density n, a constant "rest mass per particle" m_0, and an energy density ϵ by

$$\rho = nm_0 + \epsilon/c^2 . \tag{2.25}$$

However, the "rest mass per particle" is a somewhat arbitrary concept, depending on what kinds of (chemical, nuclear, or other) transformations are treated as being "allowed" in the physical context under consideration. It is always legitimate, and in highly relativistic situations will usually be most convenient, simply to set m_0 equal to *zero*. Independent of the way that the constant m_0 is chosen, the pressure tensor will be determined from the corresponding function $\epsilon(\gamma_{AB})$, defining the energy density as a function of strain, by the differential relation

$$p^{AB} = -2\frac{\partial\epsilon}{\partial\gamma_{AB}} - \epsilon\gamma^{AB} , \tag{2.26}$$

where we use capital indices to denote the components of the *natural projections* (defined by the flow lines) of ordinary orthogonal space-time tensors onto the *three-dimensional* manifold representing the idealized particles of the medium. (See Carter and Quintana,[8] and Ehlers[16]; since the projected Cauchy strain tensor is necessarily symmetric its components γ_{AB} cannot be varied independently but the tensor operator $\partial/\partial\gamma_{AB}$ can nevertheless be made well defined by imposing the condition that it be symmetric, i.e., that it be equal to $\partial/\partial\gamma_{BA}$.)

Equation (2.26) is an exact relation which is valid for *any* linear or nonlinear equation of state function $\epsilon(\gamma_{AB})$, and for *any* metric theory of gravity in which the conservation law

$$T^{ab}{}_{;b} = 0 \tag{2.27}$$

holds. In particular (2.26) would hold in the case of special relativity theory, wherein gravity is ignored so that C_{abcd} and R_{ab} could be set equal to zero in (2.22). We are concerned here with the case of general relativity wherein the Ricci tensor is given by the Einstein equations

$$R_{ab} = \frac{8\pi G}{c^4}(T_{ab} - \tfrac{1}{2}Tg_{ab}) , \tag{2.28}$$

where

$$T = T^c{}_c = p^c{}_c - \rho c^2 , \tag{2.29}$$

and wherein the conformal tensor is algebraically restricted only by the purely kinematic conditions

$$C_{abcd} = C_{cdab} = C_{[ab][cd]} , \tag{2.30}$$

$$C^a{}_{[bcd]} = 0, \quad C^{ab}{}_{ac} = 0 \tag{2.31}$$

although by the Bianchi identities its derivatives must obey

$$C^{abcd}{}_{;d} = \frac{8\pi G}{c^4}(T^{c[a;b]} - \tfrac{1}{3}g^{c[a}T^{;b]}) \tag{2.32}$$

(cf., Ellis[17]).

We shall use the abbreviation

$$C_{ab} = u^c u^d C_{acbd} = c^2 \gamma^e_a \gamma^f_b \gamma^{cd} C_{ecfd} \tag{2.33}$$

for the "electric" part of the gravitational field (with respect to the local rest frame of the medium), which by (2.30) and (2.31) will satisfy

$$C_{[ab]} = 0, \quad C_{ab}u^b = 0, \quad C_a{}^a = 0 . \tag{2.34}$$

Then using the equation of motion

$$\rho\dot{u}^a = -p^{ab}{}_{;b} + \frac{1}{c^2}u^a p^{bc}\theta_{bc} \tag{2.35}$$

[obtained from (2.27)] to eliminate the acceleration, and using (2.21) to eliminate the expansion rate tensor, we may replace the purely kinematic equations (2.15) and (2.22) by the corresponding dynamic equations

$$\mathcal{L}_\tau \omega_{ab} = \gamma_{c[a}\gamma_{b]}{}^d \frac{1}{\rho^2}(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}) + \frac{1}{\rho c^2}\omega_{ab}p^{cd}\mathcal{L}_\tau e_{cd} \tag{2.36}$$

and

$$\begin{aligned}\mathcal{L}_\tau\mathcal{L}_\tau e_{ab} = {} & \gamma_{c(a}\gamma_{b)}{}^d \frac{1}{\rho^2}(\rho_{,d}p^{ce}{}_{;e} - \rho p^{ce}{}_{;e;d}) \\ & - C_{ab} - \frac{4\pi G}{3}\rho\gamma_{ab} \\ & + \omega_{ca}\omega^c{}_b + 2\omega^c{}_{(a}\mathcal{L}_\tau e_{b)c} + g^{cd}(\mathcal{L}_\tau e_{ac})\mathcal{L}_\tau e_{bd} \\ & + \frac{1}{c^2\rho^2}[\gamma_{ac}\gamma_{bd}p^{ce}{}_{;e}p^{df}{}_{;f} + \rho p^{cd}(\mathcal{L}_\tau e_{cd})\mathcal{L}_\tau e_{ab}] \\ & + \frac{4\pi G}{c^2}(p_{ab} - \tfrac{2}{3}p_c{}^c\gamma_{ab}) .\end{aligned} \tag{2.37}$$

Bearing in mind that ρ and p^{ab} are algebraic functions of the strain components e^{cd} we see that the equations (2.36) and (2.37) together form a coupled set of nonlinear partial differential equations for ω_{ab} and e_{ab} (of first order in the former and of second order in the latter). The vorticity equation (2.36) differs from the corresponding equation in Newtonian theory only by the presence of the special relativistic correction term in the last line. The strain equation (2.37) differs from the corresponding Newtonian equation by the terms on the last two (fourth and fifth) lines. Thus in the Newtonian limit one is left with only the *first*, *second* and *third* lines, of which the first consists of ordinary Navier-Stokes terms, linearized in velocity; the second consists of terms representing the effects of Newtonian gravity (C_{ab} is the tidal force term), while the third consists of the Newtonian dynamic terms that are quadratic in the velocity. In the *special relativistic* limit wherein gravity is completely ignored, only the terms of the *first*, *third*, and *fourth* lines of (2.37) would remain; those of the fourth line are the relativistic correction terms. Thus the only truly *general*-relativistic correction terms are those of the last (fifth) line of (2.37).

A particularly simple example in which the strain dependence of the terms in (2.36) and (2.37) can be made to appear more explicit is that of an *exactly Hookean* solid (in the sense defined by Carter and Quintana[8]), wherein the pressure tensor is related to the strain tensor via the elasticity tensor by an apparently linear relation of the historically familiar form

$$p^{ab} = -E^{abcd} e_{cd}\,. \qquad (2.38)$$

However, in a rigorous treatment, the appearance of linearity in this relation is illusory since the elasticity tensor itself is necessarily strain dependent; it is governed in this particular case by the equation of motion

$$\mathcal{L}_\tau E_{abcd} = 4E_{ab}{}^{e}{}_{(c}\mathcal{L}_\tau e_{d)e} + 4E_{cd}{}^{e}{}_{(a}\mathcal{L}_\tau e_{b)e} - E_{abcd}\gamma^{ef}\mathcal{L}_\tau e_{ef}\,. \qquad (2.39)$$

In practice the Hookean idealization is physically relevant as a good approximation only in the weak-field small strain limit discussed in Sec. IV, wherein higher-order corrections such as those arising from the terms on the right-hand side of (2.39) may be neglected.

The exact equations (2.36) and (2.37) are rather too unwieldy to be of much practical use in general circumstances. They become much more amenable in situations where the motion can be considered to be approximately rigid (as when treating approximate equilibrium states of a solid star) in which circumstances the strain reference components κ_{AB} may be chosen in such a way that the strain components e_{ab} remain small. However, if we are obliged to approximate then it is advantageous to do so systematically from the outset in the manner described in the next section.

III. THE DISPLACEMENT PERTURBATION EQUATION

In Sec. II we derived an *exact* equation governing the strain variations in an elastic medium. We now describe an alternative approach using linearized perturbation theory along the lines developed by Carter.[9] In this approach one compares the "perturbed" state of the medium under the influence of acoustic and gravitational waves with a near-by "unperturbed" state which in practice will usually be required to have specially convenient simplifying properties such as stationarity. As a basic unknown variable we shall work with an *infinitesimal displacement* vector, with components ξ^a say, which specifies the position of any particle in the perturbed state relative to its position in the unperturbed state. Such a displacement is highly *gauge dependent* in that its value depends on the way one chooses to identify points in the perturbed and unperturbed space-time manifolds. In principle it is always possible to take ξ^a to be zero—identifying points by the use of a Lagrangian (comoving) coordinate system. In practice, however, it will usually be more convenient to specify the points by their coordinates with respect to some *geometrically* fixed (e.g., harmonic) Eulerian system (such Eulerian systems include standard Minkowski coordinates in special relativity and ordinary Cartesian coordinates in Newtonian theory). The difference between the Lagrangian (comoving) variation of any quantity, which we denote by Δ, and the corresponding Eulerian ("fixed point") variation, which we denote by δ, is given by the Lie derivative with respect to the displacement vector field, i.e.,

$$\Delta - \delta = \vec{\xi}\mathcal{L}\,. \qquad (3.1)$$

The freedom to alter the way in which the space-time manifolds of the perturbed and unperturbed states are identified by arbitrary infinitesimal relative displacements, with components ζ^a say (corresponding to arbitrary infinitesimal coordinate transformations), gives rise to *gauge transformations of the first kind*, where the Eulerian variation undergoes transformations of the form

$$\delta \to \delta - \vec{\zeta}\mathcal{L}\,. \qquad (3.2)$$

The Lagrangian variation is subject only to a much more restricted group of *gauge transformation of the second kind*, arising from infinitesimal displacements with components of the form σu^a (i.e., displacements leaving the world lines invariant), where σ is an arbitrary infinitesimal scalar, which gives rise to alterations of the form

$$\Delta \to \Delta - \sigma\vec{u}\mathcal{L} \qquad (3.3)$$

in the Lagrangian variation. It is to be remarked that in the special case when the flow vector $\vec{u}$ is *parallel to the generator of an invariance group* of the unperturbed motion, the Lagrangian variation of any *covariant* tensor *orthogonal* to the flow will clearly be gauge invariant. This applies in particular to the Lagrangian perturbation

$$\epsilon_{ab} = \Delta e_{ab} = \tfrac{1}{2}\Delta\gamma_{ab} \qquad (3.4)$$

of the strain, which will be *gauge invariant* whenever the unperturbed motion is *rigid* (i.e., whenever θ_{ab} is zero). The material displacement vector ξ will always be affected by gauge transformations of *both* kinds. The combined effect of the transformations (3.2) and (3.3) is to induce a change of the form

$$\xi^a \to \xi^a + \zeta^a - \sigma u^a , \tag{3.5}$$

as is evidently required for (3.1) to remain valid.

Even when one wishes to express the final results in terms of Eulerian variations, the intermediate calculations directly involving the medium are nevertheless most easily carried out in terms of Lagrangian variations. If as usual we use the abbreviation

$$h_{ab} = \delta g_{ab} \tag{3.6}$$

to denote the Eulerian variation of the metric components, then the corresponding Lagrangian variation, which we shall denote by

$$\Delta_{ab} = \Delta g_{ab} , \tag{3.7}$$

may be evaluated by means of (3.1) using the familiar formula

$$\xi\pounds g_{ab} = 2\xi_{(a;b)} . \tag{3.8}$$

This leads to the expression

$$\Delta_{ab} = h_{ab} + 2\xi_{(a;b)} , \tag{3.9}$$

which may be used for the explicit evaluation of the Lagrangian variations of functions of strain, including, in particular, the Lagrangian variation of the strain tensor itself, which will be given by

$$\epsilon_{ab} = \tfrac{1}{2}\gamma^c_{\ a}\gamma^d_{\ b}\Delta_{cd} \tag{3.10}$$

(see Carter[9]).

In order to evaluate variations of *covariant derivatives* it is also necessary to have an expression for the variation of the affine connection components Γ^a_{bc}, since for any mixed tensor with components $T^{b\cdots}_{a\cdots}$ we shall have

$$\Delta(T^{b\cdots}_{a\cdots;e}) = \Delta T^{b\cdots}_{a\cdots;e} + T^{f\cdots}_{a\cdots}\Delta\Gamma^b_{ef} + \cdots - T^{b\cdots}_{f\cdots}\Delta\Gamma^f_{ae} - \cdots . \tag{3.11}$$

When the Eulerian metric variation is expressed by (3.6) it is well known (see, e.g., Weinberg,[18] or Landau and Lifshitz[19]) that the corresponding formula for the Eulerian variation of the connection is

$$\delta\Gamma^a_{bc} = h^a_{(b;c)} - \tfrac{1}{2}h_{bc}{}^{;a} . \tag{3.12}$$

By applying (3.1) to (3.12) and using the formula

$$\xi\pounds\Gamma^a_{bc} = \xi^a{}_{;(b;c)} - \xi^d R^a{}_{(bc)d} \tag{3.13}$$

(see, e.g., Yano[20]) for the Lie derivative of the connection, or alternatively by applying (3.12) directly in a gauge with zero displacement, one obtains for the Lagrangian variation of the connection the expression

$$\Delta\Gamma^a_{bc} = \Delta^a{}_{(b;c)} - \tfrac{1}{2}\Delta_{bc}{}^{;a} . \tag{3.14}$$

An elementary example of the application of this formula is to the variation of the acceleration. Thus starting from the formula

$$\Delta u^a = \frac{1}{2c^2}u^a u^b u^c \Delta_{bc} \tag{3.15}$$

(Carter[9]) for the Lagrangian variation of the unit flow tangent vector, and using the expression

$$\Delta_{ab;c} = 2g_{d(a}\Delta\Gamma^d_{\ b)c} , \tag{3.16}$$

we obtain the formula

$$\Delta\dot{u}^a = \gamma^a_{\ b}u^c u^d \Delta\Gamma^b_{cd} + \frac{2}{c^2}\dot{u}^{(a}u^{b)}u^c\Delta_{bc} , \tag{3.17}$$

which may be evaluated in more explicit form using (3.14).

We can now obtain the fundamental equation of motion for the displacement vector ξ (in an arbitrarily chosen gauge) by taking the Lagrangian variation of both sides of the basic equation of motion in the form

$$\rho\dot{u}^a + \gamma^a_{\ c}p^{cb}{}_{;b} = 0 , \tag{3.18}$$

evaluating the variations of density and pressure by means of the formulas

$$\Delta\rho = -\tfrac{1}{2}\rho y^{cd}\Delta_{cd} \tag{3.19}$$

and

$$\Delta p^{ab} = -\tfrac{1}{2}\Big(E^{abcd} + p^{ab}\gamma^{cd} - \frac{4}{c^2}p^{c(a}u^{b)}u^d\Big)\Delta_{cd} , \tag{3.20}$$

(Carter[9]) where we have introduced the abbreviation

$$y^{cd} = \gamma^{cd} + \frac{1}{\rho c^2}p^{cd} \tag{3.21}$$

and where E^{abcd} are components of the *elasticity* tensor (see Carter and Quintana[8]) which has the symmetry and orthogonality properties

$$E^{abcd} = E^{(ab)(cd)} = E^{cdab} , \tag{3.22}$$

$$E^{abcd}u_d = 0 . \tag{3.23}$$

Introducing the further abbreviation

$$A^{ab\ d}_{\ \ c} = E^{ab\ d}_{\ \ c} - \gamma^a_{\ c}p^{bd} \tag{3.24}$$

for the modified "Hadamard" elasticity tensor previously introduced by Carter[10] in a discussion of the characteristic wave fronts of sound propagation, we may write the perturbed equation of motion derived from (3.18) in the compact form

$$(A^{ab}{}_c{}^d - \rho y^a_c u^b u^d)\Delta\Gamma^c_{bd} = -\gamma^a_c E^{cebd}{}_{;e}\epsilon_{bd} + \frac{1}{c^2}(p^{ab}\dot{u}^d - \tfrac{1}{2}\dot{u}^a p^{bd} - 2A^{a(b}{}_c{}^{e)} v^c{}_e u^d + \rho y^a_c \dot{u}^c u^b u^d)\Delta_{bd}, \tag{3.25}$$

which is arranged with the second term (in parentheses) on the right-hand side consisting exclusively of relativistic correction terms, so that in the Newtonian limit only the first term remains. The terms of (3.25) may be evaluated explicitly in terms of ξ^a and h_{ab} by using the expressions (3.9) and (3.10) for Δ_{bd} and ϵ_{bd}, and using (3.14) to express the connection perturbation components in the form

$$\Delta\Gamma^c_{bd} = \xi^c{}_{;(b;d)} + h^c{}_{(b;d)} - \tfrac{1}{2}h_{bd}{}^{;c} - \xi^e R^c{}_{(bd)e}. \tag{3.26}$$

Equation (3.25) thus takes the form of a wave equation for the displacement components ξ^a, with the second partial derivative terms all grouped on the left-hand side. It is important to notice, however, that since all the terms are identically orthogonal to the flow vector $\vec{u}$, there are only three independent equations for the four components ξ^a. The consequent indeterminacy is a manifestation of the freedom to make arbitrary gauge transformations of the second kind: It may be removed by imposing the orthogonality requirement

$$\xi^a u_a = 0 \tag{3.27}$$

as a gauge restriction on the displacement vector.

The characteristic acoustic wave fronts may be defined as hypersurfaces on which the second derivatives $\xi^a{}_{;b;c}$ may be discontinuous while ξ^a and $\xi^a{}_{;b}$ (as well as h_{ab} and $h_{ab;c}$) remain continuous. If λ_a are components of a covector normal to such a hypersurface of discontinuity then by the well-known principle of Hadamard the discontinuity in the second derivative must have the form

$$[\xi^a{}_{;b;c}]^+_- = l^a\lambda_b\lambda_c \tag{3.28}$$

for some vector l^a, which characterizes the amplitude and polarization direction of the discontinuity. Since the right-hand side of (3.25) contains only first derivatives it gives no contribution at all when we take the discontinuity, so that we obtain the characteristic equation directly, in the simple form

$$(A^{ab}{}_c{}^d - \rho y^a_c u^b u^d)l^c\lambda_b\lambda_d = 0. \tag{3.29}$$

If $\vec{l}$ is parallel to $\vec{u}$ then (3.29) will be satisfied identically for arbitrary λ_a —this is merely the mode corresponding to gauge transformations of the second kind. This trivial mode is automatically eliminated by the orthogonality condition (3.27), which automatically implies the corresponding restriction

$$l^a u_a = 0. \tag{3.30}$$

The genuine physical sound wave fronts are thus obtained by solving the purely three-dimensional eigenvalue problem determined by (3.29) subject to (3.30). [See Carter[10] for a more direct derivation of (3.29) and a more thorough discussion of its implications.]

In order to obtain a complete system of equations of motion for the perturbations, we must of course supplement the wave equation (3.25) for the ξ^a by a corresponding equation for the metric perturbation components h_{ab}. In special relativity one could simply take the h_{ab} to be zero. In general relativity, however, the h_{ab} will be governed by the dynamic wave equation given by the perturbation of the Einstein equation (2.28), which takes the form

$$\delta\hat{R}^{ab} = \frac{8\pi G}{c^4}\,\delta T^{ab} = \frac{8\pi G}{c^4}(\Delta T^{ab} - \vec{\xi}\,\pounds T^{ab}), \tag{3.31}$$

where

$$\hat{R}^{ab} = R^{ab} - \tfrac{1}{2}Rg^{ab}. \tag{3.32}$$

The Lagrangian variation of the energy-momentum tensor may easily be obtained (cf., Carter[9]) combining (3.15), (3.19), and (3.20). Following Friedman and Schutz,[12] we may express the result compactly in the form

$$\Delta T^{ab} = -\tfrac{1}{2}(\mathcal{E}^{abcd} + T^{ab}g^{cd})\Delta_{cd}, \tag{3.33}$$

where $\mathcal{E}^{abcd}$ are components of a tensor having the same symmetry properties as those [given by (3.22)] of the elasticity tensor, i.e.,

$$\mathcal{E}^{abcd} = \mathcal{E}^{(ab)(cd)} = \mathcal{E}^{cdab} \tag{3.34}$$

but without the orthogonality property (3.23); its explicit form is given by

$$\mathcal{E}^{abcd} = E^{abcd} + \frac{1}{c^2}(6u^{(a}u^b p^{cd)} - 8u^{(a}p^{b)(c}u^{d)} - \rho u^a u^b u^c u^d). \tag{3.35}$$

Combining this with the formula

$$\vec{\xi}\,\pounds T^{ab} = T^{ab}{}_{;c}\xi^c - 2T^{c(a}\xi^{b)}{}_{;c}, \tag{3.36}$$

we have all the elements necessary for the evaluation of the right-hand side of (3.31).

The evaluation of the left-hand side of (3.31) is a mere routine exercise. Starting from the expression

$$\delta R^a{}_{bcd} = 2(\delta\Gamma^a_{b[d})_{;c]} \tag{3.37}$$

and using (3.12) one obtains by contraction the familiar formula

$$\delta R_{ab} = h^c{}_{(a;b);c} - \tfrac{1}{2}(h_{ab}{}^{;c}{}_{;c} + h_c{}^c{}_{;a;b}). \tag{3.38}$$

The left-hand side of (3.31) may now be obtained in the form

$$\delta\hat{R}^{ab} = h^{c(a}{}_{;c}{}^{;b)} - \tfrac{1}{2}(h^{ab}{}_{;c}{}^{;c} + h_c{}^{c;a;b)}) - g^{ab}h^c{}_{[d;c]}{}^{;d}$$
$$- C^a{}_c{}^b{}_d h^{cd} + \tfrac{1}{3}Rh^{ab} - \tfrac{1}{2}(R^{ab} - \tfrac{1}{3}Rg^{ab})h_c{}^c. \quad (3.39)$$

In order to obtain a well-behaved wave equation for the components h_{ab} it is necessary to eliminate the indeterminacy that would result from the direct substitution of (3.39) in (3.31) as a consequence of the freedom to make arbitrary gauge transformations of the first kind. The standard way to do this is to use the harmonic gauge condition, which may be expressed in terms of the quantity

$$\hat{h}^{ab} = h^{ab} - \tfrac{1}{2}h_c{}^c g^{ab} \quad (3.40)$$

by the divergence condition

$$\hat{h}^{ab}{}_{;b} = 0 . \quad (3.41)$$

This does not entirely eliminate the freedom to make gauge transformations of the form (3.2): We are still free to make arbitrary displacement adjustments $\vec{\xi}$ on an initial hypersurface provided that their subsequent evolution is made to obey the wave equation

$$\Box\xi^a = 0 , \quad (3.42)$$

where, for *ad hoc* convenience, we have introduced a generalized wave operator whose action on a contravariant vector is defined by

$$\Box\xi^a \equiv \xi^a{}_{;c}{}^{;c} + R^a{}_c\xi^c . \quad (3.43)$$

(it is to be noted that this is *not* the same as the Lichnerowicz-De Rham operator, in which the curvature term would appear with the same magnitude but with the opposite relative sign.) When the harmonic condition (3.41) is satisfied, the expression (3.39) reduces to the form

$$\delta\hat{R}^{ab} = -\tfrac{1}{2}\Box\hat{h}^{ab} , \quad (3.44)$$

where the action of the operator $\Box$ on a symmetric contravariant tensor is defined by

$$\Box\hat{h}^{ab} \equiv \hat{h}^{ab}{}_{;c}{}^{;c} - \hat{R}^{ab}\hat{h}_c{}^c + 2C^a{}_c{}^b{}_d\hat{h}^{cd}$$
$$- \tfrac{2}{3}R(\hat{h}^{ab} - \tfrac{1}{4}g^{ab}\hat{h}_c{}^c) . \quad (3.45)$$

The resulting equation of motion for the metric perturbation may thus be expressed in the form

$$\Box\hat{h}^{ab} = -\frac{16\pi G}{c^4}\,\delta T^{ab} , \quad (3.46)$$

where by (3.32) and (3.36)

$$\delta T^{ab} = -\tfrac{1}{2}(\mathcal{E}^{abcd} + T^{ab}g^{cd})(h_{cd} + 2\xi_{(c;d)})$$
$$+ 2T^{c(a}\xi^{b)}{}_{;c} - T^{ab}{}_{;c}\xi^c . \quad (3.47)$$

The evolution of the perturbations is determined completely by solving the equations (3.25) and (3.46) simultaneously, with the gauge condition (3.41) imposed only as an initial-value constraint; the structure of the system is such that the condition (3.41) will be preserved *automatically* in the subsequent evolution. To see this we start by remarking that the equation (3.25) is (by its derivation) equivalent to the perturbation

$$\delta(T^{ab}{}_{;b}) = 0 \quad (3.48)$$

of the energy-momentum conservation condition, so that by the unperturbed Einstein equations (2.28) it also implies

$$\frac{8\pi G}{c^4}(\delta T^{ab})_{;b} = -2\hat{R}^{c(a}\delta\Gamma^{b)}_{cb} . \quad (3.49)$$

Hence taking the divergence of (3.46) subject to (3.25) and (2.28) gives

$$(\Box\hat{h}^{ab})_{;b} + \hat{R}^{c(a}\delta\Gamma^{b)}_{cb} = 0 , \quad (3.50)$$

which reduces, after substitution of the explicit expression (3.45) for the wave operation on a tensor, to the simple form

$$\Box(\hat{h}^{ab}{}_{;b}) = 0 , \quad (3.51)$$

where the operator $\Box$ (now acting only on a vector) is as defined by (3.43). Since this $\Box$ is a well-behaved hyperbolic wave operator, the automatic preservation of the gauge condition (3.41) is thus established.

In the perturbation theory of gravitational waves in a *flat*-space background it is customary to make use of the remaining gauge freedom allowed by (3.42) so as to impose further restrictions to the effect that the trace $h_c{}^c$ and the contraction of h_{ab} with a time translation Killing vector $\vec{k}$ be zero. Unlike the harmonic condition (3.41), however, such additional simplifications will *not* be compatible with the equation of motion (3.46) in a general background space. [It may be remarked, however, that the trace $h_c{}^c$ can still be set equal to zero in an *empty* curved space, since when the medium is absent the right-hand side of (3.46) vanishes, and the structure of (3.45) is such that the trace will decouple from the trace-free part of h_{ab}.]

We complete this section by observing that the presence of the background medium does not affect the second-order terms in (3.46) so that the characteristics on which the second partial derivatives of h_{ab} may be discontinuous are just the ordinary null hypersurfaces with tangent covector components specified by

$$g^{ab}\lambda_a\lambda_b = 0 . \quad (3.52)$$

Thus the discontinuity fronts of acoustic and gravitational waves move independently even though the continuous perturbations behind the fronts are inextricably coupled.

IV. THE WEAK-FIELD LIMIT

In Sec. III we imposed no restriction on the strength of the gravitational curvature fields in the *unperturbed* background space, so that the results would be applicable, e.g., to the theory of gravitational radiation by an oscillating neutron star—a problem that has been dealt with previously only be using a perfect fluid treatment (see Thorne[21] or Ipser[22]) in which many important effects (such as torsional vibration modes) could not be taken into account at all.

In the present section we shall show how the standard results of Dyson,[2] Papapetrou,[3] and others may be recovered in the linearized weak-field approximation wherein the background is treated as being approximately flat, as is appropriate in the analysis of weak gravitational wave interaction with a nonrelativistic body such as a white dwarf or planet, and hence *a fortiori* with a Weber bar detector.

It is important to emphasize that the weak-field theory to be considered here is somewhat different in spirit from the type of weak-field theory whose use for problems such as the emission of gravitational radiation by a binary system is described by Landau and Lifshitz[19] and criticized by Ehlers *et al.*[23] In the limit when the background space is approximately flat, it might at first sight appear that our fundamental gravitational radiation equation (3.46) is the same as the Landau-Lifshitz radiation equation. In fact, however, there is a fundamental difference in the interpretation of the source term on the right-hand side. In the Landau-Lifshitz theory (which was originally sponsored by Einstein himself) the δT^{ab} term would represent *in its entirety* the effect of introducing low-density matter (such as that of the sun and planets), with a motion described approximately by Newtonian theory, into a background space that was previously completely empty. On the other hand in our present theory the term δT^{ab} merely represents a small fractional adjustment in a background space in which the relevant (stellar or planetary) matter has already been introduced at the outset. Thus if we apply (3.46) to an *exactly* flat—and therefore empty—background space the term δT^{ab} will simply vanish. (Perturbing the world lines of nonexistent matter can have no effect at all.) Since the unperturbed background space is postulated to obey the exact Einstein equation (whether the fields be weak or strong) our treatment completely bypasses the well-known consistency difficulties that beset the Landau-Lifshitz type theory. (The inconsistencies arise from the fact that the Bianchi identities would require the Landau-Lifshitz energy-momentum source to be conserved relative to the *flat* background, which is incompatible with the existence of any gravitational interaction at all.) The perfect self-consistency of our present treatment is achieved at the expense of a limitation of its domain of applicability: It *cannot* be used *ab initio* for the construction of any solution of a problem such as that of a radiating binary system; but if just one (perhaps specially simple) solution for such a system is supposed to have been given in advance by other means then our method *can* be used for the construction of other (perhaps more complicated) solutions in the neighborhood of the first one. In practice the present formalism is most suitable for application to a single body, such as the earth as a whole, or an individual Weber bar, in which the unperturbed state is supposed to be one of *rigid* motion, which in practice must necessarily be supposed to be static (nonrotating) unless—as in the case of the earth—the body may be treated as being axisymmetric so that nongravitationally radiating stationary rotating states exist.

Our linearized weak-field treatment will be based on the assumption that the unperturbed background space is characterized by a small dimensionless parameter ϵ_0 such that the magnitude a say of the gravitational, centripetal, or other accelerations characterizing the motion has a maximum value given by

$$a \lesssim \frac{c^2}{L}\,\epsilon_0\,, \tag{4.1}$$

where L is a characteristic length scale of the body under consideration. This implies a corresponding upper limit

$$\omega \lesssim \frac{c}{L}\,\epsilon_0^{1/2} \tag{4.2}$$

on the magnitude of the characteristic angular velocity ω of the body, and also an upper limit

$$\rho \lesssim \frac{c^2}{GL^2}\,\epsilon_0\,, \tag{4.3}$$

on the characteristic density. If we assume that the only internal pressures and stresses are those required to maintain equilibrium against the gravitational and other acceleration forces, then their magnitude will be characterized by

$$p/\rho \lesssim c^2\epsilon_0\,. \tag{4.4}$$

The magnitude C, say, characterizing the corresponding gravitational curvature will be given by

$$C \lesssim \frac{\epsilon_0}{L^2}\,. \tag{4.5}$$

If we suppose that the unperturbed state is strictly rigid then the expansion tensor will be zero; however, for the results that follow it will be more than sufficient if we postulate that the characteristic magnitude θ, say, of the expansion rates is only restricted by

$$\theta \lesssim \frac{c}{L}\,\epsilon_0\,. \tag{4.6}$$

Finally we remark that since the characteristic magnitude v_s, say, of sound speeds will certainly be less than the speed of light (by a factor of order at least 10^4 in ordinary terrestrial matter), the order of magnitude E, say, of the elasticity components (in an orthonormal frame) will be given by

$$E \sim \rho\, v_s^{\,2}\, L \lesssim \frac{c^4}{GL}\epsilon_0. \tag{4.7}$$

As a particular example the earth as a whole satisfies all these requirements with $\epsilon_0 \sim 10^{-10}$; for a Weber bar—unless in a state of ultrarapid rotation—we could take ϵ_0 to be even smaller.

We shall suppose that the perturbations themselves are characterized by a second small dimensionless parameter ϵ_1, in terms of which the order of magnitude h say of the space-metric perturbation and ϵ, say, of the strain perturbation components (in an orthonormal frame) are given by

$$h \lesssim \epsilon_1\,, \tag{4.8}$$

$$\epsilon \lesssim \epsilon_1\,. \tag{4.9}$$

If the perturbations have a characteristic frequency ν, the characteristic wavelength will satisfy

$$\lambda \lesssim v_s/\nu \tag{4.10}$$

(since the speed of sound cannot be greater than the speed of light), and hence the magnitude ξ, say, characterizing the relative displacements will satisfy

$$\xi \lesssim \frac{v_s}{\nu}\,\epsilon_1\,. \tag{4.11}$$

For waves induced in the earth by a supernova explosion in even a very nearby star the corresponding value of ϵ_1 would be very much smaller than the value of ϵ_0, but of course the ratio would be reversed if the source were a nearby earthquake. We shall make no assumption about the absolute ratio of ϵ_1 to ϵ_0 but will consider the limiting situation in which both are postulated to tend to zero together, using the notation $O(\epsilon)$ to denote quantities that tend to zero, irrespectively, when divided by either ϵ_1 or ϵ_0.

If we apply this limiting procedure to the exact strain evolution equation (2.37), considering it as describing either the weak background alone *or* the background plus perturbation, we are left with

$$\mathcal{L}_T\mathcal{L}_T e_{ab} = -\gamma_{c\,(a}\gamma_{b)}{}^{d}(\rho^{-1}p^{ce}{}_{;e})_{;d} - C_{ab} - \frac{4\pi G}{3}\rho\gamma_{ab} + \omega_{ca}\omega^{c}{}_{b} + O(\epsilon)\,. \tag{4.12}$$

If instead we consider the evolution of the *perturbation alone* then it is easy to see that the self-gravitational and centrifugal force terms drop out and that we are left only with

$$\mathcal{L}_T\mathcal{L}_T\,\epsilon_{ab} = \gamma_{c\,(a}\gamma_{b)}{}^{d}[\rho^{-1}(E^{cefg}\epsilon_{fg})_{;e}]_{;d} - \Delta C_{ab} + O(\epsilon) \tag{4.13}$$

[having used (3.10) and (3.20)]. Since the perturbed Einstein equations give

$$\Delta R_{ab} = O(\epsilon)\,, \tag{4.14}$$

we shall have

$$\Delta C^{a}{}_{bcd} = \Delta R^{a}{}_{bcd} + O(\epsilon)\,, \tag{4.15}$$

and since

$$\Delta R^{a}{}_{bcd} = 2(\Delta\Gamma^{a}{}_{b[d})_{;c]} = \Delta^{a}{}_{[d;c];b} + \Delta_{b[c;d]}{}^{;a} + O(\epsilon)\,, \tag{4.16}$$

the definition (2.33) gives

$$\Delta C_{ab} = c^2\gamma_a^{(e}\gamma_b^{f)}\gamma^{cd}(2\epsilon_{ce;d;f} - \epsilon_{ef;c;d} - \epsilon_{cd;e;f}) + O(\epsilon)\,. \tag{4.17}$$

[Note that in the special relativistic limit where the h_{ab} are zero, the term on the right-hand side of (4.17) will vanish identically.] It is apparent that by substituting (4.17) in (4.13) we can obtain a second-order linear wave equation for the strain perturbation components ϵ_{ab} alone (cf., Papapetrou[3]). However, despite its attractive feature of containing only gauge-invariant quantities, the strain wave equation obtained in this way does not provide the most efficient way of dealing with practical problems, and in order to decouple the various components from among themselves one would be led ineluctably to break the gauge invariance. In practice it is more effective to work with the displacements ξ^a and the metric perturbation components h_{ab} from the outset as in Sec. III, using the harmonic condition (3.41) so that by (3.46) the metric perturbation is described by the effectively source-free flat-space wave equation

$$\Box h^{ab} = O(\epsilon)\,, \tag{4.18}$$

where $\Box$ is now just the flat-space wave operator given simply by

$$\Box h^{ab} = h^{ab}{}_{;c}{}^{;c} . \qquad (4.19)$$

When this decoupled equation for the h^{ab} has been solved (e.g., by taking a standard plane-wave solution) it remains only to satisfy the wave equation (3.25) for the three independent displacement components, which reduces in the present limit to

$$\gamma^a_c u^b u^d (\xi^c{}_{;b;d} + h^c{}_{b;d} - \tfrac{1}{2} h_{bd}{}^{;c}) = \rho^{-1}[E^{abcd}(\xi_{c;d} + \tfrac{1}{2}h_{cd})]_{;b} + O(\epsilon) . \qquad (4.20)$$

An equivalent displacement wave equation has been given by Papapetrou.[3] As pointed out by Papapetrou, the gauge-invariant strain wave equation [in a form equivalent to (4.13)] can be obtained by taking the symmetrized derivative of the displacement wave equation and using, instead of (4.17), the alternative expression

$$\Delta C_{ab} = u^c u^d \delta C_{acbd} + O(\epsilon) = u^c u^d (h_{a[d;b];c} + h_{c[b;d];a}) + O(\epsilon) . \qquad (4.21)$$

Dyson[2] presented the displacement wave equation in an even more highly simplified form than (4.20), effectively using the fact that under the postulated weak-field conditions, we shall have

$$\Box h_a{}^a = O(\epsilon) , \qquad (4.22)$$

and

$$\Box(u_a h^{ab}) = O(\epsilon) , \qquad (4.23)$$

which makes it possible to use the remaining gauge freedom [still consistently with (3.14)] to set

$$h_c{}^c = O(\epsilon) \qquad (4.24)$$

and

$$u_a h^{ab} = O(\epsilon) , \qquad (4.25)$$

provided we are considering only a bounded time interval. Using the last of these conditions (4.24), together with the displacement gauge condition (3.27), the displacement wave equation (4.20) may be reduced to the form

$$u^b u^d \xi^a{}_{;b;d} = \rho^{-1}[E^{abcd}(\xi_{c;d} + \tfrac{1}{2}h_{cd})]_{;b} + O(\epsilon) , \qquad (4.26)$$

and the expression (4.21) to be substituted in (4.13) may be reduced to

$$\Delta C_{ab} = \tfrac{1}{2} \mathcal{L}_\tau \mathcal{L}_\tau h_{ab} + O(\epsilon) . \qquad (4.27)$$

If the unperturbed medium is in an approximately isotropic state, we shall have

$$E^{abcd} = \beta \gamma^{ab}\gamma^{cd} + 2\mu(\gamma^{a(c}\gamma^{d)b} - \tfrac{1}{3}\gamma^{ab}\gamma^{cd}) + O(\epsilon) , \qquad (4.28)$$

where β is the bulk modulus and μ the shear modulus, so that using (4.24) we obtain

$$\rho^{-1}[E^{abcd}(\tfrac{1}{2}h_{cd})]_{;b} = \rho^{-1}\mu_{,b} h^{ab} + O(\epsilon) , \qquad (4.29)$$

which shows that (as pointed out by Dyson[2]) a passing gravitational wave has no effect on the wave equation (4.26) (i.e., the acoustic mode will be completely decoupled from the gravitational mode) unless there are inhomogeneities in the modulus of rigidity.

Most of the previous workers cited have at least by implication based their treatment of weak gravitational wave interactions on the use of a coordinate system that is comoving with the medium in the unperturbed background. Such a system may be based on the use of approximately Euclidean space coordinates x^μ (taking Greek indices to run from 1 to 3) with fixed values on given unperturbed particle world lines, together with a proper-time coordinate $x^0 = \tau$, where τ is a measure of proper time along the world lines. Under the weak-field conditions postulated here we may arrange to have

$$g_{ab} = \eta_{ab} + O(1) , \qquad (4.30)$$

$$u^a = \delta^a_0 + O(1) , \qquad (4.31)$$

in such a system, where η_{ab} are components of the flat metric defined by

$$\eta_{ab} dx^a dx^b = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - c^2 d\tau^2 , \qquad (4.32)$$

$O(1)$ denotes terms that tend to zero as ϵ_0 tends to zero. [If the unperturbed medium were sufficiently slowly rotating we could replace $O(1)$ by $O(\epsilon)$ here.] In terms of such a system the covariant equations (4.18) and (4.26) may be replaced by the specialized forms

$$h_{\mu\nu,0,0} = c^2 h_{\mu\nu;\rho}{}^{;\rho} , \qquad (4.33)$$

$$\xi_{\mu,0,0} = \rho^{-1}[E_\mu{}^{\rho\nu\sigma}(\xi_{\nu\sigma} + \tfrac{1}{2}h_{\nu\sigma})]_{,\rho} + O(\epsilon) , \qquad (4.34)$$

the gauge conditions having the form

$$h_{0a} = O(\epsilon), \quad h_\mu{}^\mu = O(\epsilon), \quad h_{\mu\nu}{}^{;\nu} = O(\epsilon), \quad \xi^0 = 0 . \qquad (4.35)$$

The corresponding specialized form for the gauge-invariant strain equation is

$$\epsilon_{\mu\nu,0,0} + \Delta C_{\mu\nu} = [\rho^{-1}(E_{(\mu|}{}^{\rho\lambda\sigma}\epsilon_{\lambda\sigma})_{;\rho}]_{;|\nu)} + O(\epsilon) \qquad (4.36)$$

(the bars indicate that only μ and ν are affected by the symmetrization), where the tidal force tensor is defined gauge invariantly by

$$\Delta C_{\mu\nu} = \tfrac{1}{2}(h_{\mu 0,0,\nu} + h_{\nu 0,0,\mu} - h_{\mu\nu,0,0} - h_{00,\mu,\nu}) + O(\epsilon) , \qquad (4.37)$$

which reduces under the conditions (4.33) to

$$\Delta C_{\mu\nu} = -\tfrac{1}{2} h_{\mu\nu,0,0} + O(\epsilon)\,. \tag{4.38}$$

APPENDIX: ERRATA

We take this opportunity to rectify some arithmetical copying errors in our original paper (Carter and Quintana[8]) that have been kindly drawn to our attention by J. Leroy and M. A. H. MacCallum. These errors are of three kinds:

(1) Misordering of indices. The covariant indices d,g should be interchanged in the first term on the right-hand side of (5.41); the covariant indices b,d should be interchanged in the last term of both the second and the third line of (6.13), and in the second term of (3.34).

(2) Error of sign. On the left-hand side of (5.7), $n_c{}^c\{\ldots$ should read $-n_c{}^c\{\ldots$; in the second term of the numerator in (5.18), $-\frac{4}{3}s_a{}^b\ldots$ should read $+\frac{4}{3}s_a{}^b\ldots$; in the first term of the second line of (6.13), $-n(2w-\frac{1}{3}\ldots$ should read $+n(2w+\frac{1}{3}\ldots$; in the first term on the right of (6.19), $-\frac{1}{3}\mu$ should read $+\frac{1}{3}\mu$.

(3) Factor of 2 missing. On the first line of (6.13) the coefficient $\frac{2}{3}$ should be $\frac{4}{3}$.

None of the sign or factor errors listed here is of practical importance: They affect only higher-order corrections that were included for the sake of exact mathematical consistency, but which are negligible in all the practical applications that we have considered.

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