# Fully-constrained formulation of Einstein's field equations using Dirac gauge

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# Outline

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2. Description of the formulation and strategy
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   - Models of rotating neutron stars
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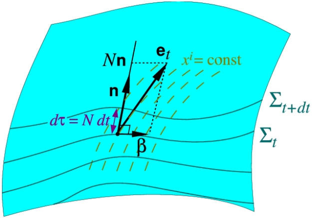
# 3+1 FORMALISM

Decomposition of spacetime and of Einstein equations




**EVOLUTION EQUATIONS:**

$$\frac{\partial K_{ij}}{\partial t} - \mathcal{L}_\beta K_{ij} =$$
$$-D_i D_j N + N R_{ij} - 2N K_{ik} K^k_{\ j} +$$
$$N\left[K K_{ij} + 4\pi((S-E)\gamma_{ij} - 2S_{ij})\right]$$
$$K^{ij} = \frac{1}{2N}\left(\frac{\partial \gamma^{ij}}{\partial t} + D^i\beta^j + D^j\beta^i\right).$$

**CONSTRAINT EQUATIONS:**

$$R + K^2 - K_{ij}K^{ij} = 16\pi E,$$
$$D_j K^{ij} - D^i K = 8\pi J^i.$$

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = -N^2\, dt^2 + \gamma_{ij}\,(dx^i + \beta^i dt)\,(dx^j + \beta^j dt)$$

# Constraint violation

If the constraints are verified for initial data, evolution should preserve them. Therefore, one could in principle solve Einstein equations without solving the constraints

$\Downarrow$

**Appearance of constraint violating modes**

However, some cures have been (are) investigated :

- solving the constraints at (almost) every time-step ...
- constraints as evolution equations (Gentle *et al.* 2004)
- constraint-preserving boundary conditions (Lindblom *et al.* 2004 & presentation by M. Scheel)
- relaxation (Marronetti 2005)
- constraint projection (presentation by L. Lindblom)

# Some reasons not to solve constraints



computational cost of usual elliptic solvers ...

few results of well-posedness for mixed systems versus solid mathematical theory for pure-hyperbolic systems

definition of boundary conditions at finite distance and at black hole excision boundary

# Motivations for a fully-constrained scheme

"Alternate" approach (although most straightforward)

- partially constrained schemes: Bardeen & Piran (1983), Stark & Piran (1985), Evans (1986)
- fully constrained schemes: Evans (1989), Shapiro & Teukolsky (1992), Abrahams *et al.* (1994), Choptuik *et al.* (2003)

⇒Rather popular for 2D applications, but disregarded in 3D

Still, many advantages:

- constraints are verified!
- elliptic systems have good stability properties
- easy to make link with initial data
- evolution of only two scalar-like fields ...

# Usual conformal decomposition

Standard definition of conformal 3-metric (e.g. Baumgarte-Shapiro-Shibata-Nakamura formalism)

## Dynamical degrees of freedom of the gravitational field:

York (1972) : they are carried by the conformal "metric"

$$\hat{\gamma}_{ij} := \gamma^{-1/3} \gamma_{ij} \qquad \text{with } \gamma := \det \gamma_{ij}$$

## Problem

$\hat{\gamma}_{ij}$ = *tensor density* of weight $-2/3$
not always easy to deal with tensor densities... not *really* covariant!

# Introduction of a flat metric





We introduce $f_{ij}$ (with $\frac{\partial f_{ij}}{\partial t} = 0$) as the asymptotic structure of $\gamma_{ij}$, and $\mathcal{D}_i$ the associated covariant derivative.

**Define:**

$$\tilde{\gamma}_{ij} := \Psi^{-4}\,\gamma_{ij} \text{ or } \gamma_{ij} =: \Psi^4\,\tilde{\gamma}_{ij}$$

with

$$\Psi := \left(\frac{\gamma}{f}\right)^{1/12}$$

$$f := \det f_{ij}$$

$\tilde{\gamma}_{ij}$ is invariant under any conformal transformation of $\gamma_{ij}$ and verifies $\det \tilde{\gamma}_{ij} = f$

$\Rightarrow$ *no more tensor densities: only tensors.*

Finally,

$$\tilde{\gamma}^{ij} = f^{ij} + h^{ij}$$

is the deviation of the 3-metric from conformal flatness.

# Generalized Dirac gauge

One can generalize the gauge introduced by Dirac (1959) to any type of coordinates:

**Divergence-free condition on $\tilde{\gamma}^{ij}$**

$$\mathcal{D}_j \tilde{\gamma}^{ij} = \mathcal{D}_j h^{ij} = 0$$

where $\mathcal{D}_j$ denotes the covariant derivative with respect to the flat metric $f_{ij}$.

Compare

- minimal distortion (Smarr & York 1978) : $D_j \left( \partial \tilde{\gamma}^{ij} / \partial t \right) = 0$
- pseudo-minimal distortion (Nakamura 1994) : $\mathcal{D}^j \left( \partial \tilde{\gamma}_{ij} / \partial t \right) = 0$

*Notice:* Dirac gauge $\iff$ BSSN connection functions vanish: $\tilde{\Gamma}^i = 0$

# Generalized Dirac gauge properties

- $h^{ij}$ is transverse
- from the requirement $\det \tilde{\gamma}_{ij} = 1$, $h^{ij}$ is asymptotically traceless
- ${}^3R_{ij}$ is a simple Laplacian in terms of $h^{ij}$
- ${}^3R$ does not contain any second-order derivative of $h^{ij}$
- with constant mean curvature ($K = t$) and spatial harmonic coordinates ($\mathcal{D}_j \left[ (\gamma/f)^{1/2} \gamma^{ij} \right] = 0$), Anderson & Moncrief (2003) have shown that the Cauchy problem is *locally strongly well posed*
- the Conformal Flat Condition (CFC) verifies the Dirac gauge ⇒possibility to easily use initial data for binaries now available

# Einstein equations

## Dirac gauge and maximal slicing ($K = 0$)

### Hamiltonian constraint

$$\Delta(\Psi^2 N) = \Psi^6 N \left(4\pi S + \frac{3}{4}\tilde{A}_{kl}A^{kl}\right) - h^{kl}\mathcal{D}_k\mathcal{D}_l(\Psi^2 N) + \Psi^2\left[N\left(\frac{1}{16}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} - \frac{1}{8}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il} + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k \ln\Psi\right) + 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N\right]$$

### Momentum constraint

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\left(\mathcal{D}_j\beta^j\right) = 2A^{ij}\mathcal{D}_j N + 16\pi N\Psi^4 J^i - 12 N A^{ij}\mathcal{D}_j \ln\Psi - 2\Delta^i{}_{kl} N A^{kl} - h^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i - \frac{1}{3}h^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l$$

### Trace of dynamical equations

$$\Delta N = \Psi^4 N\left[4\pi(E+S) + \tilde{A}_{kl}A^{kl}\right] - h^{kl}\mathcal{D}_k\mathcal{D}_l N - 2\tilde{D}_k \ln\Psi\, \tilde{D}^k N$$

# Einstein equations
## Dirac gauge and maximal slicing ($K = 0$)



### Evolution equations

$$\frac{\partial^2 h^{ij}}{\partial t^2} - \frac{N^2}{\Psi^4}\Delta h^{ij} - 2\pounds_\beta \frac{\partial h^{ij}}{\partial t} + \pounds_\beta \pounds_\beta h^{ij} = \mathcal{S}^{ij}$$

6 components - 3 Dirac gauge conditions - $\left(\det \tilde{\gamma}^{ij} = 1\right)$

### 2 degrees of freedom

$$-\frac{\partial^2 \chi}{\partial t^2} + \Delta \chi = S_\chi$$

$$-\frac{\partial^2 \mu}{\partial t^2} + \Delta \mu = S_\mu$$

# Spherical coordinates and components

## Choice for $f_{ij}$ : spherical polar coordinates

- stars and black holes are of spheroidal shape
- compactification made easy (only $r$)
- use of spherical harmonics
- grid boundaries are smooth surfaces

## Use of spherical orthonormal triad (tensor components)

- Dirac gauge can easily be imposed
- boundary conditions for excision might be better formulated
- asymptotically, it is easier to extract gravitational waves

# Representation of $h^{ij}$

Introduction of $\bar{h}^{ij}$ as the transverse-traceless part of $h^{ij}$, with only two degrees of freedom:

## First Dirac condition $\mathcal{D}_i h^{ir} = 0$

$$\frac{\partial \bar{h}^{rr}}{\partial r} + \frac{3\bar{h}^{rr}}{r} + \frac{1}{r}\Delta_{\theta\varphi}\eta = 0$$

with

$$\bar{h}^{r\theta} = \frac{1}{r}\left(\frac{\partial \eta}{\partial \theta} - \frac{1}{\sin\theta}\frac{\partial \mu}{\partial \varphi}\right)$$

$$\bar{h}^{r\varphi} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial \eta}{\partial \varphi} + \frac{\partial \mu}{\partial \theta}\right)$$

Knowing $\bar{h}^{rr}$ and $\mu$, it is possible to deduce $\bar{h}^{r\theta}$ and $\bar{h}^{r\varphi}$ from the first Dirac condition.
$\Rightarrow \bar{h}^{\theta\varphi}$ and $\bar{h}^{\varphi\varphi}$ from the other two gauge conditions
$\Rightarrow \bar{h}^{\theta\theta}$ from the trace-free condition.

# Integration procedure

Everything is know on slice $\Sigma_t$

$\Downarrow$

Evolution of $\chi = r^2 \bar{h}^{rr}$ and $\mu$ to next time-slice $\Sigma_{t+dt}$ (+ hydro)

$\Downarrow$

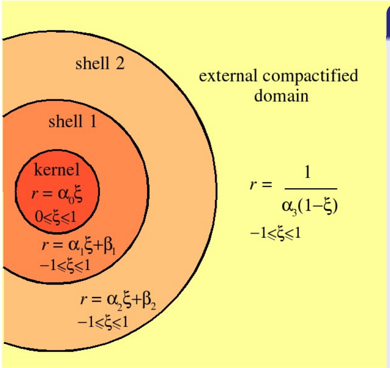
Deduce $\bar{h}^{ij}(t+dt)$ from Dirac and trace-free conditions

$\Downarrow$

Deduce the trace from $\det \tilde{\gamma}^{ij} = 1$; thus $h^{ij}(t+dt)$ and $\tilde{\gamma}^{ij}(t+dt)$.

$\Downarrow$

Iterate on the system of elliptic equations for $N, \Psi^2 N$ and $\beta^i$ on $\Sigma_{t+dt}$



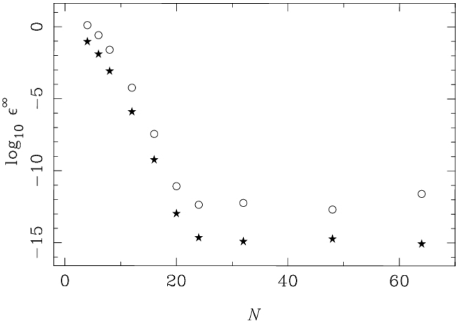

### Decomposition:

Chebyshev polynomials for $\xi$, Fourier or $Y_l^m$ for the angular part $(\theta, \phi)$,

- symmetries and regularity conditions of the fields at the origin and on the axis of spherical coordinate system
- compactified variable for elliptic PDEs $\Rightarrow$boundary conditions are well imposed

Drawback: Gibbs phenomenon!

$f(x) = \cos^3\left(\frac{\pi}{2}x\right) - \frac{1}{8}(x+1)^3$ over $[-1;1]$, represented by a series of $N$ Chebyshev polynomials ($T_n(x) = \cos(n \arccos x)$)




Error decaying as $e^{-N}$; as for comparison, more than $10^5$ points are necessary with a third-order finite-difference scheme ...

# Solutions of Poisson and wave equations

The angular part of any field $\phi$ is decomposed on a set of spherical harmonics $Y_\ell^m(\theta,\varphi)$, which are eigenvectors of the angular part of the Laplace operator

$$\Delta_{\theta\varphi} Y_\ell^m = -\ell(\ell+1) Y_\ell^m$$

**$\Delta\phi = \sigma$**

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\phi_{\ell m}(r) = \sigma_{\ell m}(r)$$

Accuracy on the solution $\sim 10^{-13}$ (exponential decay)

**$\Box\phi = \sigma$**

$$\left[1 - \frac{\delta t^2}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)\right]\phi_{\ell m}^{J+1} = \sigma_{\ell m}^J$$

Accuracy on the solution $\sim 10^{-10}$ (time-differencing)

$\forall(\ell, m)$ the operator inversion $\iff$ inversion of a $\sim 30 \times 30$ matrix

Non-linear parts are evaluated in the physical space and contribute as sources to the equations.

# Boundary conditions



## Poisson-type PDEs

Thanks to the compactification of type $u = 1/r$, it is possible to impose asymptotic flatness "exactly"

## Wave-type PDEs

Standard compactification cannot apply ⇒choice of transparent boundary conditions for $\ell \leq 2$ at finite distance, in the linear regime.

$$\forall \ell, \left. \frac{\partial \phi_{\ell m}}{\partial r} + \frac{\partial \phi_{\ell m}}{\partial t} + \frac{\phi_{\ell m}}{r} \right|_{r=R} = \zeta_{\ell m}(t)$$

with $\zeta(t, \theta, \varphi)$ being a function verifying a wave-like equation on the outer-boundary sphere

$$\frac{\partial^2 \zeta}{\partial t^2} - \frac{3}{4R^2} \Delta_{\theta\varphi} \zeta + \frac{3}{R} \frac{\partial \zeta}{\partial t} + \frac{3\zeta}{2R^2} = \frac{1}{2R^2} \Delta_{\theta\varphi} \left( \frac{\phi}{R} - \left. \frac{\partial \phi}{\partial r} \right|_{r=R} \right)$$

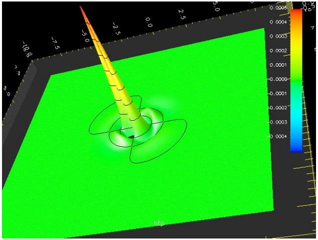
# Results with a pure gravitational wave spacetime

## Initial data

Similar to Baumgarte & Shapiro (1999), namely a momentarily static ($\partial\tilde{\gamma}^{ij}/\partial t = 0$) Teukolsky (1982) wave $\ell = 2$, $m = 2$:

$$\begin{cases} \chi(t=0) = \dfrac{\chi_0}{2}\, r^2 \exp\left(-\dfrac{r^2}{r_0^2}\right) \sin^2\theta \, \sin 2\varphi \\ \mu(t=0) = 0 \end{cases} \quad \text{with } \chi_0 = 10^{-3}$$

Preparation of the initial data by means of the *conformal thin sandwich* procedure



Evolution of $h^{\phi\phi}$ in the plane $\theta = \frac{\pi}{2}$

# How well are equations solved?



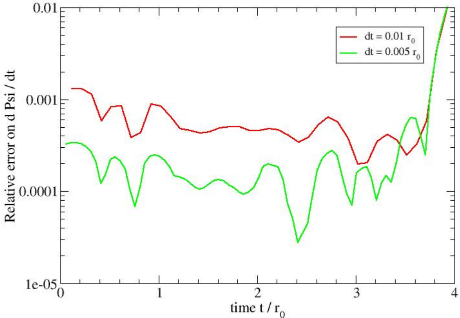
## Constraints

- Imposed numerically at every time-step!
- depends on spectral resolution & number of iterations
- keep the error below $10^{-6}$

## Evolution equations

- only two out of six are solved
- check on the others: equation for $\Psi$

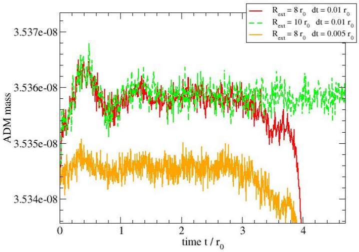
$$\frac{\partial \Psi}{\partial t} = \beta^k \mathcal{D}_k \Psi + \frac{\Psi}{6} \mathcal{D}_k \beta^k$$

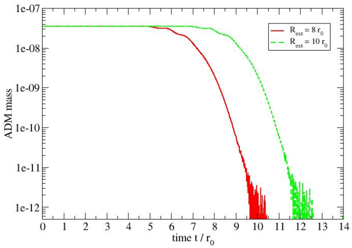


# Evolution of ADM mass

## Are the boundary conditions efficient?

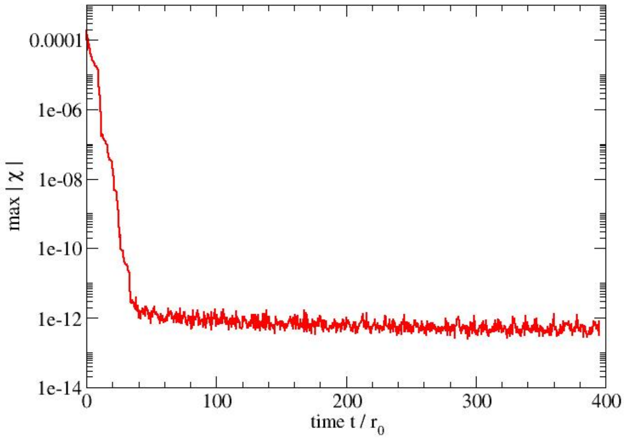





- ADM mass is conserved up to $10^{-4}$
- main source of error comes from time finite-differencing
- the wave is let out at better than $10^{-4}$

# Long-term stability




40000 time-steps

# Physical model of rotating neutron stars

Code developed for

- self-gravitating perfect fluid in general relativity
- two Killing vector fields (axisymmetry + stationarity)
- Dirac gauge
- equilibrium between matter and gravitational field
- equation of state of a relativistic polytrope $\Gamma = 2$

## Considered model here:

- central density $\rho_c = 2.9\rho_{nuc}$
- rotation frequency $f = 641.47$ Hz $\simeq f_{\text{Mass shedding}}$
- gravitational mass $M_g \simeq 1.51 M_\odot$
- baryon mass $M_b \simeq 1.60 M_\odot$

Equations are the same as in the dynamical case, replacing time derivative terms by zero

# Comparison with quasi-isotropic gauge



Other code using quasi-isotropic gauge has been used for a long time and successfully compared to other codes in Nozawa *et al.* (1998).

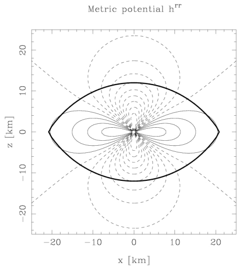
## Global quantities

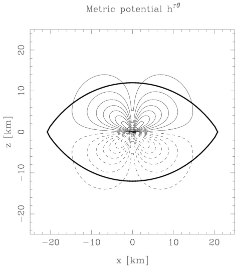
| Quantity | q-isotropic | Dirac | rel. diff. |
|---|---|---|---|
| $N(r=0)$ | 0.727515 | 0.727522 | $10^{-5}$ |
| $M_g$ $[M_\odot]$ | 1.60142 | 1.60121 | $10^{-4}$ |
| $M_b$ $[M_\odot]$ | 1.50870 | 1.50852 | $10^{-4}$ |
| $R_{\rm circ}$ [km] | 23.1675 | 23.1585 | $4 \times 10^{-4}$ |
| $J$ $\left[GM_\odot^2/c\right]$ | 1.61077 | 1.61032 | $3 \times 10^{-4}$ |
| Virial 2D | $1.4 \times 10^{-4}$ | $1.5 \times 10^{-4}$ | |
| Virial 3D | $2.5 \times 10^{-4}$ | $2.1 \times 10^{-4}$ | |

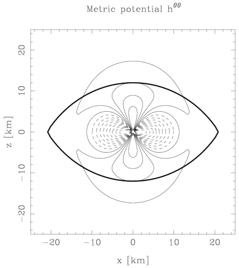
Virial identities (2 & 3D) are covariant relations that should be fulfilled by any stationary spacetime; they are not imposed numerically.

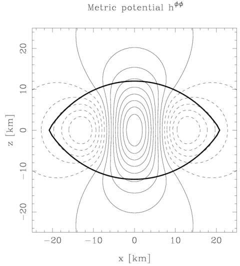
# Stationary axisymmetric models

## Deviation from conformal flatness




For all components (except $h^{r\varphi}$ and $h^{\theta\varphi}$, which are null), $h^{ij}_{\mathrm{max}} \sim 0.005$
$\Rightarrow$comparable with $\gamma_{\theta\theta} - \gamma_{\varphi\varphi}$ in quasi-isotropic gauge

# Problems for large amplitudes

In both gravitational wave evolution and neutron star models, there are convergence problems for

$$h^{ij} \gtrsim 0.01$$

In the neutron star case, these are very compact (in practice on the unstable branch) models *and* close to mass-shedding limit

## Possible reason:

1. if $\chi$ and $\mu$ are supposed regular (obtained from the PDE solver)
2. $h^{r\theta}, h^{r\varphi} \sim \partial_r \chi$
3. $h^{\theta\theta}, h^{\theta\varphi} \sim \partial_r^2 \chi$
4. $S^{ij}$ (source of the equation for $h^{ij}$) $\sim \partial_r^4 \chi$ !

# New integration strategy

Taking tensor spherical harmonics from Zerilli (1970), it is possible to express the tensor wave/Poisson equation for $h^{ij}$ and the Dirac gauge condition in terms of quantities expandable on $Y_\ell^m$.

1. solve for potentials linked with $h^{\theta\theta}$ and $h^{\theta\varphi}$
2. from the $\theta$- and $\varphi$- components of the Dirac gauge, integrate to get $\eta$ and $\mu$
3. from the remaining Dirac condition, integrate to get $h^{rr}$
4. the trace is obtained from the requirement that $\det \tilde{\gamma}^{ij} = 1$ as before

Works fine for the simplified case of divergence-free vector Poisson equation.

# Summary

- We have developed and implemented a fully-constrained evolution scheme for solving Einstein equations, using a generalization of Dirac gauge and maximal slicing
- Easy to extract gravitational radiation (asymptotical TT gauge + spherical grid)
- Well tested for $h^{ij} \lesssim 0.01$, corresponding to most of relevant astrophysical scenarios without a black hole

- Ongoing work and outlook
  - Try to reduce the number of radial derivatives in our scheme and replace them by integrations
  - Already possible applications to core collapse ("Mariage Des Maillages" project) or study of oscillations of relativistic stars
  - Compatible with no-radiation approximations: *e.g.* Schäfer & Gopakumar (2004); useful for slow evolution studies of inspiralling compact binaries

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