AGRO

Institut Henri Poincaré Trimester on

GRAVITY

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course on

ADVANCED GENERAL RELATIVITY

by

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1. Gravitational Radiation
2. General Relativistic N-body Problem
3. Motion of Strongly Self-gravitating Bodies
4. Binary Pulsars
5. Coalescing Binary Black Holes
6. String Theory and Gravity

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GRAVITATIONAL RADIATION:

STF Multipole Expansions

Most relevant background references:

[T80] K.S. Thorne, 'Multipole expansions of gravitational radiation', Rev. Mod. Phys 52, 299 (1980)

[BD86] L. Blanchet, T. Damour, 'Radiative gravitational fields in general relativity I General structure of the field outside the source', Philo. Trans. R. Soc. London, Ser. A, 320, 379 (1986)

[BD89] L. Blanchet, T. Damour 'Post-Newtonian generation of gravitational waves', Ann. Inst. H. Poincaré, A, 50, 377 (1989)

[DI91] T. Damour, B.R. Iyer, 'Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors', Phys. Rev. D 43, 3259 (1991)

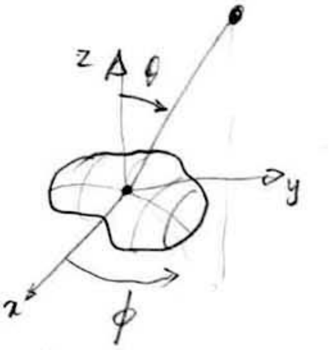
1.1 Multipole Expansions : Stationary Scalar Case

$$\Delta \varphi(\vec{x}) = -4\pi \rho(\vec{x})$$

usual multipole expansion :

$$\varphi(\vec{x}) = 4\pi \sum_{l \geq 0} \sum_{-l \leq m \leq l} \frac{Q_{lm}}{2l+1} \frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

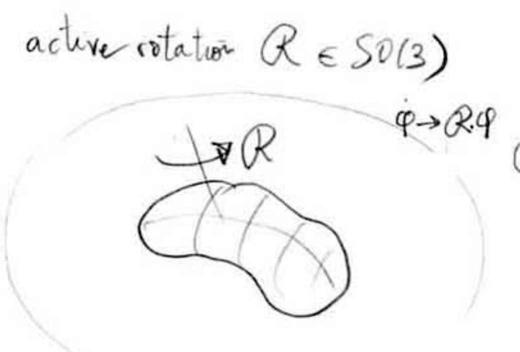
$$Q_{lm} = \int d^3x \; Y^*_{lm}(\theta,\phi) \; r^l \rho(\vec{x})$$



two features : $\varphi(\vec{x})$ decomposed in $\sum_l \sum_m \varphi_{lm}(r,\theta,\phi)$

- each piece $\varphi_{lm}(r\theta\phi)$ is a solution of $\Delta\varphi = 0$
- the decomposition $\varphi(\theta\phi) = \sum_{l,m} \varphi_{lm}(\theta,\phi)$ is a decomposition in irreducible representations of $SO(3)$ acting on $\varphi(\theta,\phi)$

active rotation $\mathcal{R} \in SO(3)$



$\varphi \to \mathcal{R}\cdot\varphi$

$$\mathcal{R}\cdot Q_{lm} = \sum_{m'} D^{(l)}_{mm'} Q_{lm'}$$

$$Q_{lm} = \begin{pmatrix} Q_{ll} \\ \vdots \\ Q_{l0} \\ \vdots \\ Q_{l-m} \end{pmatrix}$$ carries a $(2l+1)$-dim irrep of $SO(3)$

$\boxed{1.2}$ Symmetric Trace-Free tensors, see [T'80], App. A of [BD86], [DI91]

A useful representation of the $(2\ell+1)$-dim irrep of $SO(3)$

replace $\{Q_{\ell m} \mid -\ell \leq m \leq \ell\}$ by $\left\{ \hat{Q}_{i_1 i_2 \cdots i_\ell} \mid \;\hat{}\; \equiv \langle\cdots\rangle \equiv \text{STF}, \; i_j = 1,2,3 \right\}$

$$\hat{Q}_{i_1 i_2 \cdots i_\ell} \equiv Q_{\langle i_1 i_2 \cdots i_\ell \rangle}$$

STF = Symmetric Trace-Free

$$\hat{Q}_{\cdots i_p \cdots i_q \cdots} = \hat{Q}_{\cdots i_q \cdots i_p \cdots}$$

$$\delta^{i_p i_q} \hat{Q}_{\cdots i_p \cdots i_q \cdots} = 0$$

The set of STF_ℓ is $(2\ell+1)$-dim and is $\leftrightarrow \{Q_{\ell m}\}$

It carries a rep of $SO(3)$: $\hat{Q}'_{i_1 i_2 \cdots i_\ell} = R_{i_1 j_1} R_{i_2 j_2} \cdots \hat{Q}_{j_1 j_2 \cdots j_\ell}$

- Multi-index notation

$$i_1 i_2 \cdots i_\ell \equiv L$$
$$i_1 i_2 \cdots i_p \equiv P$$

Einstein's summation convention

$$A_{i_1 i_2 \cdots i_\ell} B^{i_1 i_2 \cdots i_\ell} \equiv A_L B^L$$

$$A_{i i_1 \cdots i_\ell} B^{i_1 \cdots i_\ell} \equiv A_{iL} B^L$$

- STF projection:

$$T_{ij} \rightarrow T_{(ij)} \equiv \tfrac{1}{2}(T_{ij} + T_{ji}) \rightarrow T_{\langle ij \rangle} \equiv \hat{T}_{ij} \equiv T_{(ij)} - \tfrac{1}{3}\delta_{ij} T_{(ss)}$$

$$T_{ijk} \rightarrow T_{(ijk)} \rightarrow \hat{T}_{ijk} \equiv T_{(ijk)} - \tfrac{1}{5}\left(\delta_{ij} T_{(kss)} + \delta_{jk} T_{(iss)} + \delta_{ki} T_{(jss)}\right)$$

General: $T_{i_1 \cdots i_\ell} \rightarrow S_{i_1 \cdots i_\ell} \equiv T_{(i_1 \cdots i_\ell)} \rightarrow \hat{T}_{i_1 \cdots i_\ell} \equiv T_{\langle i_1 \cdots i_\ell \rangle}$

$$\equiv \sum_{k=0}^{[\ell/2]} a_k^\ell \, \delta_{(i_1 i_2} \cdots \delta_{i_{2k-1} i_{2k}} S_{i_{2k+1} \cdots i_\ell) s_1 s_1 \cdots s_k s_k}$$

coefficients: $a^l_k = (-)^k \frac{l!}{(l-2k)!} \frac{(2l-2k-1)!!}{(2l-1)!!\,(2k)!!}$

where e.g. $5!! \equiv 5.3.1$
$6!! \equiv 6.4.2$

1.3 STF form of multipole expansion

two ways of deriving it from $\Delta \varphi = -4\pi \rho$

1. use explicit Green function $\Delta \frac{1}{|\vec{x}|} = -4\pi \delta^{(3)}(\vec{x})$

$$\varphi(\vec{x}) = \int d^3y\; G(\vec{x}-\vec{y})\,\rho(\vec{y}) = \int d^3y\; \frac{1}{|\vec{x}-\vec{y}|}\,\rho(\vec{y})$$

Taylor expand $\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{|\vec{x}|} - y^i \partial_i \frac{1}{|\vec{x}|} + \frac{1}{2} y^{i_1} y^{i_2} \partial_{i_1 i_2} \frac{1}{|\vec{x}|} + \dots$

and use the fact that $\Delta \frac{1}{|x|} = 0$ for $\vec{x} \neq \vec{0}$

2. replace the continuous source by a distributional skeleton

$$\longrightarrow \sum_{l \geq 0} * = \delta(\vec{x}) + \partial\delta + \dots + \partial^l \delta + \dots$$

$$\rho(\vec{x}) \overset{\text{distrib}}{=} \sum_{l=0}^{\infty} \frac{(-)^l}{l!}\, \rho_L\, \partial_L\, \delta(\vec{x}) \qquad \boxed{\partial_L \equiv \partial_{i_1 i_2 \dots i_l}}$$

Writing $\int d^3x\, x^L\, LHS(\vec{x}) = \int d^3x\, x^L\, RHS(\vec{x})$ $\qquad \boxed{x^L \equiv x^{i_1} x^{i_2} \dots x^{i_l}}$

$$\rho_L = \int d^3x\;\; x^L \rho(\vec{x}) \equiv \int d^3y\;\; y^L \rho(\vec{y})$$

Then STF decompose the source tensors ρ_L

e.g. $\rho_{ij} \equiv \rho_{<ij>} + \frac{1}{3}\delta_{ij}\,\rho_{ss}$

$\int d^3x\, \rho(\vec{x})\, x^i x^j$

$$\frac{(-)^2}{2!}\rho_{ij}\,\partial_{ij}\delta(\vec{x}) = \frac{1}{2}\rho_{<ij>}\hat{\partial}_{ij}\delta(\vec{x}) + \frac{1}{2}\cdot\frac{1}{3}\rho_{ss}\,\Delta\delta(\vec{x})$$

but $\Delta\varphi \sim \rho_{ss}\,\Delta\delta(\vec{x})$

$\rightarrow \varphi \sim \rho_{ss}\,\delta(\vec{x})$
$= 0$ outside source

$$\boxed{\hat{Q}_L \equiv \rho_{<L>} \equiv \int d^3x\, x^{<i_1}\dots x^{i_\ell>}\,\rho(\vec{x})}$$

$$\rho(\vec{x}) \overset{\text{distr}}{=} \sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\hat{Q}_L\,\hat{\partial}_L\,\delta(\vec{x}) + \mathcal{O}(\Delta^k\delta(\vec{x}))$$

$$\boxed{\varphi^{\text{outside}}(\vec{x}) = \sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\hat{Q}_L\,\hat{\partial}_L\frac{1}{r}}$$

$Q_{i_1 i_2\dots i_\ell}\,\partial_{i_1\dots i_\ell}\frac{1}{|\vec{x}|}$

$\boxed{r \equiv |\vec{x}|}$

1.4 STF multipole expansion for relativistic scalar field

see [BD89]

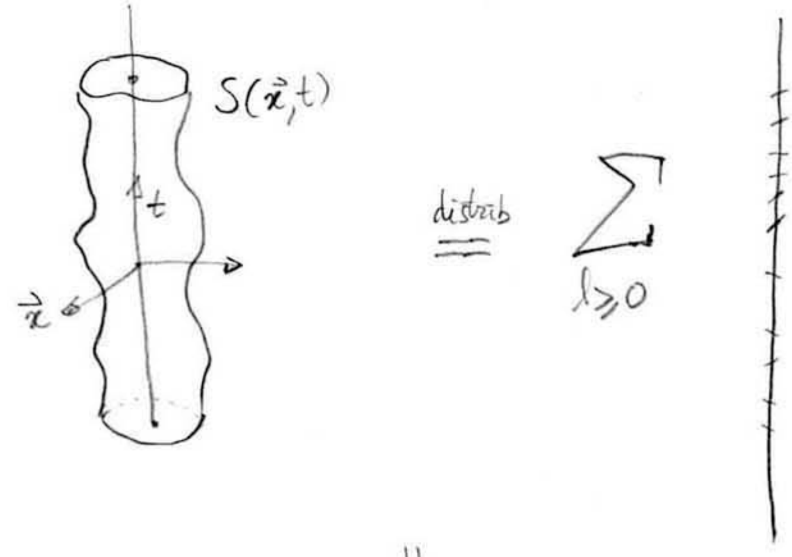
signature mostly plus

$$\boxed{\Box\,\varphi(\vec{x},t) = -4\pi\,S(\vec{x},t)}$$

time-dependent source

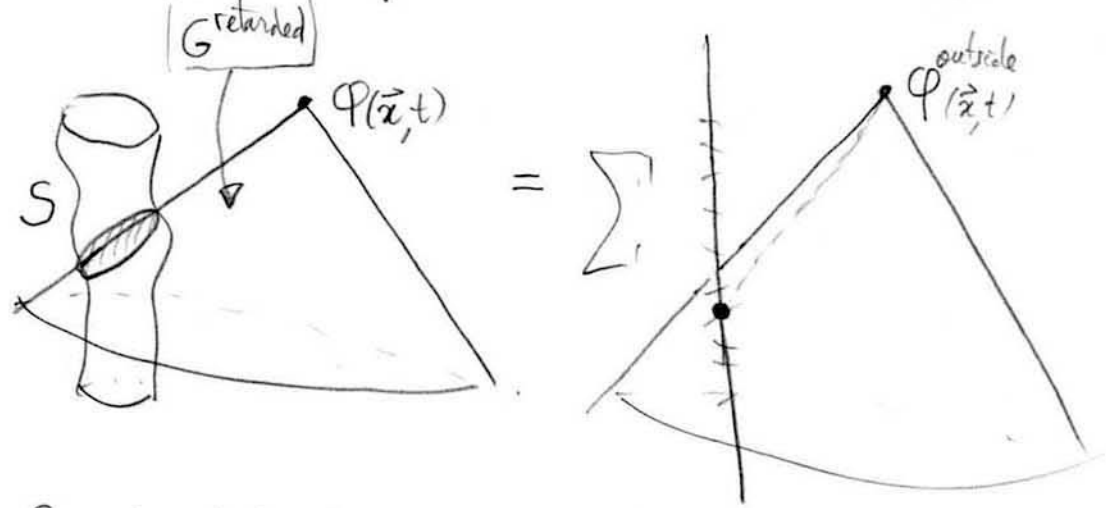
$\Box \equiv \Delta - \frac{1}{c^2}\partial_t^2$

Basic idea: skeletonize continuous source $S(\vec{x},t)$



$$= S(t)\,\delta(\vec{x}) - S^i(t)\,\partial_i\delta(\vec{x}) + \cdots S^L(t)\,\partial_L\delta$$

$\Downarrow$



Second method (easier in dimensions $D = d+1 \neq 3+1$)

$$S(\vec{x},t) \overset{\text{distrib}}{=} S^{\text{skel}}(\vec{x},t) = \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!}\, S_L(t)\ \partial_L\,\delta(\vec{x})$$

$$S_L(t) = \int d^3x\ x^L\, S(\vec{x},t)$$

$$S^{\text{skel}} = S(t)\,\delta(\vec{x}) - S_i(t)\,\partial_i\delta(\vec{x}) + \frac{1}{2}\,\underbrace{S_{ij}(t)\,\partial_{ij}\,\delta(\vec{x})} - \cdots$$

$$\frac{1}{2} S_{ij}(t)\,\partial_{ij}\delta(\vec{x}) = \frac{1}{2}\Big[\hat{S}_{ij}(t) + \frac{1}{3}\delta_{ij} S_{ss}(t)\Big]\partial_{ij}\delta$$

$$= \frac{1}{2}\hat{S}_{ij}\,\hat{\partial}_{ij}\delta(\vec{x}) + \underbrace{\frac{1}{6} S_{ss}(t)\,\Delta\delta(\vec{x})}_{\Delta\left\{\frac{1}{6}S_{ss}(t)\,\delta(\vec{x})\right\}}$$

$$\equiv \Big(\Box + \frac{1}{c^2}\partial_t^2\Big)\Big\{\frac{1}{6} S_{ss}(t)\,\delta(\vec{x})\Big\}$$

$$S^{skel}(\vec{x},t) = \Big(S(t) + \frac{1}{6}\frac{1}{c^2}\partial_t^2 S_{ss}(t) + \dots\Big)\delta(\vec{x}) - \big(S_i(t) + \dots\big)\partial_i\delta(\vec{x})$$

$$+ \frac{1}{2}\Big(S_{\langle ij\rangle}(t) + (\dots)\frac{1}{c^2}\partial_t^2 S_{\langle ij\rangle ss} + \dots\Big)\,\hat{\partial}_{ij}\delta(\vec{x}) + \dots$$

$$+ \mathcal{O}\Big(\Box\,\big(F(t)\,\partial^k\delta(\vec{x})\big)\Big)$$

corrections to usual

$$\hat{S}_{ij} \equiv S_{\langle ij\rangle}$$

solution $\varphi^{\text{outside}} \sim F(t)\,\partial^k\delta(\vec{x}) = 0$

of form $$\int d^3x\, \frac{1}{c^2}\partial_t^2\big(S(\vec{x},t)\, x^{\langle i}x^{j\rangle} x^s x^s\big)$$

$$= \int d^3x\, x^{\langle ij\rangle}\, \frac{\vec{x}^2}{c^2}\,\partial_t^2 S(\vec{x},t)$$

Finally, modulo $\mathcal{O}\big(\Box(F(t)\,\partial^k\delta(x))\big)$

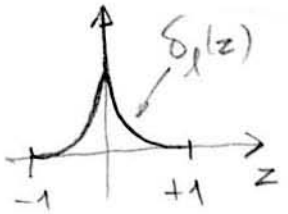
$$S^{skel}(\vec{x},t) = \sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\,\hat{\bar{S}}_L(t)\,\partial_L\delta(\vec{x}) = \sum\frac{(-)^\ell}{\ell!}\partial_L\Big\{\hat{\bar{S}}_L(t)\,\delta(\vec{x})\Big\}$$

$$\hat{\bar{S}}_L(t) = \int d^3x\,\hat{x}^L\,\bar{S}_\ell(\vec{x},t)$$

$$\bar{S}_\ell(\vec{x},t) = S(\vec{x},t) + \frac{1}{2(2\ell+3)}\frac{\vec{x}^2}{c^2}\partial_t^2 S(\vec{x},t) + \frac{(\vec{x}^2)^2/c^4}{8(2\ell+3)(2\ell+5)}\partial_t^4 S(\vec{x},t) + \dots$$

Exact, resummed form

$$\overline{S}_\ell(\vec{x},t) = \int_{-1}^{+1} dz\ \delta_\ell(z)\ S\left(\vec{x}, t + \frac{r z}{c}\right) \qquad r = |\vec{x}|$$



$$\delta_\ell(z) \equiv \frac{(2\ell+1)!!}{2^{\ell+1}\,\ell!}\,(1-z^2)^\ell \quad , \int_{-1}^{+1} dz\ \delta_\ell(z) = 1$$

$$\lim_{\ell\to\infty} \delta_\ell(z) = \delta(z)$$

Finally: STF multipole expansion of the solution of

$$\Box\, \varphi(\vec{x},t) = -4\pi\, S(\vec{x},t) = -4\pi \sum_\ell \frac{(-)^\ell}{\ell!}\, \partial_L \left\{ \hat{\overline{S}}_L(t)\, \delta(\vec{x}) \right\}$$

enough to solve $\quad \Box\, \varphi_L = -4\pi\, \hat{\overline{S}}_L(t)\, \delta(\vec{x})$ and to $\partial_{x^{i_1}} \cdots \partial_{x^{i_\ell}}$

$$\boxed{\varphi(\vec{x},t) = \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!}\, \partial_L \left(\frac{\hat{S}_L\left(t - \frac{|\vec{x}|}{c}\right)}{|\vec{x}|} \right)}$$

relativistic STF multipole moments

$$\hat{S}_L(t) = \int d^3x\ \hat{x}^L \left\{ S(\vec{x},t) + \frac{1}{2(2\ell+3)} \frac{\vec{x}^{\,2}}{c^2} \partial_t^2 S(\vec{x},t) + \mathcal{O}(r^4/c^4) \right\}$$

each 'block' $\varphi^{(\ell)}$ in the decomposition $\sum_\ell$

- is a solution of $\Box\, \varphi^{(\ell)} = 0$, because $\varphi^{(\ell)} \sim \partial_L \left(\frac{F_L(t - r/c)}{r} \right)$
- belongs to the $(2\ell+1)$-dim irrep of $SO(3)$: $\hat{S}_L \xrightarrow[\mathcal{R}]{\text{rotates by}}$

$\boxed{1.5}$ STF multipole expansion for linearized gravity

see [DI91]

Linearized gravity: $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad \boxed{h \equiv \eta^{\alpha\beta} h_{\alpha\beta}}$$

(linearized) Harmonic gauge: $\partial_\nu \bar{h}^{\mu\nu} = 0$

Harmonically relaxed linearized Einstein's eqs :

$$\boxed{\Box \bar{h}^{\mu\nu}(\vec{x},t) = -\frac{16\pi G}{c^4} T^{\mu\nu}(\vec{x},t)}$$

Source: $T^{\mu\nu}$

a set of 'scalar' wave eqs

$\angle$ beware of use of <u>gothic</u> metric deviation: $\sqrt{g}\, g^{\mu\nu} = \eta^{\mu\nu} + \mathfrak{h}^{\mu\nu}$

$$\Rightarrow \quad \bar{h}^{\mu\nu} = -\mathfrak{h}^{\mu\nu} + O(\mathfrak{h}^2)$$

- <u>Step 1</u> : Use scalar STF multipole decomposition of each separate $\bar{h}_{\mu\nu}$

field point

$$\bar{h}^{00}(\vec{X},T) = \frac{4G}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l!}\,\partial_L\left(\frac{F_L(U)}{R}\right) \qquad \boxed{\begin{array}{c} U \equiv T - R/c \\ \text{retarded 'field' time}\end{array}}$$

$$\bar{h}^{0i}(\vec{X},T) = \frac{4G}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l!}\,\partial_L\left(\frac{G_{iL}(U)}{R}\right)$$

$$\bar{h}^{ij}(\vec{X},T) = \frac{4G}{c^4}\sum_{l\geq 0}\frac{(-)^l}{l!}\,\partial_L\left(\frac{H_{ijL}(U)}{R}\right)$$

$$F_L(U) \equiv \int d^3x\, \hat{x}^L \int_{-1}^{+1} dz\, \delta_\ell(z)\, T^{00}\left(\vec{x}, U + \frac{rz}{c}\right)$$

$$G_{iL}(U) \equiv \int d^3x\, \hat{x}^L \int dz\, \delta_\ell(z)\, T^{0i}\left(\vec{x}, U + \frac{rz}{c}\right)$$

$$H_{ijL}(U) \equiv \int d^3x\, \hat{x}^L \int dz\, \delta_\ell(z)\, T^{ij}\left(\vec{x}, U + \frac{rz}{c}\right)$$

- <u>Step 2</u> decompose $(\bar{h}^{\mu\nu})_\ell$ in irreducible pieces under SO(3)

e.g. under active rotation of source  : $T^{00\prime}(\vec{x}') = T^{00}(\vec{x})$ is scalar

but $T^{0i\prime}(\vec{x}') = R^i{}_j\, T^{0j}(\vec{x})$

$x'^i = R^i{}_j\, x^j$

$$G_{iL} \sim \underset{\text{vector 3-dim}}{x^i}\ \underset{(2\ell+1)\text{-dim irreducible tensor rep.}}{x^{\langle L\rangle}}$$

does not belong to an irrep, but rather to the tensor product of two irreps :

$$G_{iL} \in \underset{\text{vector: } \ell=1}{D_1} \otimes \underset{(2\ell+1)\text{-dim}}{D_\ell}$$

From QM or Group Theory :

$$D_s \otimes D_\ell = D_{|\ell-s|} \oplus D_{|\ell-s|+1} \oplus \cdots \oplus D_{\ell+s}$$

If we were using $Y_{\ell m}$-type multipole expansion, one would need to introduce tensorial spherical harmonics and related Clebsch-Gordan coefficients.

STF tensors allow for a lighter treatment

General rule: Any tensor T_P (not STF $\Rightarrow$ reducible) can be decomposed as $\sum \gamma_P^L \hat{R}_L$

SO(3)-invariant tensor made of δ_{ij} and ε_{ijk}

irreducible STF $\hat{R}_L$

where $\hat{R}_L \sim \delta\, \varepsilon\, T$

e.g. for $D_1 \otimes D_\ell$

$L-1 \equiv i_1 i_2 \cdots i_{\ell-1}$

$$\boxed{G_{i\langle L\rangle} = G^{(+1)}_{iL} + \varepsilon_{ai\langle i_\ell} G^{(0)}_{L-1\rangle a} + \delta_{i\langle i_\ell} G^{(-1)}_{L-1\rangle}}$$

$$G^{(+1)}_{L+1} \equiv G_{\langle i_{\ell+1} L\rangle}$$

STF projection of $G_{i_{\ell+1} L}$

$$G^{(0)}_{L} \equiv \frac{\ell}{\ell+1} G_{ab\langle L-1} \varepsilon_{i_\ell\rangle ab}$$

$$G^{(-1)}_{L-1} \equiv \frac{2\ell-1}{2\ell+1} G_{aaL-1}$$

$D_p \equiv (2p+1)$-dim irrep $\sim \{\hat{R}_P\}$

Under rotations $G^{(+1)}$-piece rotates like $G^{(+1)}_{\langle L+1\rangle}$, i.e. $D_{\ell+1}$

$G^{(0)}$-piece " " $G^{(0)}_{L}$, i.e. D_ℓ

$G^{(-1)}$-piece " " $G^{(-1)}_{L-1}$, i.e. $D_{\ell-1}$

Similarly for $H_{ijL} \sim (D_2 \oplus D_0) \otimes D_\ell = D_2 \otimes D_\ell + D_\ell$

$$\boxed{\begin{aligned} H_{ijL} = H^{(+2)}_{ijL} &+ \underset{L}{\mathrm{STF}}\,\underset{ij}{\mathrm{STF}} \Big[\varepsilon_{aii_\ell} H^{(+1)}_{ajL-1} + \delta_{ii_\ell} H^{(0)}_{jL-1} \\ &+ \delta_{ii_\ell} \varepsilon_{aji_{\ell-1}} H^{(-1)}_{aL-2} + \delta_{ii_\ell} \delta_{ji_{\ell-1}} H^{(-2)}_{L-2} \Big] \\ &+ \delta_{ij} K_L \end{aligned}}$$

$$\begin{cases} H^{(+2)}_{L+2} \equiv H_{\langle L+2 \rangle} \\ H^{(+1)}_{L+1} \equiv \frac{2l}{l+2} \underset{L+1}{\mathrm{STF}} \{ H_{\langle c i_l \rangle d L-1} \varepsilon_{i_{l+1} cd} \} \\ H^{(0)}_{L} \equiv \frac{6l(2l-1)}{(l+1)(2l+3)} \underset{L}{\mathrm{STF}} \{ H_{\langle a i_l \rangle a L-1} \} \\ H^{(-1)}_{L-1} \equiv \frac{2(l-1)(2l-1)}{(l+1)(2l+1)} \underset{L-1}{\mathrm{STF}} \{ H_{\langle ca \rangle bc L-2} \varepsilon_{i_{l-1} ab} \} \\ H^{(-2)}_{L-2} \equiv \frac{2l-3}{2l+1} H_{\langle ac \rangle ac L-2} \\ K_L \equiv \frac{1}{3} H_{aaL} \end{cases}$$

- Step 3 Substitute the irrep decompositions of the scalar-type multipole moments F_L, G_{iL}, H_{ijL} of $\bar{h}_{\mu\nu}$.

Use:
$$\delta_{ij}\,\partial_{ij}\left(\frac{f(T-\frac{R}{c})}{R}\right) = \Delta\left(\frac{f(T-\frac{R}{c})}{R}\right) = \frac{1}{c^2}\partial_T^2 \frac{f(T-R/c)}{R} + \underbrace{\Box\left(\frac{f}{R}\right)}_{=0}$$
$$= \frac{1}{c^2}\frac{\ddot{f}(T-R/c)}{R}$$

$$G^{-1}\,\bar{h}^{00}(\vec{X},T) = \sum_{l\geq 0} \partial_L\left(R^{-1}\mathcal{A}_L(U)\right)$$

$$G^{-1}\,\bar{h}^{0i}(\vec{X},T) = \sum_{l\geq 0} \partial_{iL}\left(R^{-1}\mathcal{B}_L(U)\right) + \sum_{l\geq 1}\left\{ \partial_{L-1}\left(R^{-1}\mathcal{C}_{iL-1}(U)\right) + \varepsilon_{iab}\partial_{aL-1}\left(\frac{\mathcal{D}_{bL-1}(U)}{R}\right)\right\}$$

$$G^{-1}\,\bar{h}^{ij}(\vec{X},T) = \sum_{l\geq 0}\left\{ \partial_{ijL}\left(R^{-1}\mathcal{E}_L(U)\right) + \delta_{ij}\,\partial_L\left(R^{-1}\mathcal{F}_L(U)\right)\right\}$$
$$+\sum_{l\geq 1}\left\{ \partial_{L-1(i}\left(R^{-1}\mathcal{G}_{j)L-1}(U)\right) + \varepsilon_{ab(i}\,\partial_{j)aL-1}\left(R^{-1}\mathcal{H}_{bL-1}(U)\right)\right\}$$
$$+\sum_{l\geq 2}\left\{ \partial_{L-2}\left(R^{-1}\mathcal{I}_{ijL-2}(U)\right) + \partial_{aL-2}\left(R^{-1}\varepsilon_{ab(i}\,\mathcal{J}_{j)bL-2}(U)\right)\right\}$$

Here each one of the 10 sequences of tensors $\mathcal{A}_L, \mathcal{B}_L, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}$ is STF and is a combination of the irreducible blocks of F_L, G_{iL}, H_{ijL}, and of their time derivatives.

e.g. $$\mathcal{C}_L = \frac{4}{c^4}\frac{(-)^l}{l!}\left[-l\, G_L^{(+1)} + \frac{l}{(l+1)(2l+1)}\frac{1}{c^2}\,\ddot{G}_L^{(-1)}\right]$$

came from $\partial_{L-1}\left(\frac{G_{iL-1}}{R}\right)$

came from $\partial_{L+1}\left(\delta_{i<i_{l+1}}\frac{G^{(-1)}_{L>}}{R}\right)$ in $\partial_{L+1}\left(\frac{G_{iL+1}}{R}\right)$

- Step 4 Introduce 6 new sequences of STF tensors

$$\begin{cases}
M_L(U) \equiv -(\mathcal{A}_L + 2\dot{\mathcal{B}}_L + \ddot{\mathcal{E}}_L + \mathcal{F}_L) & l \geq 0 \\
S_L(U) \equiv +(\mathcal{D}_L + \frac{1}{2}\dot{\mathcal{H}}_L) & l \geq 1 \\
W_L(U) \equiv -(\mathcal{B}_L + \frac{1}{2}\dot{\mathcal{E}}_L) & l \geq 0 \\
X_L(U) \equiv -\frac{1}{2}\mathcal{E}_L & l \geq 0 \\
Y_L(U) \equiv +(\dot{\mathcal{B}}_L + \ddot{\mathcal{E}}_L + \mathcal{F}_L) & l \geq 0 \\
Z_L(U) \equiv -\frac{1}{2}\mathcal{H}_L & l \geq 1
\end{cases}$$

- Step 5 Prove (from $\partial_\nu T^{\mu\nu} = 0$ or $\partial_\nu \bar{h}^{\mu\nu} = 0$) 4 sequences of identities

$$\boxed{\begin{array}{ll} \mathcal{C}_L = \dot{M}_L + \dot{Y}_L & ; \ \mathcal{I}_L = -\ddot{M}_L \\ \mathcal{G}_L = -2Y_L & ; \ \mathcal{J}_L = -2\dot{S}_L \end{array}} + \boxed{\begin{array}{l} \dot{M} = 0 ; \ \ddot{M}_i = 0 \\ \dot{Y} = 0 ; \ \dot{S}_i = 0 \end{array}}$$

• <u>Step 6</u> Express $\bar{h}^{\mu\nu}$ in terms of M, S, W, X, Y, Z

+ Redefine

$$M_L^{new} = -\frac{1}{4} c^2 \, \ell! \, (-)^\ell M_L^{old}$$

$$S_L^{new} = -\frac{1}{4} c^3 \frac{(\ell+1)!}{\ell} (-)^\ell S_L^{old}$$

$$\bar{h}^{\mu\nu}[M,S,W,X,Y,Z] = \bar{h}^{\mu\nu}_{can}[M,S] + \underbrace{\partial^\mu w^\nu + \partial^\nu w^\mu - \eta^{\mu\nu}\partial_\sigma w^\sigma}_{\text{effect of coordinate transformation on } \bar{h}^{\mu\nu}}$$

$$G^{-1} w^0[W] = \sum_{\ell\geq 0} \partial_L \left(R^{-1} W_L(U)\right)$$

$$G^{-1} w^i[X,Y,Z] = \sum_{\ell\geq 0} \partial_{iL}\left(R^{-1} X_L(U)\right) + \sum_{\ell\geq 1}\left\{\partial_{L-1}\left(R^{-1} Y_{iL-1}(U)\right) + \varepsilon_{iab}\,\partial_{aL-1}\left(\frac{Z_{bL-1}}{R}\right)\right\}$$

"Gauge-fixed" "canonical" metric outside the source

$$\boxed{\begin{aligned}
G^{-1}\bar{h}^{00}_{can}(\vec{X},T) &= +\frac{4}{c^2}\sum_{\ell\geq 0}\frac{(-)^\ell}{\ell!}\partial_L\left(R^{-1} M_L(U)\right) \\
G^{-1}\bar{h}^{0i}_{can}(\vec{X},T) &= -\frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell}{\ell!}\partial_{L-1}\left(R^{-1}\dot{M}_{iL-1}\right) - \frac{4}{c^3}\sum_{\ell\geq 1}\frac{(-)^\ell \ell}{(\ell+1)!}\varepsilon_{iab}\,\partial_{aL-1}\left(R^{-1} S_{bL-1}\right) \\
G^{-1}\bar{h}^{ij}_{can}(\vec{X},T) &= +\frac{4}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell}{\ell!}\partial_{L-2}\left(R^{-1}\ddot{M}_{ijL-2}\right) + \frac{8}{c^4}\sum_{\ell\geq 2}\frac{(-)^\ell \ell}{(\ell+1)!}\partial_{aL-2}\left(R^{-1}\varepsilon_{ab(i}\dot{S}_{j)bL-2}\right)
\end{aligned}}$$

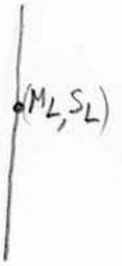
$U \equiv T - R/c$, everywhere in multipole moments

satisfies $\partial_0\bar{h}^{00} + \partial_i\bar{h}^{0i} = 0$; $\partial_0\bar{h}^{i0} + \partial_j\bar{h}^{ij} = 0$

$\boxed{1.6}$ Multipole moments of a source in linearized gravity

Source $T^{\mu\nu}$

$\longrightarrow$ reduced skeleton :



$$\begin{pmatrix} M \\ M_i(U) \\ M_{\langle ij\rangle}(U) \\ M_{\langle ijk\rangle}(U) \\ \vdots \\ M_L(U) \\ \vdots \end{pmatrix} + \begin{pmatrix} S_i \\ S_{\langle ij\rangle}(U) \\ S_{\langle ijk\rangle}(U) \\ \vdots \\ S_L(U) \\ \vdots \end{pmatrix}$$

'mass multipole moments of source' 'spin multipole moments of source'

with $\dot{M}=0, \ddot{M}_i=0 \; ; \; \dot{S}_i=0$

roughly : sequence $\{ \hat{F}_L(U) \,|\, l=0,1,2\ldots\} \longleftrightarrow$ scalar field $\varphi(\vec{x},T)=\sum_l \partial_L\left(\frac{\hat{F}_L(U)}{R}\right)$

$\Box\varphi^{\text{outside}}=0$

i.e. gauge-invariant part of $\bar{h}^{\mu\nu}$. $\longleftrightarrow$ 2 'scalar' degrees of freedom

$\Box\varphi_M=0 \; , \; \Box\varphi_S=0$

OK usual counting h_{ij}^{TT}

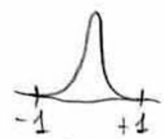
Explicit expressions for the multipole moments of a source

short-hand notation $\boxed{\tilde{T}^{\mu\nu} \equiv T^{\mu\nu}\left(\vec{x},\, U+\frac{rz}{c}\right)}$

$$\boxed{\begin{aligned} \sigma &\equiv \frac{T^{00}+T^{ss}}{c^2} \\ \sigma^i &\equiv \frac{T^{0i}}{c} \\ \sigma^{ij} &\equiv T^{ij} \end{aligned}}$$

$$\delta_\ell(z) \equiv c_\ell \left(1-z^2\right)^\ell \theta(1-z^2)$$

$$c_\ell = \frac{(2\ell+1)!!}{2^{\ell+1}\,\ell!}$$



$$\int \delta_\ell(z)\, dz = 1$$

$$M_L(U) = \int d^3x \int dz \left[\delta_\ell\, \hat{x}^L \tilde{\sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)}\, \delta_{\ell+1}\, \hat{x}^{aL}\, \dot{\tilde{\sigma}}^a + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)}\, \delta_{\ell+2}\, \hat{x}^{abL}\, \ddot{\tilde{\sigma}}^{ab} \right]$$

STF projection on L

$$S_L(U) = \underset{L}{\mathrm{STF}} \int d^3x \int dz\; \varepsilon^{i_\ell ab} \left[\delta_\ell\, \hat{x}^{aL-1}\, \tilde{\sigma}^b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)}\, \delta_{\ell+1}\, \hat{x}^{acL-1}\, \dot{\tilde{\sigma}}^{bc} \right]$$

NB: Very similar (but simpler) results for electromagnetism, see [DI91]

time-dependent electric moment: $Q_L(U) = M_L(U)$ | $\sigma \to J^0$, $\sigma^a \to J^a$; omit 4 in $\frac{1}{c^2}$ term, omit $\frac{1}{c^4}$ term

magnetic moment $\mathcal{M}_L(U) = S_L(U)$ | $\sigma^a \to J^a$; omit $\frac{1}{c^2}$ term

'Newtonian' limit: $\frac{1}{c^2} \to 0$

$$M_L(U) \simeq \int d^3x \int dz\, \delta_\ell(z)\, \hat{x}^L \frac{\tilde{T}^{00} + \tilde{T}^{ss}}{c^2}$$

fluid
$$\frac{\rho c^2 + 3p}{c^2} = \rho + \frac{3p}{c^2}$$
if $p \ll \rho c^2$

$$\simeq \int d^3x \int dz\, \delta_\ell(z)\, \hat{x}^L\, \rho(\vec{x}, U + \tfrac{rz}{c})$$

$$\simeq \int d^3x\, \hat{x}^L \rho(\vec{x}, U) + \mathcal{O}\left(\tfrac{1}{c^2}\right) = \int d^3x\, x^{\langle i_1} \ldots x^{i_\ell \rangle} \rho + \mathcal{O}\left(\tfrac{1}{c^2}\right)$$

usual Newtonian multipole moment of mass density

$$S_L(U) \simeq \underset{L}{\mathrm{STF}} \int d^3x \int dz\, \varepsilon^{i_\ell ab}\, \hat{x}^{aL-1}\, \tilde{\sigma}^b + \mathcal{O}(1/c^2)$$

$\tilde{\sigma}^b$: $\sigma^b + \mathcal{O}(1/c + 1/c^2 + \cdots)$

$$\simeq \underset{L}{\mathrm{STF}} \int d^3x\, \varepsilon^{i_\ell ab}\, \hat{x}^{aL-1} \sigma^b$$

$$= \underset{L}{\mathrm{STF}} \int d^3x\, \varepsilon^{i_\ell ab}\, x^{L-1} x^a \sigma^b$$

σ^b: $\frac{T^{0b}}{c}$ = momentum density $\sim \rho v^b$

$$= \int d^3x\, x^{\langle i_1} \ldots x^{i_{\ell-1}} j^{i_\ell \rangle} + \mathcal{O}(1/c^2)$$

j: angular momentum density

$$j^i = \varepsilon^{iab} x^a \sigma^b = \vec{x} \times \vec{\sigma}$$

Lowest moments

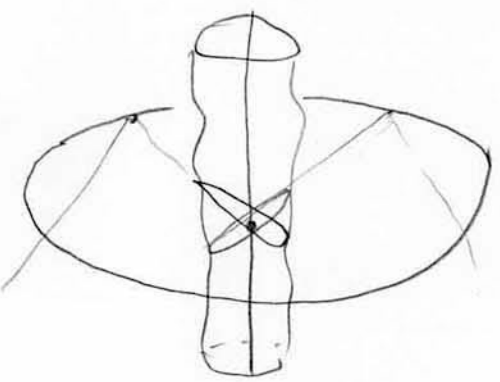
$$M^{\ell=0}(U) = \int d^3x \int dz \left[\delta_0 \tilde{\sigma} - \frac{4}{3} \delta_1 x^a \dot{\tilde{\sigma}}^a + \frac{1}{5} \delta_2 \hat{x}^{ab} \ddot{\tilde{\sigma}}^{ab} \right]$$

$$\equiv \int d^3x\, \frac{T^{00}}{c^2}(\vec{x}, U) \qquad \text{after using } \partial_\nu T^{\mu\nu} = 0$$

also $$M_i^{\ell=1}(U) \equiv \int d^3x\, x^i \frac{T^{00}}{c^2}(\vec{x}, U)$$

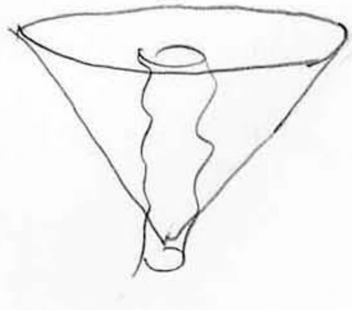
$$S_i^{\ell=1}(U) \equiv \int d^3x\, \varepsilon^{iab} x^a \frac{T^{0b}}{c}(\vec{x}, U)$$

For higher moments, necessity of $\tilde{T}^{\mu\nu} = T^{\mu\nu}(\vec{x}, U + \frac{rz}{c})$



$\boxed{1.7}$ Radiative multipole moments [T80]

At $R \to \infty$



TT gauge $\quad h_{ij}^{TT} = \bar{h}_{ij}^{TT} \quad , \delta^{ij} h_{ij}^{TT} = 0 = \partial_i h_{ij}^{TT}$

$(\theta, \varphi) \sim \vec{N} = \frac{\vec{X}}{R}$

asymptotically $\quad h_{ij}^{TT}(R, \theta, \phi, U) = \dfrac{k_{ij}(\vec{N}, U)}{R} + O\left(\frac{1}{R^2}\right)$

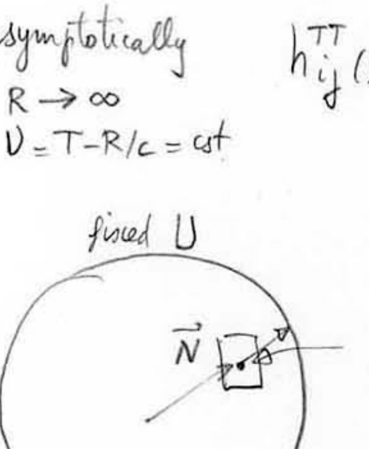
$R \to \infty$

$U = T - R/c = cst$

$k_{ss} = 0 \quad , \quad N_i \dot{k}_{ij} = 0$

$N_i k_{ij} = 0$ mod gauge

fixed U



$k_{ij}(\vec{N})$ = Symmetric Transverse Trace-free 2-tensor on Sphere at ∞

General decomposition in irreducible pieces under $SO(3)$:

must be made of δ_{ij}, ε_{ijk}, N_i and some $\hat{U}_L$

$$k_{ij}(\vec{N}) \sim \underbrace{N_i N_j}_{\text{not transverse}} N_L \hat{U}_L + \dots + \hat{U}_{ij\,L-2} N^{L-2}$$

must be projected in

plane $\perp \vec{N}$

$$+ \ \varepsilon_{iab} N^a \hat{V}_{bj\,L-2} N^{L-2} + \dots$$

for h_{ij}^{TT}

$$\boxed{h_{ij}^{TT}(R,\vec{N},U) = \frac{4G}{c^2 R} \sum_{l \geq 2} \frac{1}{c^l l!} \left[N^{L-2} \hat{U}_{ijL-2}(U) - \frac{2l}{(l+1)c} N^{aL-2} \varepsilon_{ab(i} \hat{V}_{j)bL-2}(U) \right]^{TT}}$$

algebraic TT projection: on sphere at ∞

$$(F_{ij})^{TT} \equiv \underbrace{\left\{ P_{ik}(\vec{N}) P_{jl}(\vec{N}) - \frac{1}{2} P_{ij}(\vec{N}) P_{kl}(\vec{N}) \right\}}_{P_{ijkl}(\vec{N})} F_{kl}(\vec{N})$$

$$P_{ik}(\vec{N}) \equiv \delta_{ik} - N_i N_k$$

Two types of STF radiative moments : $\hat{U}_L$ and $\hat{V}_L$, $l \geq 2$

examples:

U-type quadrupole: $h_{ij}^{TT}(\vec{N}) \propto P_{ijkl}(\vec{N})\, U_{kl}$ — polynomial in $\vec{N}$ up to $\vec{N}^4$

U-type octupole: $h_{ij}^{TT}(\vec{N}) \propto P_{ijkl}(\vec{N})\, U_{klm} N^m$ — extra dependence on $\vec{N}$

Energy flux: $\frac{d\dot{E}}{d\Omega}(\vec{N}) \propto \left(\dot{h}_{ij}^{TT}\right)^2 \propto P_{ijkl}(\vec{N}) \left(\dot{U}_{ij} + \dot{U}_{ijm} N^m + \dots\right)\left(\dot{U}_{kl} + \dot{U}_{klm} N^n + \dots\right)$ — rather intricate dependence on $\vec{N}$

Integrated E flux $\int d\Omega\, (d\dot{E}/d\Omega) \propto G \sum_{l \geq 2} \left(a_l c^{-(2l+1)} \dot{U}_L \dot{U}_L + b_l c^{-(2l+3)} \dot{V}_L \dot{V}_L \right)$

$\boxed{1.8}$ Link between Radiative Moments and Source Moments

$$R \to \infty \qquad h_{ij}^{TT} = \left[\bar{h}^{ij}_{can}\right]^{TT} = P_{ijkl}(\vec{N})\, \bar{h}^{ij}_{can}$$

in algebraic sense

M_L-piece:
$$\bar{h}^{ij\,(M)}_{can}(\vec{X},T) = \frac{4G}{c^4} \sum_{\ell \geq 2} \frac{(-)^\ell}{\ell!}\, \partial_{L-2}\left(\frac{\ddot{M}_{ijL-2}(U)}{R}\right)$$

leading
$$\partial_i\left(\frac{F(T-R/c)}{R}\right) = -\frac{\partial_i R}{c R}\,\dot{F}(U) = -\frac{N_i}{c}\,\frac{\dot{F}(U)}{R}$$

$$\bar{h}^{ij\,(M)}_{can}(\vec{X},T) \simeq \frac{4G}{c^4}\sum_{\ell \geq 2}\frac{(-)^\ell}{\ell!}\,\frac{(-)^{\ell-2}}{c^{\ell-2}}\,\frac{N_{i_1}\cdots N_{i_{\ell-2}}}{R}\, M^{(2+\ell-2)}_{ij\,i_1\cdots i_{\ell-2}}$$

$$= \frac{4G}{c^2 R}\sum_{\ell \geq 2}\frac{1}{\ell!}\, N_{L-2}\, M^{(\ell)}_{ijL-2}$$ to be TT projected

in linearized theory

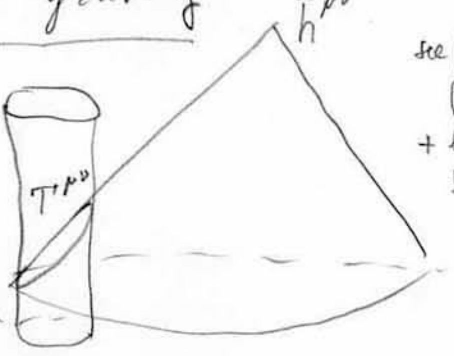
$$\boxed{\begin{aligned} U_L(U) &= M_L^{(\ell)}(U) \equiv \left(\frac{d}{dU}\right)^\ell M_L(U) \\ V_L(U) &= S_L^{(\ell)}(U) \equiv \left(\frac{d}{dU}\right)^\ell S_L(U) \end{aligned}}$$

Radiative Moments — Source Moments

Gives, in linearized theory, an exact evaluation (to all multipole and $1/c$ orders) of the radiative field in terms of the source: $T^{\mu\nu}$.
Usual 'quadrupole formula' is the lowest order truncation: only U_{ij} and $\frac{1}{c^2} \to 0$

1.9 Beyond linearized gravity

see [BD86] [BD89] + lectures by Blanchet



In linearized gravity the same set of multipole moments M_L, S_L allowed one to achieve three things at once:

(1) $$\begin{cases} M_L = \mathcal{F}_L^{(M)}[T^{\mu\nu}] \\ S_L = \mathcal{F}_L^{(S)}[T^{\mu\nu}] \end{cases}$$ explicit functionals of the source $T^{\mu\nu}$

(2) $$\bar{h}^{\mu\nu}_{\text{outside source}}(\vec{x}, t) = \mathcal{F}^{\mu\nu}[M_L(u), S_L(u), \overbrace{W_L, X_L, Y_L, Z_L}^{\text{gauge}}]$$ h: explicit functional of the multipole moments

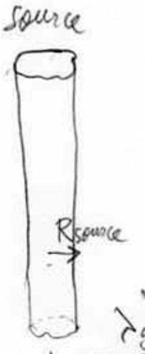
(3) at ∞ $$\bar{h}_{ij}^{TT\,\text{rad}} \overset{\text{parametrization}}{=} \mathcal{F}_{ij}[U_L, V_L]$$ and $$\begin{aligned} U_L &= \mathcal{F}^{(U)}[M_L] \\ V_L &= \mathcal{F}^{(V)}[S_L] \end{aligned}$$ explicit link between radiative and source multipoles

In full General Relativity, the non linearities render the problem of relating $h_{ij}^{TT\,\text{rad}}$ to the source much more intricate.

However, STF multipole expansions, and the explicit solution (1) of linearized gravity, still play a useful role.

Matched Multipolar Post-Minkowskian Approach

source



$H_{ij}^{TT\,rad}$

WAVE ZONE

λ_{gw} EXTERIOR ZONE $r \gg R_{source}$

NEAR ZONE

use an adapted radiative coordinate system $T\vec{X} \leftrightarrow U, R, \Theta, \Phi$ and define radiative multipole moments

U_L, V_L

$$H_{ij}^{TT\,rad} = \frac{4G}{c^2 R} \sum_{\ell \geq 2} (U_L^{\cdot\cdot} + V_L^{\cdot\cdot})$$

PN expansion

$r \ll \lambda_{gw}$

parametrize general solution of $R_{\mu\nu}(g) = 0$ by some seed multipole moments

I_L, J_L

parametrizing the first step of a multipolar expanded non-linearity expansion:

$$\sqrt{g}\, g^{\mu\nu} = \eta^{\mu\nu} + G h_1^{\mu\nu}[I_L, J_L, WXYZ] + G^2 h_2^{\mu\nu}[I, J \ldots] + G^3 h_3^{\mu\nu}[\;]$$

harmonic coords $\partial_\nu h^{\mu\nu} = 0$

$$\Box h_1^{\mu\nu} = 0$$

$$\Box h_2^{\mu\nu} = \Lambda_2^{\mu\nu}(h_1, h_1)$$

$\vdots$

seed multipoles

- parametrize $G h_1^{\mu\nu} \equiv -\bar{h}_{lin}^{\mu\nu}[M_L \to I_L, S_L \to J_L, WXYZ]$
- iterate in a unique manner by using analytic continuation in $B \in \mathbb{C}$:

$$h_2^{\mu\nu} = \underset{B=0}{FP}\, \Box_{ret}^{-1}\left[\left(\frac{r}{r_0}\right)^B \Lambda_2^{\mu\nu}(h_1[I, J])\right] + v_2^{\mu\nu}$$

etc ... $\Rightarrow$ this defines $h_{MPM}^{\mu\nu}[I_L, J_L, W \ldots]$

$\boxed{\text{MATCH}}$ between large-r behaviour of $h_{MPM}^{\mu\nu}[IJ]$ and $H_{ij}^{TT\,rad}[U_L, V_L]$

$\boxed{\text{MATCH}}$ in overlap domain

$R_{source} \ll r \ll \lambda_{gw}$

to relate $h_{MPM}^{\mu\nu}[I_L, J_L]$ to PN expansion of $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G c^{-4} T_{\mu\nu}$

- Useful formal link between seed moments of MPM exterior zone, and PN-expansion of near-zone metric (Blanchet)

$$\boxed{\begin{aligned} I_L(u) &= \mathrm{FP}\; M_L^{lin}\left[T^{\mu\nu} \to \overline{\tau}^{\mu\nu}\right] \\ J_L(u) &= \mathrm{FP}\; S_L^{lin}\left[T^{\mu\nu} \to \overline{\tau}^{\mu\nu}\right] \end{aligned}}$$

Finite Part
$B \to 0$

simply replace $T^{\mu\nu}$ of linearized gravity by

$$\boxed{\overline{\tau}^{\mu\nu} = \overline{\left(|g|\, T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}(h)\right)}}$$

Near-zone PN expansion (formally to all orders in $1/c$)

$$R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu} \Leftrightarrow \boxed{\begin{aligned} \Box h^{\mu\nu} &= \frac{16\pi G}{c^4} \tau^{\mu\nu}(h) \\ \partial_\nu h^{\mu\nu} &= 0 \end{aligned}}$$

Using it iteratively $\to$ $\boxed{\begin{cases} I_L(u) = \mathcal{F}^{(I)}[T^{\mu\nu}] \\ J_L(u) = \mathcal{F}^{(J)}[T^{\mu\nu}] \end{cases}}$ link seed moments $\updownarrow$ source

- Study of non-linear 'tail effects' in far wave zone:

$T - R/c$ of radiative coords

$$\boxed{\begin{aligned} U_L(U) &= I_L^{(l)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau\, I_L^{(l+2)}(U-\tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \kappa_l\right] + \cdots \\ V_L(U) &= J_L^{(l)}(U) + \frac{2GM}{c^3} \int_0^{+\infty} d\tau\, J_L^{(l+2)}(U-\tau) \left[\ln\left(\frac{c\tau}{2r_0}\right) + \pi_l\right] + \cdots \end{aligned}}$$

link observable radiative moments $\leftrightarrow$ seed moments

AGR 1.23

$\boxed{1.10}$ Generation of gravit. radiation at 1PN level

[BD89]

Simplification:

$$g_{00} = -e^{-\frac{2V}{c^2}} + O\left(\tfrac{1}{c^6}\right) = -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + O\left(\tfrac{1}{c^6}\right)$$

$$\Box V = -4\pi G\left(\frac{T^{00} + T^{ss}}{c^2}\right)$$

$$\tau^{00} = g\, T^{00} - \frac{7}{8\pi G}\,\partial_i V \partial_i V + \dots$$

$$\tau^{0i} = g\, T^{0i} + \dots = T^{0i} + \dots$$

$$\tau^{ij} = g\, T^{ij} + \frac{1}{4\pi G}\left[\partial_i V \partial_j V - \frac{1}{2}\delta_{ij}\,\partial_k V \partial_k V\right] + \dots$$

$$g = 1 + \frac{4}{c^2}V + \dots$$

$\Rightarrow$ crucial $$\tau^{00} + \tau^{ss} = T^{00} + T^{ss} - \frac{1}{2\pi G}\Delta\left(V^2\right) + O\left(\tfrac{1}{c^2}\right)$$

does not contribute at 1PN in I_L after integration by parts

Simple result

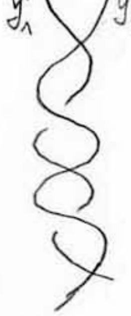
$$\boxed{I_L = M_L^{\text{lin}}\left[T^{\mu\nu}\right] + O\left(\tfrac{1}{c^4}\right)}$$

2PN: $\left(\frac{v}{c}\right)^4$ smaller

and $$\boxed{U_L = I_L^{(l)} + O\left(\tfrac{1}{c^3}\right)}$$ tail

Hence, surprisingly, for the 'mass type' moments linearized gravity is even 1PN accurate (when expressed in terms of $T^{\mu\nu}$).
This is not true for the 'spin type' moments (Damour Iyer AIHP 1991)

$\boxed{1.11}$ Example: 1PN radiation from binary system

Source $T^{\mu\nu}$  $=$ 

$$T^{\mu\nu}(\vec{x},t) = \sum_{A=1,2} m_A \frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\frac{1}{\sqrt{g}}\frac{dt}{d\tau}\delta(\vec{x}-\vec{y}_A(t))$$

$$\equiv \sum_A \mu_A(t) \frac{dy_A^\mu}{dt}\frac{dy_A^\nu}{dt}\delta(\vec{x}-\vec{y}_A)$$

$$\mu_A(t) = m_A\left(\frac{1}{\sqrt{g}}\frac{dt}{d\tau}\right)_{\vec{x}=\vec{y}_A(t)}$$

$$\begin{cases} g_{00}^{1PN} = -e^{-2V/c^2} + \mathcal{O}(1/c^6) \\ g_{0i}^{1PN} = -\frac{4}{c^3} V_i \\ g_{ij}^{1PN} = e^{+\frac{2V}{c^2}}\delta_{ij} + \mathcal{O}(1/c^4) \end{cases}$$

$$\begin{cases} \Box V = -4\pi G\sigma & ; \quad \sigma \equiv \frac{1}{c^2}(T^{00}+T^{ss}) \\ \Box V_i = -4\pi G\sigma_i & ; \quad \sigma_i \equiv \frac{1}{c}T^{0i} \end{cases}$$

eg $V(\vec{x},t) = G\int \frac{d^3y}{|\vec{x}-\vec{y}|}\,\sigma(\vec{y}, t - |\vec{x}-\vec{y}|/c)$

$$\boxed{v_A^i \equiv \frac{dy_A^i}{dt}}$$

successively

$$\sigma(\vec{x},t) = \sum_A \mu_A(t)\left(1+\frac{\vec{v}_A^{\,2}}{c^2}\right)\delta(\vec{x}-\vec{y}_A(t))$$

$$\sigma_i(\vec{x},t) = \sum_A \mu_A(t)\, v_A^i\, \delta(\vec{x}-\vec{y}_A(t))$$

keep only $\frac{Gm_B}{|\vec{y}_A-\vec{y}_B|}$

$$\mu_A = m_A\left[1 + \frac{1}{c^2}\left(\frac{\vec{v}_A^{\,2}}{c^2} - [V]_A\right)\right]$$

Then

$$I_L^{1PN} = \int d^3x \int dz \left[\delta_l \, \hat{x}^L \tilde{\sigma} - \frac{4(2l+1)}{c^2(l+1)(2l+3)} \delta_{l+1} \, \hat{x}^{aL} \dot{\tilde{\sigma}}^a + \mathcal{O}\left(\frac{1}{c^4}\right) \right]$$

expand $\sigma(\vec{x}, u + \frac{rz}{c})$ to $\mathcal{O}(1/c^2)$

$$= \int d^3x \, \hat{x}^L \sigma(\vec{x}, u) + \frac{1}{2(2l+3)} \frac{1}{c^2} \frac{d^2}{du^2} \int d^3x \, \hat{x}^L \vec{x}^2 \sigma(\vec{x}, u)$$

$$- \frac{4(2l+1)}{(l+1)(2l+3)} \frac{1}{c^2} \frac{d}{du} \int d^3x \, \hat{x}^{aL} \sigma^a(\vec{x}, u) + \mathcal{O}\left(\frac{1}{c^4}\right)$$

replace $\sigma = \sum_A \mu_A \left(1 + \frac{\vec{v}_A^2}{c^2}\right)$

$\mu_A = m_A\left(1 + \frac{1}{c^2}\left(\frac{v_A^2}{c^2} - V_A\right) + \right)$

already 1PN

For simplicity :

- consider only a circular orbit
- go to center of mass frame: $I_i^{1PN} = 0$
- focus on $l=2$: 1PN mass quadrupole

Denote: $m \equiv m_1 + m_2$; $X_1 \equiv \frac{m_1}{m}$; $X_2 \equiv \frac{m_2}{m}$; $\boxed{\nu \equiv X_1 X_2 \equiv \frac{m_1 m_2}{(m_1+m_2)^2}}$

Use $X_1 + X_2 = 1$; $X_1^2 + X_2^2 = 1 - 2\nu$; $X_1^3 + X_2^3 = 1 - 3\nu$; ...

$$\boxed{I_{ij}^{1PN} = \underset{ij}{\mathrm{STF}} \, \nu m \left[x^{ij} - \frac{1+39\nu}{42} \frac{Gm}{c^2 r} x^{ij} + \frac{11}{21}(1-3\nu) \frac{r^2}{c^2} v^{ij} \right]}$$

Then : energy loss at ∞

$$\frac{dE}{dT} = -\frac{G}{c^5}\left\{ \frac{1}{5}\left(\overset{\circ}{U}_{ij}\right)^2 + \frac{1}{c^2}\left[\frac{1}{189}\left(\overset{\circ}{U}_{ijk}\right)^2 + \frac{16}{45}\left(\overset{\circ}{V}_{ij}\right)^2\right] + \frac{1}{c^4}\cdots \right\}$$

replace $U_{ij}^{1PN} = \ddot{I}_{ij}^{1PN} + O^{\text{tail}}\left(\frac{1}{c^3}\right)$

replace $U_{ijk} \simeq \dddot{I}^{N}_{ijk}$ $\quad V_{ij} \simeq \ddot{J}^{N}_{ij}$

$$\left(\frac{dE}{dT}\right)^{1PN} \simeq -\frac{G}{c^5}\left\{ \frac{1}{5}\left(\dddot{I}^{1PN}_{ij}\right)^2 + \frac{1}{c^2}\left[\frac{1}{189}\left(\overset{N}{I}{}^{(4)}_{ijk}\right)^2 + \frac{16}{45}\left(\overset{N}{J}{}^{(3)}_{ij}\right)^2\right]\right\}$$

compute third time derivative using the 1PN accurate orbital motion

e.g. $\boxed{\omega^2_{\text{orbit}} \simeq \frac{Gm}{r^3}\left[1-(3-\nu)\gamma\right]}$

$\boxed{\gamma \equiv \frac{Gm}{c^2 r}}$

$$\boxed{\left(\frac{dE}{dT}\right)^{1PN} = -\frac{32}{5}\frac{c^5}{G}\nu^2\gamma^5\left\{1-\left(\frac{2927}{336}+\frac{5}{4}\nu\right)\gamma\right\}}$$

$\boxed{r \equiv |\vec{y}_1 - \vec{y}_2|}$

result of usual 'quadrupole formula' (in harmonic coords)

rather large coefficient

1PN correction

$$\simeq \left\{1 - 9\,\frac{Gm}{c^2 r}\right\}$$

becomes larger than leading term when

m_1 ---- m_2 $\qquad r \lesssim 9\,G(m_1+m_2)$

A serious problem for using such straightforward PN-expanded results in the case of coalescing binary black holes.