5

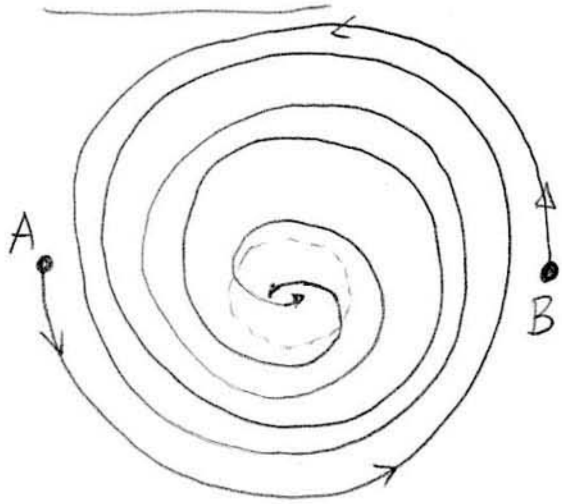
COALESCING BINARY BLACK HOLES:

Need for resummation methods, Padé, Effective One Body

Some relevant references

Test Mass Case: Davis, Ruffini, Press, Price PRL 27, 1466 (1971)
Davis, Ruffini, Tiomno PRD 5, 2932 (1972)
Regge-Wheeler, PR 108, 1063 (1957); Zerilli, PRD 2, 2141 (1970)
Moncrief, Ann Phys. 88, 323 (1974),
Reviews: Nagar, Rezzolla, CQG 22 R167 (2005); Sasaki, Tagoshi, Living Review
Quasi Normal Modes: Kokkotas, Schmidt Living Review; Chandrasekhar: book on BHoles
Padé resummation: Damour Iyer Sathyaprakash PRD 57, 885 (1998), PRD 62, 084036 (2000)
Effective One Body: Buonanno, Damour PRD 59, 084006 (1999), PRD 62, 064015 (2000)
Damour Jaranowski Schäfer, PRD 62, 084011 (2000) [Damour Schäfer Nuov Cim 101B 127 (88)]
Damour, PRD 64, 124013 (2001), Buonanno Chen Damour, gr-qc/0508067
Close Limit Approximation Price Pullin PRL 72, 3297 (1994)
Numerical Results: Pretorius PRL 95, 121101 (2005), Baker et al PRD 73, 104002 (2006)

$\boxed{5.1}$ The Problem



ADIABATIC INSPIRAL

↓

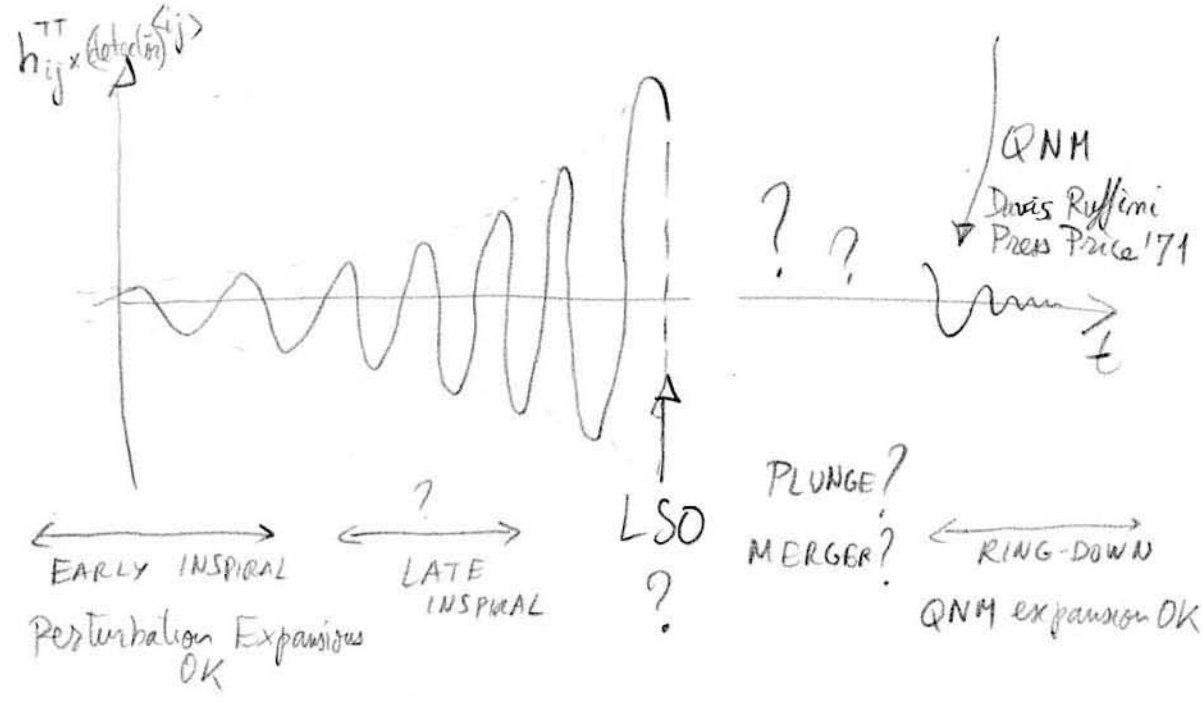
PLUNGE

↓

MERGER

↓

RING-DOWN



PHILOSOPHY: USE <u>RESUMMATION</u> OF PERTURBATIVE RESULTS

AND TRY TO <u>ANALYTICALLY</u> DESCRIBE WHOLE PROCESS

AGR 5.2

$\boxed{5.2}$ The 'Most Useful Cycles' for Coalescing BBH's

see e.g. (Damour, Iyer, Sathyaprakash 2000)

Signal to Noise ratio in coalescing binaries

$$\boxed{\left(\frac{S}{N}\right)^2 \simeq \frac{1}{15\pi}\left(\frac{G\sqrt{m_1 m_2}}{c^2 d}\right)^2 \int \frac{df}{|f|}\,\frac{1}{v(f)}\,\frac{1}{h_n^2(f)}}$$

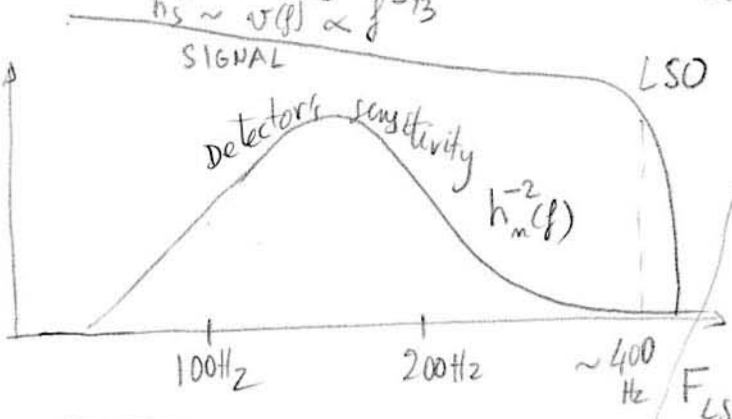
EFFECTIVE SIGNAL $h_s^2(f) \simeq N(f)\, a^2(f)$

FAVORS large masses and small distance d

ORBITAL VELOCITY @ orbit. frequency f
$v(f) \simeq (\pi G M f)^{1/3}$

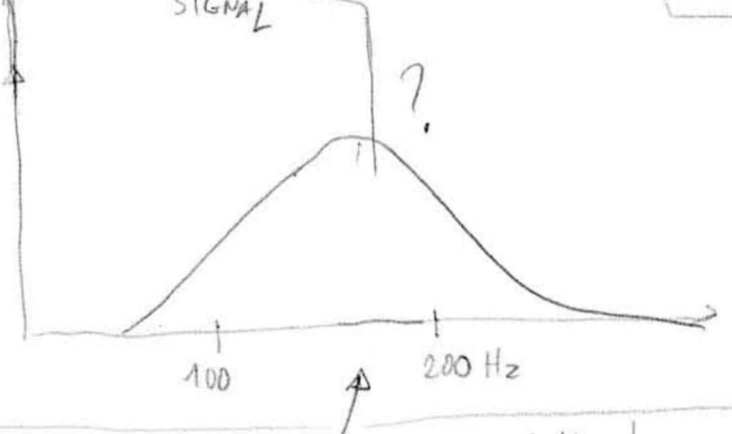
SENSITIVITY OF Detector to h^{GW}
$h_n^2(f) \equiv |f|\, S_n(f)$
in octave of f · per Hz

$\begin{pmatrix} BH, & NS \\ 10\,M_\odot, & 1.4\,M_\odot \end{pmatrix}$



EXPECTED $\gtrsim$ LSO CUT-OFF
$F_{LSO} \sim 4400 \frac{M_\odot}{M}$ Hz

$\begin{pmatrix} BH, & BH \\ 15\,M_\odot, & 15\,M_\odot \end{pmatrix}$



Last few cycles are most useful, and they happen around LSO!

5.3 Theoretical Challenges

- need eqs of motion to very high perturbation order
- need GW generation formalism to high perturb. order
- need Resummation methods because
  - perturbation expansion of phasing slowly and erratically convergent.
  - transition inspiral → plunge triggered by non-perturbative relativistic effects
  - useful to be able to describe plunge.
- need quasi-analytical description of inspiral + plunge + merger signal because of need of (nearly) continuous family of accurate GW templates depending on $\gtrsim 6$ intrinsic real parameters
- need analytical description of the whole process which can be matched and/or fitted to upcoming numerical results

5.4 Sketch of results from perturbation calculations

Eqs of Motion of Binary System

$$\frac{d^2\vec{z}_1}{dt^2} = \vec{A}_1^{CONS}(z,v) + \vec{A}_1^{RR}(z,v)$$
$$\frac{d^2\vec{z}_2}{dt^2} = \vec{A}_2^{CONS}(z,v) + \vec{A}_2^{RR}(z,v)$$

CONSERVATIVE DYNAMICS: known to 3PN — Damour, Jaranowski, Schäfer '01; Itoh, Futamase '03 '04; Blanchet, Damour, Esposito-Farèse '04

$$\vec{A}_A^{CONS} \sim \frac{GM}{r^2}\left(1 + \frac{v^2}{c^2} + \frac{v^4}{c^4} + \frac{v^6}{c^6}\right) \text{ or } H(x,p) = H_0 + \frac{H_2}{c^2} + \frac{H_4}{c^4} + \frac{H_6}{c^6}$$

RADIATION REACTION; known to very high order (see lectures of Blanchet)

$$\vec{A}_A^{RR} \sim \frac{GM}{r^2}\left(\underbrace{\frac{v^5}{c^5} + \frac{v^7}{c^7}}_{\text{direct calculations}} + \underbrace{\frac{v^8}{c^8} + \frac{v^9}{c^9} + \frac{v^{10}}{c^{10}} + \frac{v^{11}}{c^{11}} + \frac{v^{12}}{c^{12}}}_{\text{heuristically obtained by assuming balance of } E \text{ and } \vec{J}}\right)$$

Gravitational Wave Emission

$$r h_{ij}^{TT} = \left[U_{ij} + U_{ijk} n^k + \dots \right]^{TT} \qquad \text{(see lectures of Blanchet)}$$

$$U_{ij} = \nu M x^{\langle i} x^{j\rangle}\left(1 + \frac{v^2}{c^2} + \frac{v^3}{c^3} + \frac{v^4}{c^4} + \frac{v^5}{c^5} + \frac{v^6}{c^6}\right)$$
$$U_{ijk} = \nu M x^{\langle ijk\rangle}\left(1 + \frac{v^2}{c^2} + \frac{v^3}{c^3} + \frac{v^4}{c^4}\right)$$

Using $U_{ij}(U) = I_{ij}^{(2)}(U) + \frac{2GM}{c^3}\int d\tau\, I_{ij}^{(4)}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right] + \frac{\dots}{c^5} + \frac{\dots}{c^6}$

and

$$I_{ij} = \int d^3x\, \hat{x}^{ij}\, \frac{\overline{\tau}^{00} + \overline{\tau}^{ss}}{c^2} + \frac{1}{14}\int d^3x\, \hat{x}^{ij}\, \frac{\vec{x}^2}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\overline{\tau}^{00} + \overline{\tau}^{ss}}{c^2}\right) + \cdots$$

$\boxed{5.5}$ PN-expanded energy flux from circular binaries

$$\frac{dE}{dt} = \frac{32}{5}\frac{c^5}{G}\nu^2\,\gamma^5\left\{1-\left(\frac{2927}{336}+\frac{5}{4}\nu\right)\gamma + 4\pi\gamma^{3/2} + \left(\frac{293383}{9072}+\frac{380}{9}\nu\right)\gamma^2+\dots\right\}$$

At the Last Stable Orbit (of test-mass limit $\nu \to 0$, $m_1 \ll m_2$)

$$\boxed{\gamma = \frac{GM}{c^2 r_h}} = \frac{1}{5} \text{ @LSO} \qquad \frac{dE}{dt} \propto \gamma^5\left\{1-1.74(5\gamma)+1.12(5\gamma)^{3/2}+1.29(5\gamma)^2+\dots\right\}$$

NO CONVERGENCE AT ALL AT LSO

First improvement: express Flux in terms of invariants

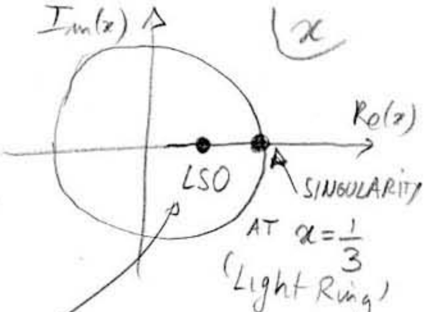
orbital frequency $n = \frac{2\pi}{P_b}$

$$\text{replace } \gamma = \frac{GM}{c^2 r_h} \longrightarrow \boxed{x \equiv \left(\frac{GMn}{c^3}\right)^{2/3}} = \left(\frac{GM}{c^2 R}\right)_{Schwarz} \sim \frac{1}{6} \text{ @LSO}$$

$$\frac{dE}{dt} \propto x^5\left\{1-0.619(6x)+0.855(6x)^{3/2}-0.137(6x)^2+\dots\right\}$$

SLOW CONVERGENCE AT LSO

BUT STILL CONVERGENCE BECAUSE



EXPECT TO IMPROVE CONVERGENCE BY USING

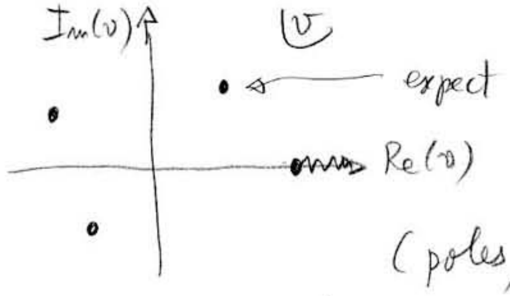
PADÉ RESUMMATION

$\boxed{5.6}$ Padé resummation

If you know Taylor expansion as $v \to 0$: $T_N(x) = a_0 + a_1 v + a_2 v^2 + \cdots + a_N v^N$

and expect 'exact function' $T(v)$ to be analytic in v

$\Rightarrow$



expect that "singularities" in complex v-plane (poles, cuts, ...) determine radius of convergence

And that the radius of convergence increases if one allows for poles

i.e. $$T(v) = \sum_i \frac{r_i}{v - v_i} + \text{better convergent Taylor series}$$

rational fraction

Polynomial of degree m

Padé approximant: $$\boxed{P^m_n\,[T(v)] = \frac{P_m(v)}{P_n(v)} = T_N(v) + O(v^{N+1})}$$

uniquely determined if $\boxed{m + n = N}$

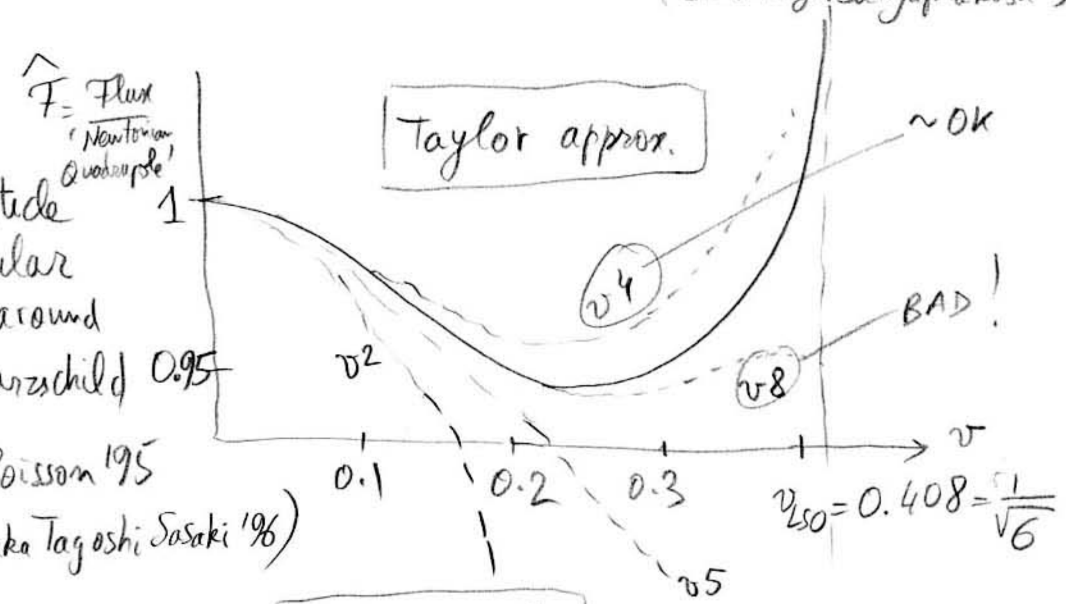
- Generally expect $\boxed{m \sim n \sim \frac{N}{2}}$ to give best approximant

(NEAR DIAGONAL)

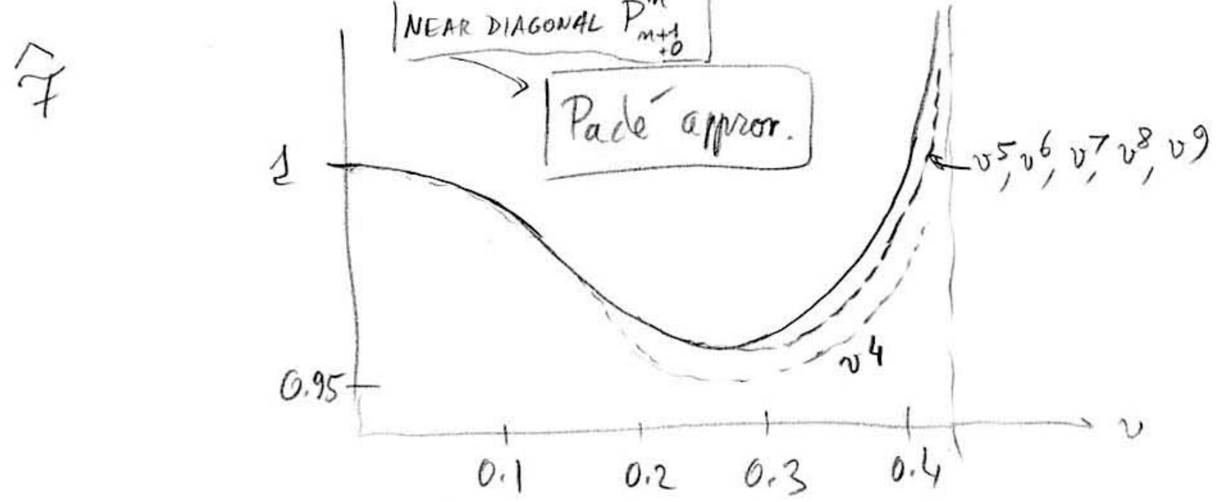
Taylor versus Padé approximants of E flux

(Damour Iyer Sathyaprakash '98)

Test particle in circular orbit around Schwarzschild



(using Poisson '95
(and Tanaka Tagoshi Sasaki '96)



$\boxed{\text{CONVERGENCE IS IMPROVED and MORE MONOTONOUS}}$

Not perfect, (smaller than exact answer) but probably adequate, if it holds also when $\nu = \frac{m_1 m_2}{M^2} \sim \frac{1}{4}$

$\boxed{5.7}$ 'EFFECTIVE ONE BODY' approach to resumming the conservative dynamics

(Buonanno, Damour '99)

Basic idea:

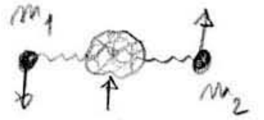
| QUANTUM ENERGY STATES OF TWO DIFFERENT SYSTEMS

REAL SYSTEM
(in the CoM FRAME)

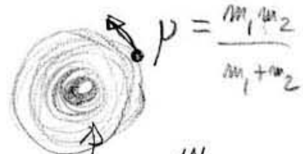
EFFECTIVE SYSTEM

ONE PARTICLE

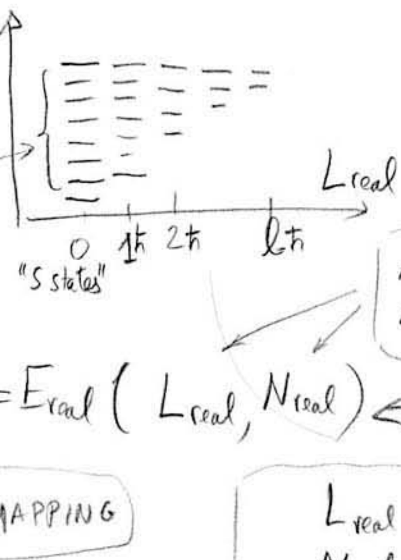
TWO BODIES



Real
Non linear
interaction

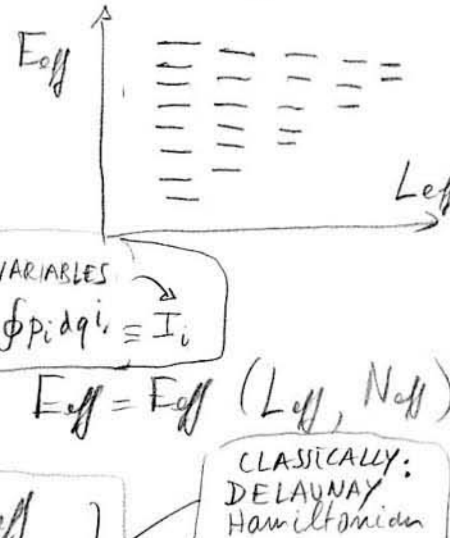


IN
Some EXTERNAL
Relativistic gravitational field: $g^{eff}_{\mu\nu} + \dots$



RADIAL QUANTUM NUMBER

"S states"



ACTION VARIABLES
$n_i + \frac{1}{2} = \frac{1}{2\pi}\oint p_i\, dq^i \equiv I_i$

$$E^{CM}_{real} = E_{real}(L_{real}, N_{real}) \longleftrightarrow E_{eff} = E_{eff}(L_{eff}, N_{eff})$$

| REQUIRE $\exists$ MAPPING |

$$\left.\begin{aligned} L_{real} &= L_{eff} \\ N_{real} &= N_{eff} \end{aligned}\right\}$$

$$E_{real} = f(E_{eff})$$

CLASSICALLY: DELAUNAY Hamiltonian $H(I_r, I_\theta)$

Damour Schäfer '88

$$H \sim -\frac{1}{2(I_r+I_\theta)^2} + \frac{1}{c^2}(\dots) + \frac{1}{c^4}(\dots)$$

| DETERMINES BOTH $g^{eff}_{\mu\nu}(x)$ AND f |

Computation of Delaunay Hamiltonians $H(I_r, I_\theta)$

Real dynamics in Center of Mass

ADM Hamiltonian $\longrightarrow$ $H^{ADM}_{real}(\vec{x}_1-\vec{x}_2, \vec{p}_1, \vec{p}_2)$

C of Mass: $\vec{r} = \vec{x}_1 - \vec{x}_2 \Leftrightarrow \vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$

with $\vec{\mathcal{P}} = \vec{p}_1 + \vec{p}_2 = \vec{0}$

$$H^{C.M.}_{real}(\vec{r}, \vec{p})$$

Hamilton-Jacobi eq.

$$E_{real} = H^{C.M.}_{real}\left(\vec{r}, \frac{\partial S_0}{\partial \vec{r}}\right)$$

$$S_0 = L\theta + \int p_r\, dr$$

$$p_r = \sqrt{\mathcal{R}(r, E, L)} = \sqrt{A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5} + \cdots}$$

$$N^{real}_{radial}\, h \simeq I^{real}_r = \frac{2}{2\pi}\int_{r_1}^{r_2} p_r\, dr$$

Effective dynamics

$$g^{eff}_{\mu\nu}(x)dx^\mu dx^\nu = -A(R)c^2 dT^2 + B(R)dR^2 + R^2 d\Omega^2$$

$$A(R) = 1 + \frac{a_1}{c^2 R} + \frac{a_2}{c^4 R^2} + \frac{a_3}{c^6 R^3} + \cdots$$

$$B(R) = 1 + \frac{b_1}{c^2 R} + \frac{b_2}{c^4 R^2} + \cdots$$

Hamilton-Jacobi eq

$$0 = \mu^2 + g^{\mu\nu}_{eff}(x) P_\mu P_\nu + A^{\mu\nu\rho\sigma}(x) P_\mu P_\nu P_\rho P_\sigma + \cdots$$

$$P_\mu = \frac{\partial S}{\partial x^\mu}$$

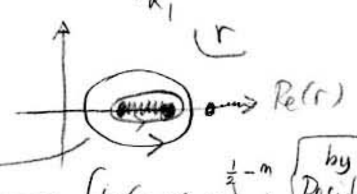
$$S = -E_{eff}T + L_{eff}\theta + \int p^{eff}_R\, dR$$

$$P_R = \sqrt{\frac{E^2_{eff}}{c^2}\frac{B(R)}{A(R)} - B(R)\left(\mu^2 c^2 + \frac{L^2_{eff}}{R^2} + \cdots\right)}$$

$$N^{eff}_{radial}\, h \simeq I^{eff}_R = \frac{2}{2\pi}\int_{R_1}^{R_2} P_R\, dR$$

Sommerfeld's method for $\displaystyle\int_{r_1}^{r_2} dr \sqrt{A + \frac{2B}{r} + \frac{C}{r^2} + \sum \frac{D_m}{r^{m+2}}}$

(see Damour Schäfer '88)



Larger Contour integral $\longrightarrow$ Expand in powers of D_m $\longrightarrow \displaystyle\int dr \left(A + \frac{2B}{r} + \frac{C}{r^2}\right)^{\frac{1}{2}-m} \frac{1}{r^m}$ $\boxed{\text{by Residues}}$

$$\longrightarrow H^{C.M.}_{real} = H_{real}(I^{real}_r, I^{real}_\theta)$$

$$= \underbrace{-\frac{1}{2(I_r + I_\theta)^2} + \frac{\cdots}{c^2} + \frac{\cdots}{c^4} + \frac{\cdots}{c^6}}_{\text{UNIQUELY DETERMINED}}$$

$\boxed{\text{MATCH}}$

$$H_{eff} = H_{eff}(I^{eff}_R, I^{eff}_\theta)$$

$$= \underbrace{-\frac{1}{2(I_R + I_\theta)^2} + \frac{f_2(a_i, b_i)}{c^2} + \frac{f_4(a_i, b_i)}{c^4} + \cdots}$$

$$H_{real}(I_r, I_\theta) = f(H_{eff}(I_r, I_\theta))$$

$$E^{NR}_{eff} = E^{NR}_{real}\left(1 + \alpha_1 \frac{E^{NR}}{c^2} + \frac{\alpha}{2}\frac{(E^{NR})^2}{c^4} + \cdots\right)$$

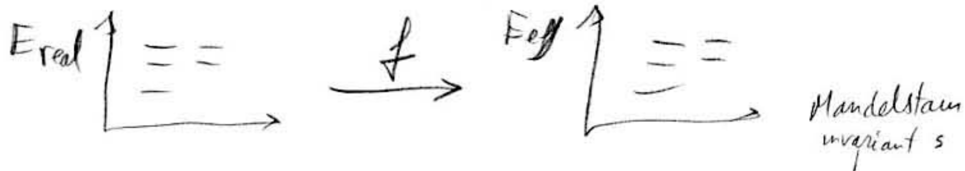
DEPENDS ON $a_i, b_i, \alpha_i, A^{\mu\nu\rho\sigma}$,

$\boxed{\text{MATCHING DETERMINES COEFFICIENTS } a_i, b_i, \alpha_i, A^{ijkl}}$

AGR 5.10

EOB: enormous simplification of dynamics

(1)



UNIVERSAL ENERGY MAP:

$$\boxed{\frac{E_{eff}}{\mu} = \frac{E_{real}^2 - m_1^2 - m_2^2}{2 m_1 m_2} = \frac{s - m_1^2 - m_2^2}{2 m_1 m_2}}$$

(2) Writing $\boxed{g^{eff}_{\mu\nu}(x)\, dx^\mu dx^\nu = -A(r) c^2 dt^2 + \frac{D(r)}{A(r)} dr^2 + r^2 d\Omega^2}$

$$\underset{CofM}{H^{1PN}_{(x,p)}} = \frac{\vec{p}^2}{2} - \frac{1}{r} + \frac{1}{c^2}\underbrace{\left(a_1 \vec{p}^4 + a_2 \frac{\vec{p}^2}{r} + a_3 \frac{(\vec{n}\cdot\vec{p})^2}{r} + \frac{a_4}{r^2}\right)}_{4 \text{ nontrivial 1PN coeffs}} \longleftrightarrow \boxed{\begin{aligned} A^{1PN}(r) &= 1 - \frac{2GM}{c^2 r} \\ D^{1PN}(r) &= 1 \end{aligned}}$$

already determined by Newtonian limit

ALL SUBSUMED IN $f(E)$ and Schwarzschild metric

$$H^{2PN}_{(x,p)} = H_0 + \frac{4 \text{ coeffs}}{c^2} + \frac{7 \text{ coeffs}}{c^4} \longleftrightarrow \boxed{\begin{aligned} A^{2PN}(r) &= 1 - \frac{2GM}{c^2 r} + 2\nu\left(\frac{GM}{c^2 r}\right)^3 \\ D^{2PN}(r) &= 1 - 6\nu\left(\frac{GM}{c^2 r}\right)^2 \end{aligned}}$$

ADM COORDS AND HAMILTONIAN

ALL CONDENSED in $+2\nu$, -6ν

$$H^{3PN}_{(x,p)} = H_0 + \frac{4 \text{ coeffs}}{c^2} + \frac{7 \text{ coeffs}}{c^4} + \frac{11 \text{ coeffs}}{c^6}$$

all condensed in 3 coeffs

$$\boxed{\begin{aligned} A^{3PN}(r) &= 1 - \frac{2GM}{c^2 r} + 2\nu\left(\frac{GM}{c^2 r}\right)^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu\left(\frac{GM}{c^2 r}\right)^4 \\ D^{3PN}(r) &= 1 - 6\nu\left(\frac{GM}{c^2 r}\right)^2 + 2(3\nu - 26)\nu\left(\frac{GM}{c^2 r}\right)^3 \\ &+ p^2 \longrightarrow p^2\left[1 + 2(4 - 3\nu)\nu\left(\frac{GM}{c^2 r}\right)^2\left(\frac{\vec{n}\cdot\vec{p}}{p}\right)^4\right] \end{aligned}}$$

Damour, Jaranowski, Schäfer '00

5.8 EOB RESUMMED REAL HAMILTONIAN

At the end of the day, the EOB approach defines a specific <u>resummation</u> of $H_{real}(\vec{r}\,', \vec{p}\,')$ after a specific change of variables $(\vec{r}_{ADM}, \vec{p}_{ADM}) \to \underbrace{(\vec{r}\,', \vec{p}\,')}_{\text{effective}}$

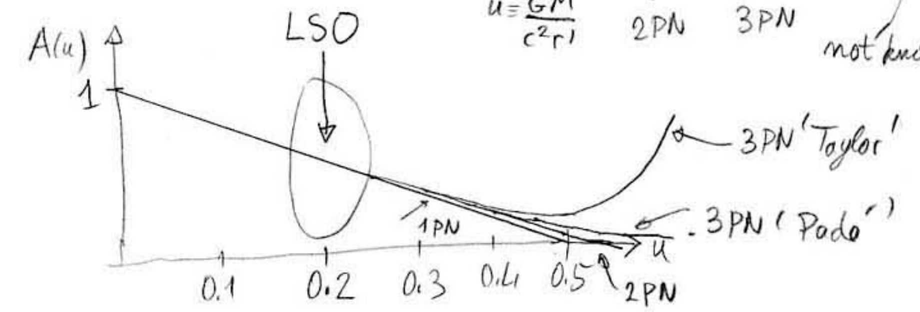
$$\boxed{H_{real}(\vec{r}\,', \vec{p}\,') = M\sqrt{1 + 2\nu\left(\sqrt{A(r')\left[1 + \vec{p}\,'^2 + \left(\frac{A(r')}{D(r')} - 1\right)(\vec{n}\,'.\vec{p}\,')^2 + z_3\frac{(\vec{n}\,'.\vec{p}\,')^4}{r'^2}\right]} - 1\right)}}$$

special $\underbrace{\sqrt{\ \sqrt{\ \ }}}$ structure

a specific resummation

crucial 'Radial Potential'

$$A(r') = 1 - 2\underset{\substack{\uparrow \\ u \equiv \frac{GM}{c^2 r'}}}{u} + \underset{\substack{\uparrow \\ 2PN}}{2\nu u^3} + \underset{\substack{\uparrow \\ 3PN}}{\hat{a}_4 \nu u^4} + \underset{\substack{\uparrow \\ \text{not known}}}{(?)\nu u^5} + \dots$$

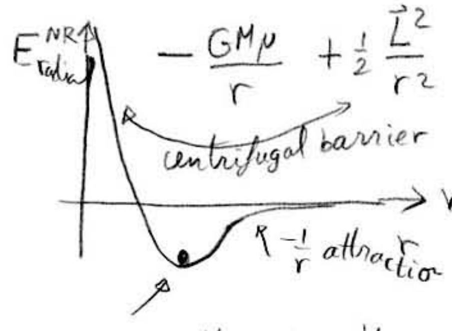


$+$ Padé resum

$$\boxed{A^{Padé}_{3PN}(u) = P^1_3\left[1 - 2u + 2\nu u^3 + \hat{a}_4 \nu u^4\right] = \frac{1 + c_1 u}{1 + d_1 u + d_2 u^2 + d_3 u^3}}$$

5.9 LAST STABLE (circular) ORBIT (LSO)

Newtonian Case

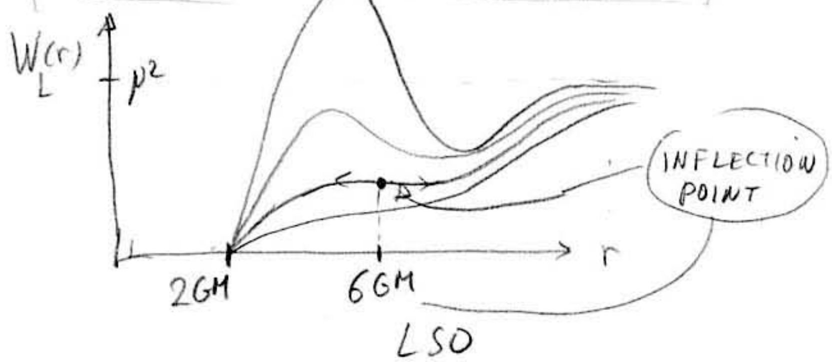


∃ STABLE ORBIT ∀L, i.e. ∀r

Test particle around Schwarzschild

attraction centrifugal barrier

$$E^2_{radial} = W_L(r) = \left(1 - \frac{2GM}{r}\right)\left(\mu^2 + \frac{L^2}{r^2}\right)$$



Two Comparable masses

$$E^2_{radial} = A(r)\left(\mu^2 + \frac{L^2}{r^2}\right)$$

$u = GM/c^2 r$

+ Padé resum $A^{Padé}(r) = P^1_3\left[1 - 2u + 2\nu u^3 + \hat{a}_4 \nu u^4\right]$

Predicted Characteristics of 3 PN LSO $m_1 = m_2$

$$f^{LSO}_{GW} = 2 f^{LSO}_{orbit} \simeq \frac{5700\,\text{Hz}}{(M/M_\odot)} \quad ; \quad \frac{E^{real} - Mc^2}{Mc^2} \simeq -1.67\%$$

1.296 larger than $\simeq \frac{4400\,\text{Hz}}{(M/M_\odot)}$ expected from Schwarzschild

instead of −1.43%

$\boxed{5.10}$ INCLUSION OF SPIN EFFECTS

(Damour '01)

Convenience of Hamiltonian approach:

From Quantum Mechanics: $[x^i, P_j] = i\hbar \delta^i_j$; $[S^i_A, S^j_A] = i\hbar \varepsilon^{ijk} S^k_A$

$\rightarrow$ Poisson brackets: $\{x^i, P_j\} = \delta^i_j$, $\boxed{\{S^i_A, S^j_A\} = \varepsilon^{ijk} S^k_A \quad A=1,2}$

Real dynamics

$$H^{CM}_{real}(\vec{r}, \vec{P}, \vec{S}_1, \vec{S}_2) =$$

$$H_{orbital}(\vec{r}, \vec{P}) +$$

$$\frac{2G}{c^2 r^3}\left[\left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\vec{S}_1 + \cdots\right](\vec{x}\times\vec{P}) + \cdots$$

$$+ \mathcal{O}(S_1 S_2) + \mathcal{O}(S_1^2) + \mathcal{O}(S_2^2)$$

SPIN-ORBIT (Barker O'Connell) SPIN-SPIN QUADRUPOLE $\propto S_1^2$ OF KERR BLACK HOLE

Effective dynamics

Particle of mass μ in

$g^{eff}_{\mu\nu} dx^\mu dx^\nu$

Consider a ν-deformation of some Kerr metric with $\vec{S}_{eff}$

MATCH $\rightarrow \vec{S}_{eff} = \left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\vec{S}_1 + \left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\vec{S}_2$

+ extra ~ quadrupole contribs

$\rightarrow$ **EOB RESUMMED HAMILTONIAN FOR TWO SPINNING BLACK HOLES**

$$\boxed{H_{real}(\vec{r}, \vec{P}, \vec{S}_1, \vec{S}_2) = M\sqrt{1+2\nu\left(\beta^i(\vec{S}_A)P_i + \alpha(r, S_A)\sqrt{1+\gamma^{ij}(\cdot)P_iP_j + \mathcal{O}(\vec{P}^4)} - 1 + \cdots\right)}}$$

'gravito-magnetic-type coupling'

Comparison between EOB and numerical results

Analytical

$H^{EOB}_{real}(\vec{r}, \vec{p}, \vec{S}_1, \vec{S}_2)$

$\downarrow$

predicts **invariant** characteristics of close orbits

when neglecting RADIATION DAMPING

Using Christodoulou-Ruffini

$$m_A[\vec{S}_A] = \sqrt{(m_A^{irr})^2 + \frac{\vec{S}_A^{\,2}}{4(m_A^{irr})^2}}$$

$$dE_{real} = \Omega_0\, dL + \Omega_1\, dS_1 + \Omega_2\, dS_2$$

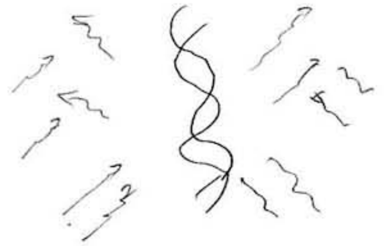
predicts links between

E^{tot}_{real}, $E_{binding}$,

$J = L + S_1 + S_2$, Ω_0, Ω_1, Ω_2

Numerical

('RADIATION-LESS ORBITS')



$\rightarrow \exists$ 'HELICAL KILLING VECTOR'

Detweiler, Iriondo,

Gourgoulhon, Grandclément, Bonazzola '02

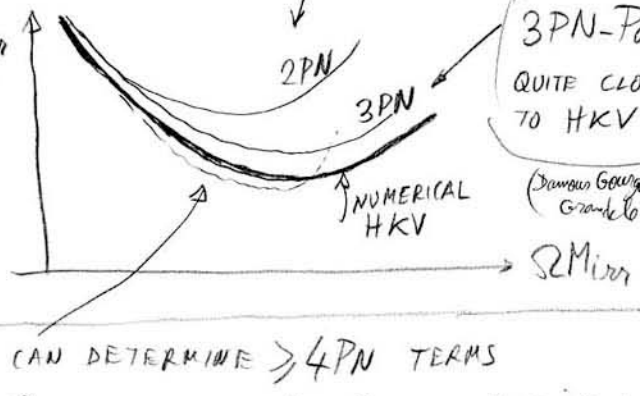
$\downarrow$

especially CO-ROTATING CIRCULAR BINARY ORBITS

$\Omega_1 = \Omega_2 = \Omega_0$

COMPARE RESULTS

e.g. $\frac{M_{ADM} - M_{irr}}{M_{irr}}$



3PN-Padé-EOB QUITE CLOSE TO HKV RESULTS

(Damour, Gourgoulhon, Grandclément '02)

IN PRINCIPLE CAN DETERMINE $\geq 4PN$ TERMS

$A^{4PN} = \text{Padé}[1 - 2u \cdots + \nu\underline{\underline{\hat{a}_5}}u^5]$ BY FITTING TO NUMERICAL

$\boxed{5.11}$ COALESCING BINARY BH's IN EOB

(Buonanno Damour '00)
(Buonanno Chen Damour '05)

BASIC EOB INPUTS / $H_{real}(\vec{r}, \vec{P}, \vec{S}_1, \vec{S}_2) = \sqrt{\quad \sqrt{A^{Padé}}}$

\ Padé-resummed Radiation Reaction $\mathcal{F}_i$

quasi-circular motions: $\boxed{\mathcal{F}_i = \frac{1}{\omega |L|} \left(\frac{dE}{dt}\right)^{\text{Padé resummed}} P_i + [c_1 \vec{P}.\vec{S}_1 + c_2 \vec{P}.\vec{S}_2]\, L_i}$

DIS '98

$$\left(\frac{dE}{dt}\right)^{Padé} \propto \frac{1}{1 - \frac{v}{v_{pole}}} \; \frac{1}{1 + \frac{c_1(\nu) v}{1 + \frac{c_2(\nu) v}{1 + \frac{c_3(\nu) v}{1 + \frac{c_4(\nu) v}{1 + c_5(\nu) v}}}}}$$

CONTINUOUS FRACTION VERSION OF NEAR-DIAGONAL PADÉ

Universal Hamiltonian Evolution Equation : $\frac{d}{dt} f(\vec{r}, \vec{P}, \vec{S}_A) = \{f, H_{real}\}$.

$$\frac{d\vec{S}_1}{dt} = \{\vec{S}_1, H\} = \vec{\Omega}_1 \times \vec{S}_1 \quad ; \quad \frac{d\vec{S}_2}{dt} = \vec{\Omega}_2 \times \vec{S}_2$$

$$\frac{dx^i}{dt} = \frac{\partial H_{real}}{\partial p_i}$$

EOB resummed

$$\frac{dp_i}{dt} = -\frac{\partial H_{real}}{\partial x^i} + \mathcal{F}_i$$

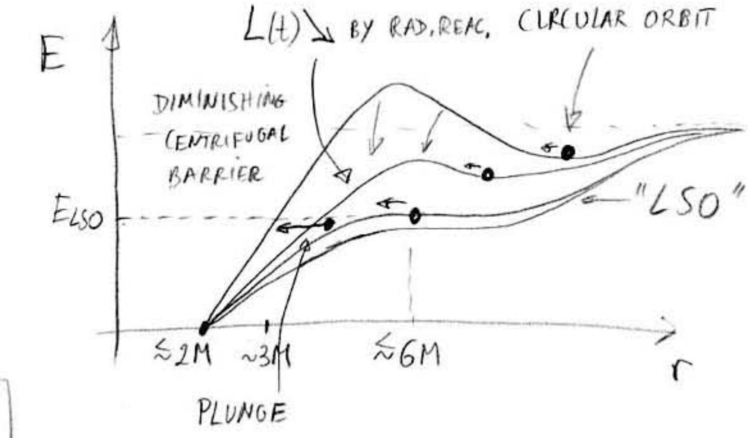
Padé resummed

<u>Conjecture</u> : This analytical description is $\approx$ valid, not only around the LSO, but also during most of the plunge

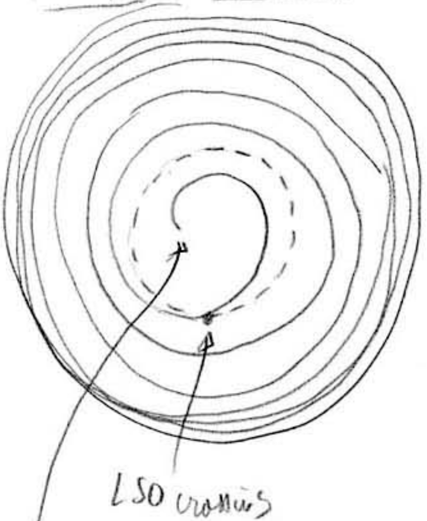
<u>BOLD</u>, but one can check
- inner consistency
- comparison to numerical results

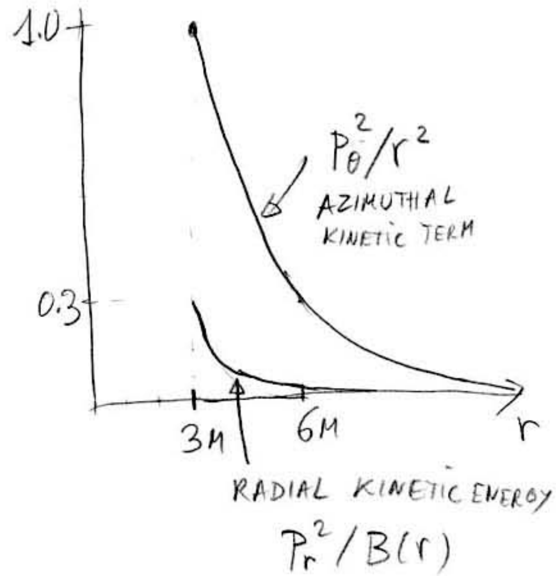
$\boxed{5.12}$ Results of EOB dynamics

Qualitatively



CASE $\boxed{\nu = \frac{1}{4}}$ i.e. $\boxed{m_1 = m_2}$





ONLY 0.6 ORBIT TO PLUNGE FROM $\sim 6M \to \sim 3M$

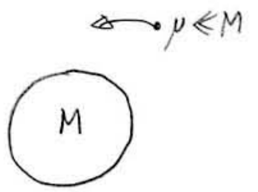
'PLUNGE' REMAINS QUASI-CIRCULAR

MOTION IS STILL 'NON-RELATIVISTIC' AT LSO: $\frac{\vec{p}^2}{\mu^2} \sim 0.3$

'MILDLY RELATIVISTIC' AT $\sim 3M$: $\frac{\vec{p}^2}{\mu^2} \sim 1$

[5.13] Waveforms from test-masses plunging in Schw. BH 5.17

Davis Ruffini Press Price '71
Davis Ruffini Tiomno '72
Sasaki Tagoshi Living Rev.



$$g_{\mu\nu}(x) = g^{SCHWARZ}_{\mu\nu}(x) + h_{\mu\nu}(x)$$

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \sim \int \frac{ds}{\sqrt{g}} \mu u_\mu u_\nu \, \delta(x - z(s))$$

- Regge-Wheeler-Zerilli: Exploit symmetries of Background

$$h_{\mu\nu}(r,\theta,\phi,t) = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \sum_{I=1}^{10} h^{(I,\pi)}_{\ell m}(r,t) \, Y^{\ell m (I,\pi)}_{\mu\nu}(\theta,\phi)$$

tensorial spherical harmonics

suitable combination of radial functions satisfy simple decoupled equations

Useful to work with gauge-invariant combinations of $h_{\ell m}$ (Moncrief '74 Nagar Rezzolla '05)

- For each ℓm and each parity π there is one gauge-invariant scalar degree of freedom ($\sim$ analogs of $\sum_\ell \partial_L\left(\frac{M_L(t-r)}{r}\right)$ and $\sum_\ell \partial_L\left(\frac{S_L(t-r)}{r}\right)$)

$$\boxed{\partial_t^2 \Psi^{(\pi)}_{\ell m} - \partial^2_{r_*} \Psi^{(\pi)}_{\ell m} + V^{(\pi)}_\ell(r_*) \Psi^{(\pi)}_{\ell m} = S^{(\pi)}_{\ell m}}$$

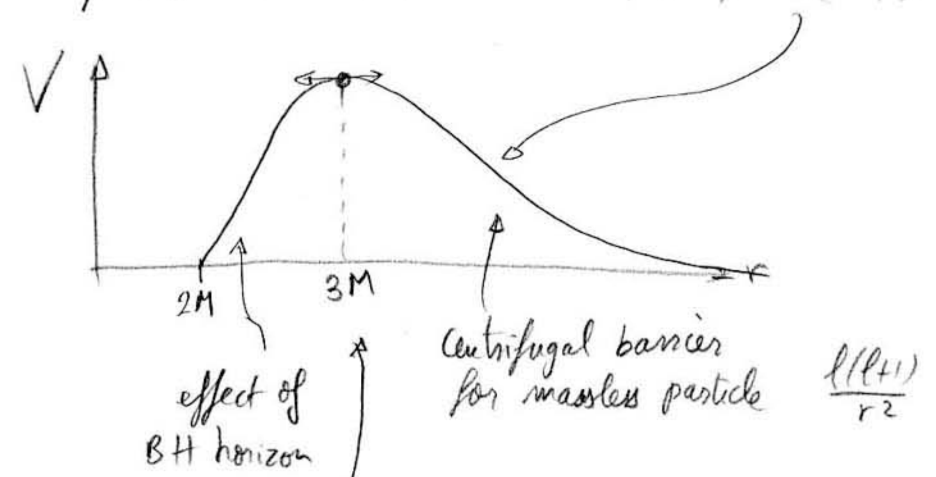
$\boxed{r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right)}$ 'EFFECTIVE POTENTIAL'

SOURCE TERM $\sim T^{\mu\nu}$ and $\partial_{r\alpha} T^{\mu\nu}$

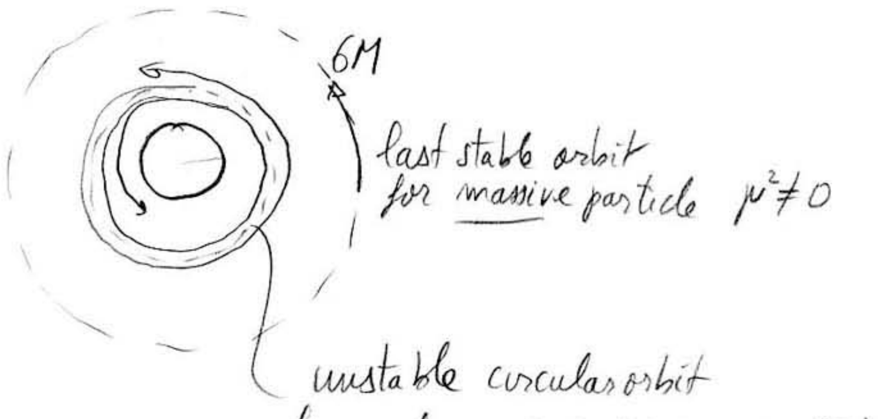
eg. $$\boxed{V^{(odd)}_\ell = \left(1 - \frac{2M}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right)} \simeq V_\ell^{(even)}$$

N.B. Effective Potential $V_\ell(r)$ is approximately

$$\lim_{\mu^2 \to 0} \left(1 - \frac{2M}{r}\right)\left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right) = \left(1 - \frac{2M}{r}\right)\frac{\ell(\ell+1)}{r^2}$$



| unstable null geodesic: 'light ring' @ 3M |



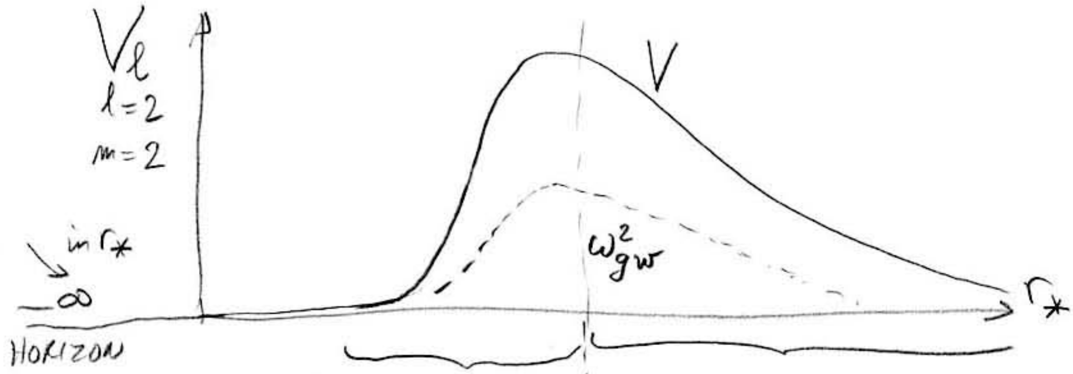
Qualitative Behaviours of Wave Generation

If $\Psi_{\ell m}(t) \sim e^{-i\omega_{gw} t}$ $\qquad \omega_{gw} = m\,\omega_{orbit}$

of (ℓ, m)

$$-\partial^2_{r_*}\Psi_{\ell m} + \underbrace{\left(V_\ell - \omega^2_{gw}\right)}\Psi_{\ell m} = S_{\ell m}$$

compare $\omega^2_{gw} = m^2\omega^2_{orbit}$ to V_ℓ



here the effect of Source term is filtered by a potential barrier

here $\underbrace{\omega^2_{gw} \ll V \simeq \frac{\ell(\ell+1)}{r^2}}$

i.e. ~Long wave length propagating essentially on flat spacetime and generated by Source

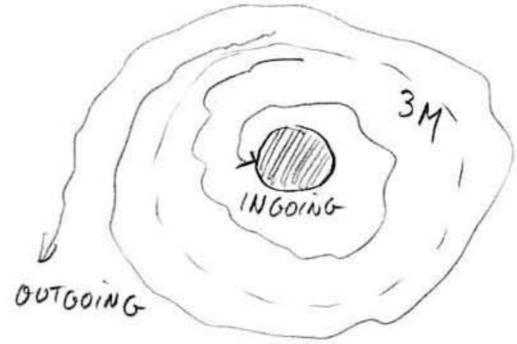
physically

Only modes that tunnel out are crucial

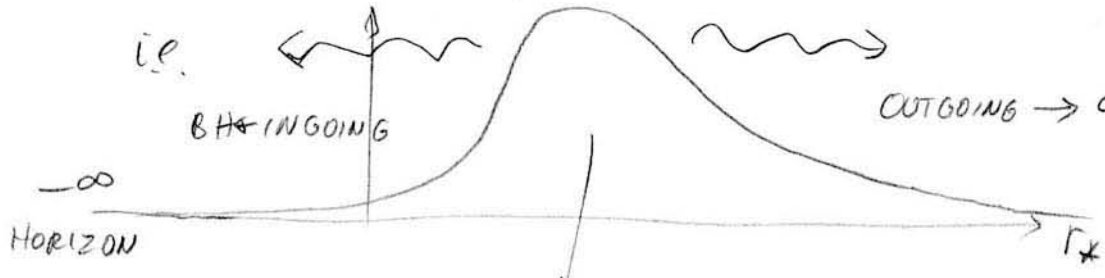
'Source free'

5.14 Quasi Normal Modes of BH's

'Resonances' of BH



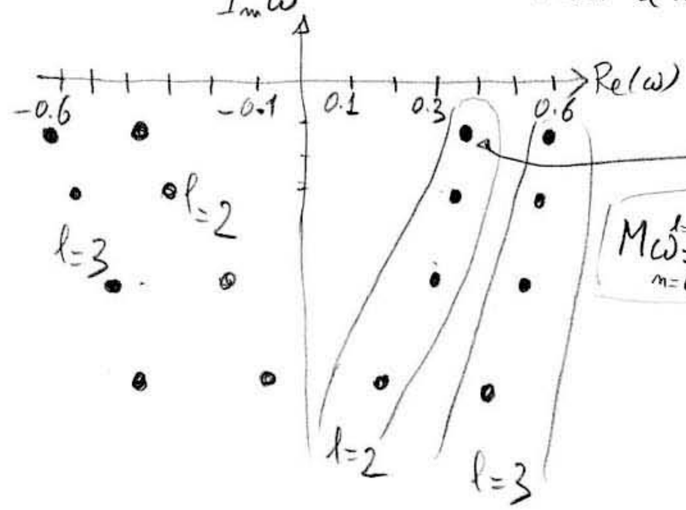
i.e.



source-free $\psi \propto e^{-i\omega_{QNM} t}$

$$\boxed{-\partial_{r_*}^2 \psi_{\ell m} + V_\ell \psi_{\ell m} = \omega_{QNM}^2 \psi_{\ell m}}$$

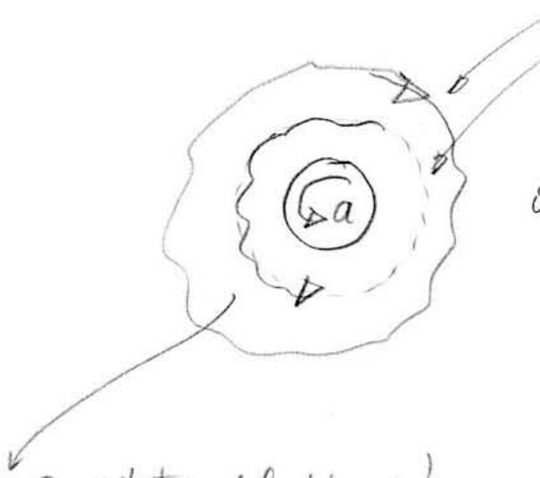
COMPLEX QNM frequencies



LEAST DAMPED QUADRUPOLAR QNM

$$\boxed{M\omega^{l=2}_{n=0} = 0.3737 - i\,0.0890}$$

QNM's of ROTATING (KERR) B HOLES



$\omega^{lm} = \sigma^{lm} - i\,\alpha^{lm}$

depends strongly on whether

$m > 0$ co-rotating

$m < 0$ counter-rotating

$m > 0$ Co-rotating 'light ring' has higher E/L than Schwarzschild $\rightarrow$ $Re(\omega^{ll}_{QNM}) \nearrow$ when $a \nearrow$

$-Im(\omega^{ll}_{QNM}) \searrow$ when $a \nearrow$

$m < 0$ counter-rotating 'light ring' $\rightarrow$ $Re(\omega^{l-l}_{QNM}) \searrow$

$-Im(\omega^{l-l}_{QNM}) \sim cst$

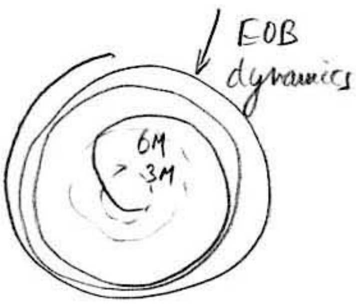
Regge-Wheeler-Zerilli ('metric perturbation formalism')

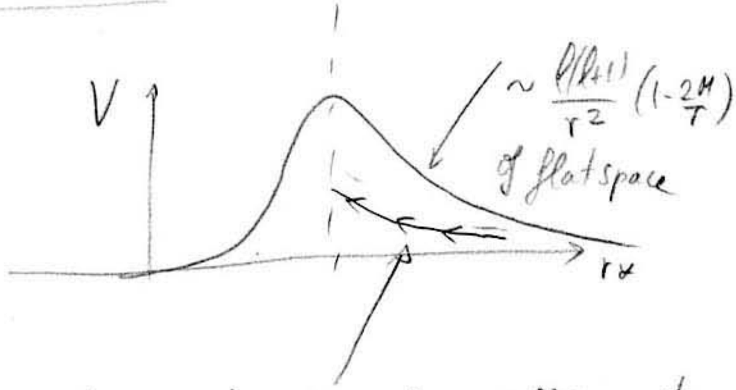
$\longrightarrow$ Teukolsky ('Weyl perturbation formalism')

Null extreme components: Ψ_0, Ψ_4 : more complex (see Sasaki, Tagoshi ; Chandrasekhar)

$\boxed{5.15}$ EOB waveforms from BH coalescence

- INSPIRAL + PLUNGE





COMPUTE GW generation by perturbation thy / flat spacetime

$\left(+ \text{ effect of } \left(1-\frac{2M}{r}\right)_x + \cdots \right)$

e.g. $U_{ij} \simeq I_{ij}^{(2)}$

rotating quadrupole $\sim \mu x^{\langle ij \rangle}$

$\boxed{\begin{array}{l} l = m = +2 \\ \text{leading term} \end{array}}$

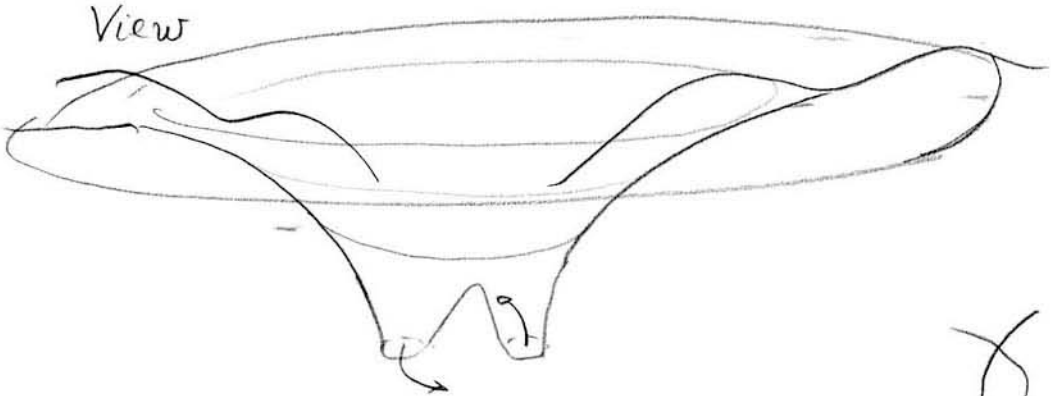
the $\boxed{\text{PHASING}}$ is more important thant the GW amplitude

$\vec{x} = (r\cos\varphi,\ r\sin\varphi,\ 0)$

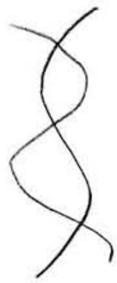
$\rightarrow \left[x^{\langle ij \rangle}\right]^{l=m=2} \propto r^2 e^{-i2\varphi}$ ← 2x orbital phase

- ROUGH TRANSITION TO MERGER

View

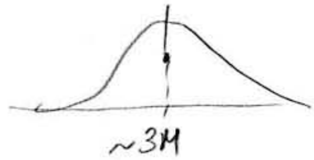


DURING INSPIRAL + PLUNGE : SKELETONIZE BY TWO POINT MASSES



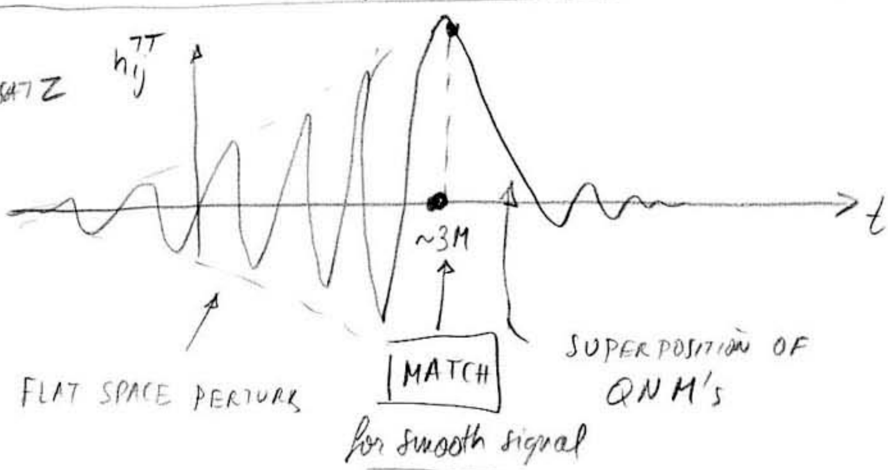
SEEN WITH BLURRED VISION OF GWAVES $\lambda_{gw} \gg r$

things change drastically when crossing 'light ring'



After that the two 'point sources' are strongly filtered and excite a combination of QNM RINGING

FOR ROUGH ANSATZ

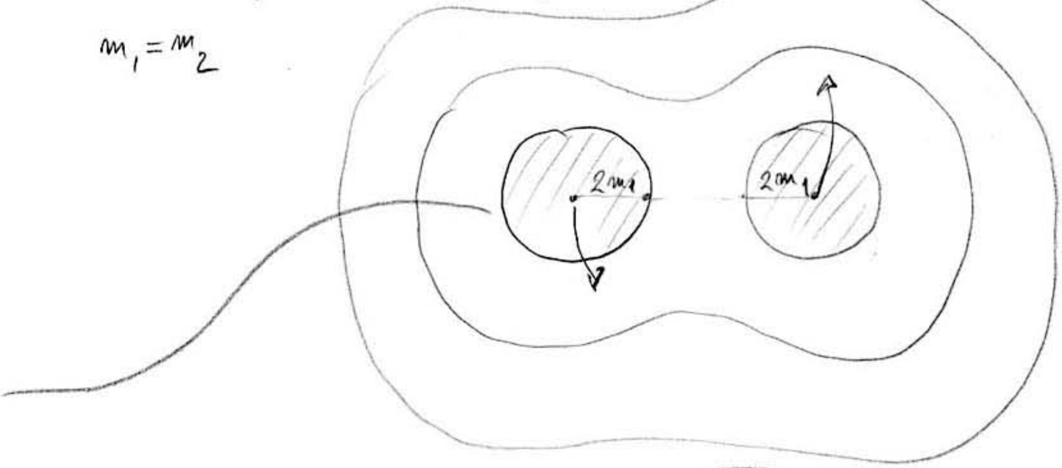


This rough match is suggested both

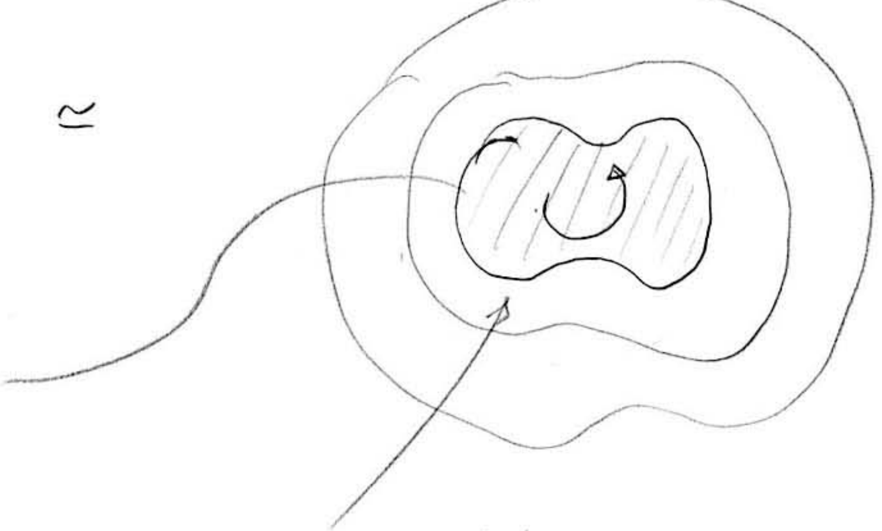
- by numerical results for test particles (Davis Ruffini Press Price ....)
- by results of 'Close Limit Approximation' (Price Pullin '94)

about matching 2 BHs → 1 deformed BH

$m_1 = m_2$



$\simeq$



deformed rotating BH

→ Estimate mass and spin of BH formed by merging m_1, m_2 by H_{real}, $\vec{J}_{real}$ around the 'light ring' ~ $3M$

→ Use QNM's for $M, \vec{a}$ OF MERGED BLACK HOLE

5.16 Comparison EOB waveforms and numerical results

- Numerical results for comparable-mass coalescing BH's have been available only very recently (Pretorius '05, Baker et al '06)

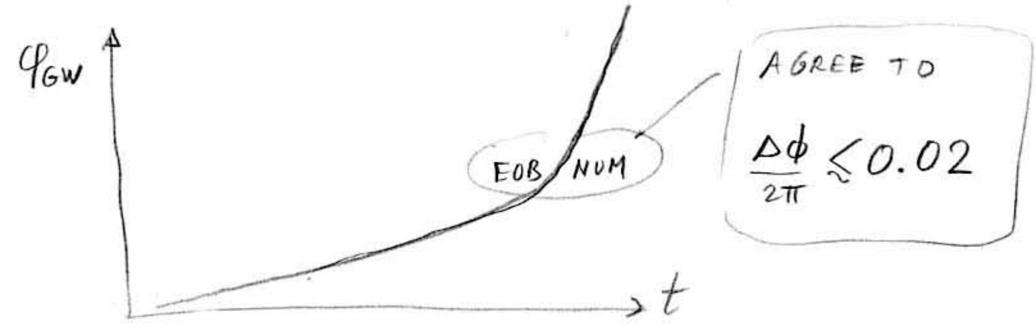
  Comparison with EOB is quite encouraging (Buonanno '06)

  Basically confirms the EOB picture

  Also spin parameter $\hat{a} \equiv \frac{a}{M}$ $\boxed{\text{for } m_1 = m_2 \,;\, \nu = 1/4 \quad \hat{a}^{final}_{EOB} \simeq 0.8 \;;\; \hat{a}^{final}_{NUM} \simeq 0.7}$ ~ OK

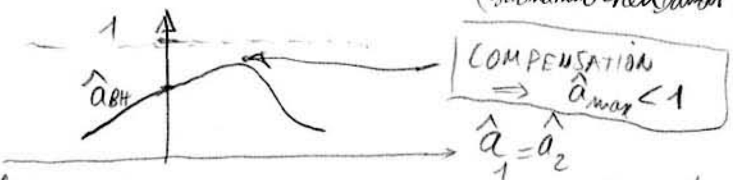
- Numerical results when $\nu \ll 1$: extreme mass ratio (Damour Nagar.. '06)

  Excellent agreement in computing the PHASE of GW



- Spin effects important and need further work (Buonanno Chen Damour)

  e.g. prediction



- Need precise matching between EOB $\leftrightarrow$ various Numerical Results

$\boxed{5.17}$ MAIN CONCLUSION

One is entering an interesting era where

- ANALYTICAL TECHNIQUES IN GR
- NUMERICAL RELATIVITY
- SOON: GW OBSERVATIONS (LIGO, VIRGO, GEO, ...)

can be all compared.

Very probably analytical techniques will be needed for quite a long time because

- need very accurate phasing of long inspiral phase
- need GW templates in multi-dimensional space: ν, $\vec{S}_1$, $\vec{S}_2$
- very flexible: e.g. $A^{EOB}(u) = 1 - 2u + \cdots + \hat{a}_5 \nu u^5$

can be fitted to numerical results