Gravitational Waves from Inspiralling Compact Binaries in General Orbits

Bala Iyer

Raman Research Institute
Bangalore India

IHP
2006

PART I

Based on

3PN Gravitational wave fluxes of energy and angular momentum from inspiralling eccentric binaries

K. Arun, L. Blanchet, B. R. Iyer and M. S. S. Qusailah

Part I, II, III (2006) - To be submitted

Introduction

- Inspiralling Compact Binaries (ICB) are considered to be the most probable sources of detectable gravitational radiation for laser interferometric gravitational-wave detectors. ICB are usually modeled as point particles in *quasi-circular* orbits.
- For long lived compact binaries, the quasi-circular approximation is quite appropriate. Gravitational Radiation Reaction (GRR) decreases the orbital eccentricity to negligible values by the epoch the emitted gravitational radiation enters the sensitive bandwidth of the interferometers. For a lower cutoff of 40 Hz, the binary enters detector when orbital freq is 20 HZ. For NS-NS binary corresponds to orbital radius of 290 km and orbital vel $.12c$ For an isolated binary, the eccentricity goes down roughly by a factor of three, when its semi-major axis is halved since $e/e_0 = (a/a_0)^{19/12}$ - (Peters, 64)

Eccentric Binaries

Based on Koenigdorffer and Gopakumar

- Stellar-mass compact binaries in eccentric orbits are excellent sources for LISA.
- LISA will "hear" GW from intermediate-mass black holes moving in highly eccentric orbits

  K. Gültekin, M. C. Miller, and D. P. Hamilton (2005), T. Matsubayashi, J. Makino, and T. Ebisuzaki (2005), M. A. Gürkan, J. M. Fregeau, and F. A. Rasio (2005)
- Several papers indicate that SMBHB formed from galactic mergers, may coalesce with orbital eccentricity

  S. J. Aarseth (2003), P. Berczik, D. Merritt, R. Spurzem, and H.-P. Bischof (2006), O. Blaes, M. H. Lee, and A. Socrates(2002), P. J. Armitage and P. Natarajan (2005), M. Iwasawa, Y. Funato, and J. Makino (2005)

  These investigations employ different techniques and astrophysical scenarios to reach the above conclusion.

Kozai Mechanism

- One proposed astrophysical scenario, involves hierarchical triplets modeled to consist of an inner and an outer binary. If the mutual inclination angle between the orbital planes of the inner and of the outer binary is large enough, then the time averaged tidal force on the inner binary may induce oscillations in its eccentricity, known in the literature as the Kozai mechanism

Kozai (1962),M. C. Miller and D. P. Hamilton (2002), E. B. Ford, B. Kozinsky, and F. A. Rasio (2000), Wen (2003)

Kozai..SMBHB

- Cosmological SMBBH embedded in surrounding stellar populations, would be powerful GW sources for detectors like LISA

  K. S. Thorne and V. B. Braginsky, 1976)

  These SMBBH, in highly eccentric orbits, would merge within the Hubble time

  O. Blaes, M. H. Lee and A. Socrates, (2003)

- Raises the interesting possibility of being able to detect GW from SMBBH in eccentric orbits using LISA.

Kozai Mechanism, Globular Clusters

- In globular clusters (GC), the inner binaries of hierarchical triplets undergoing Kozai oscillations can merge under GRR

  M. C. Miller and D. P. Hamilton (2002).

  A good fraction of such systems will have eccentricity~ 0.1, when emitted GW from these binaries passes through $10\,\mathrm{Hz}$

  Wen (2003).

- Such scenarios involving compact *eccentric* binaries are being suggested as potential GW sources for the terrestrial GW detectors.

NS-BH...GRB

- During the late stages of BH–NS inspiral the binary can become eccentric

  M. B. Davies, A. J. Levan, and A. R. King (2005).

  In general NS is not disrupted at the first phase of mass transfer and what remains of NS is left on a wider eccentric orbit from where it again inspirals back to the black hole. Scenario invoked to explain the light curve of the short gamma-ray burst GRB 050911

  Page (2006)

- At least partly short GRBs are produced by the merger of NS–NS binaries, formed in GC by exchange interactions involving compact objects

  J. Grindlay, S. P. Zwart, and S. McMillan, (2006)

  A distinct feature of such binaries is that they have high eccentricities at short orbital separation.

Kicks, Eccentricity

- Compact binaries that merge with some residual eccentricities may be present in galaxies too. Chaurasia and Bailes demonstrated that a natural consequence of an asymmetric kick imparted to neutron stars at birth is that the majority of NS–NS binaries should possess highly eccentric orbits

  H. K. Chaurasia and M. Bailes (2005).

- Observed deficit of highly eccentric short-period binary pulsars was attributed to selection effects in pulsar surveys.
- Conclusions are applicable to BH–NS and BH–BH binaries.

Compact star clusters

- Yet another scenario that can create inspiralling eccentric binaries with short periods involves compact star clusters. It was noted that the interplay between GW-induced dissipation and stellar scattering in the presence of an intermediate-mass black hole can create short-period highly eccentric binaries

  C. Hopman and T. Alexander (2005)

- A very recent attempt to model realistically compact clusters that are likely to be present in galactic centers indicates that compact binaries usually merge with eccentricities

  G. Kupi, P. Amaro-Seoane, and R. Spurzem (2006),

Related Earlier Work

∗ Peters and Mathews (1963,64), Seminal work

∗ Wahlquist (1987), Spacecraft Doppler detection of GW from CB, Newtonian GW polarization

∗ Lincoln Will (1990), Osculating orbital elements, Numerical integration, effects of eccentricity and dominant radiation damping on GW polarisations

∗ Wagoner Will (1976), Blanchet Schäfer(1989,1993), Junker Schäfer (1992), Rieth-Schäfer (1997) 1PN and 1.5PN FZ fluxes, GW polarizations for CB in eccentric orbits

∗ Moreno-Garrido, Buitrago and Mediavilla, (1994, 95) effect on GW polarizations of introducing by hand some secular effects either in the longitude of the periastron or in the semi-major axis and eccentricity.

Earlier Work

∗ MBM (1994), Martel Poisson (1999), Pierro et al (2001), Influence of eccentricity on the SNR in GWDA,
∗ Gusev et al (2002), Seto (1991) LISA will be sensitive to eccentric Galactic binary neutron stars and that by measuring their periastron advance, accurate estimates for the total mass of these binaries may be obtained
∗ Damour Schäfer (1987), Schäfer Wex (1994): 2PN GQKR Will Wiseman (1996), Gopakumar Iyer (1997) 2PN GWF, Energy Flux, AM Flux, Evoln of Orbital elts under 2PN GRR
∗ Gopakumar Iyer (2002), 2PN GW Polarisations. Effects of eccentricity, advance of periastron and orbital inclination on power spectrum of the dominant Newtonian part of the polarizations

Earlier Work

∗ GW from an eccentric binary, during that stage of inspiral where the GRR is so small that the orbital parameters can be treated as essentially unchanging over a few orbital periods (‘adiabatic approx’).

∗ Damour, Gopakumar and Iyer (2004) Analytic method for constructing high accuracy templates for the GW signals from the inspiral phase of compact binaries moving on quasi-elliptical orbits by improved “method of variation of constants” to combine the three time scales involved in the elliptical orbit case, namely, orbital period, periastron precession and radiation reaction time scales, without making the usual approximation of treating the radiative time scale as an adiabatic process.

Implication of Eccentricity for GWDA

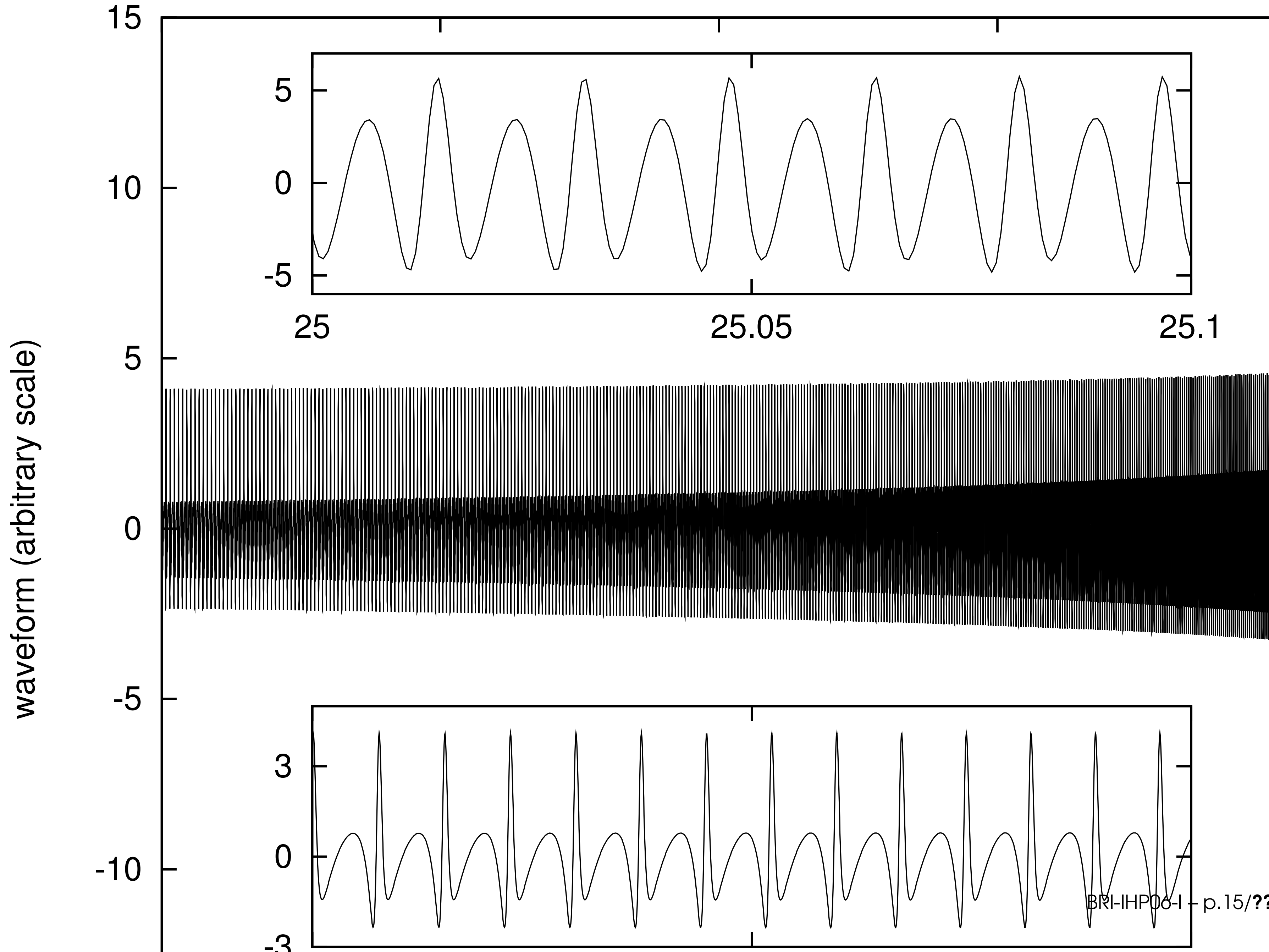
Martel Poisson

- Investigated reduction in SNR if eccentric signals are recd but searched for in data by circular templates - nonoptimal signal processing
- Found that for a binary system of given total mass, the *loss increases* with *increasing eccentricity*
- For a given eccentricity, *loss decreases* as *total mass is increased*
- Fitting factor (FF) (Apostolatos) to measure degree of optimality of a given set of templates. FF is the ratio of the *actual* signal-to-noise ratio, obtained when searching for eccentric signals using circular templates, to the SNR that would be obtained if eccentric templates were used.
- FF close to unity indicates that the circular templates are quite *effective* at capturing an eccentric signal. FF much smaller than unity indicates that the circular templates do poorly, and a set of eccentric templates would be required for a successful search.
- The loss in event rate caused by using nonoptimal templates is given by $1 - \mathrm{FF}^3$.

FF as fn of initial eccentricity e_0

e_0	$1.0 + 1.0$	$1.4 + 1.4$	$1.4 + 2.5$	$1.4 + 5.0$	$1.4 + 10$	$3.0 + 6.0$	$6.0 + 6.0$
.00	0.998	0.997	0.999	0.998	0.998	0.998	0.998
.05	0.960	0.976	0.985	0.992	0.992	0.998	0.996
.10	0.898	0.931	0.947	0.965	0.976	0.984	0.993
.15	0.836	0.879	0.902	0.930	0.946	0.961	0.975
.20	0.762	0.822	0.854	0.893	0.913	0.934	0.955
.25	0.695	0.761	0.802	0.852	0.885	0.903	0.930
.30	0.630	0.637	0.749	0.805	0.850	0.868	0.900
.35	0.569	0.581	0.693	0.753	0.811	0.829	0.867
.40	0.513	0.520	0.635	0.698	0.765	0.783	0.827
.45	0.454	0.460	0.574	0.637	0.714	0.732	0.781
.50	0.397	0.402	0.513	0.576	0.656	0.675	0.728
.55	0.348	0.350	0.452	0.513	0.595	0.614	–
.60	0.297	0.303	0.396	0.452	0.534	–	–
.65	0.257	0.231	0.344	–	–	–	–

Eccentric Signal



Eccentric Signal

- Plots of $s(t)$ (up to an overall scaling) for a $1.4 + 1.4$ binary system with initial eccentricity $e_0 = 0.5$. The main figure shows the waveform for its entire duration. The bottom inset shows the waveform at early times, when the eccentricity is still large. The top inset shows the waveform at late times, when the eccentricity is much reduced. Notice the monotonic increase of both the amplitude and frequency.

Biases

Table 1: Value of $\mathcal{M}/\mathcal{M}_{\rm actual}$ that maximizes the reduced ambiguity function.

e_0	$1.0+1.0$	$1.4+1.4$	$1.4+2.5$	$1.4+5.0$	$1.4+10$	$3.0+6.0$	$6.0+6.0$	8
.00	1.0000	0.9999	0.9999	0.9999	0.9997	0.9997	0.9994	
.05	1.0007	1.0006	1.0007	1.0007	1.0007	1.0006	1.0004	
.10	1.0012	1.0016	1.0017	1.0022	1.0030	1.0031	1.0033	
.15	1.0024	1.0027	1.0031	1.0039	1.0053	1.0056	1.0076	
.20	1.0037	1.0042	1.0048	1.0060	1.0083	1.0083	1.0113	
.25	1.0059	1.0059	1.0071	1.0087	1.0122	1.0122	1.0165	
.30	1.0088	1.0095	1.0106	1.0121	1.0172	1.0170	1.0228	
.35	1.0125	1.0134	1.0149	1.0167	1.0234	1.0232	1.0312	
.40	1.0182	1.0194	1.0210	1.0223	1.0319	1.0314	1.0418	
.45	1.0260	1.0288	1.0302	1.0307	1.0430	1.0416	1.0562	
.50	1.0404	1.0412	1.0447	1.0466	1.0586	1.0574	1.0774	
.55	1.0598	1.0654	1.0640	1.0680	1.0842	1.0749	–	
.60	1.0986	1.0946	1.0986	1.0976	1.1138	–	–	
.65	1.1508	1.1542	1.1484	–	–	–	–	

The Generation Modules

- Generation problem for GW at any PN order requires solution to two independent problems
- First relates to the equation of motion of the binary
- Second to FZ fluxes of energy, angular momentum
- Latter requires the computation of the relativistic mass and current multipole moments to appropriate PN orders.
- Unlike at earlier PN orders, the 3PN contribution to energy flux come not only from the 'instantaneous' terms but also include 'hereditary' contributions arising from the *tail of tails* and *tail-square* terms.

Present Work

- For binaries moving in general orbits, we compute *all* the instantaneous contributions to the 3PN accurate GW energy flux.
- Flux averaged over an elliptical orbit using 3PN quasi-Keplerian parametrization of the binary's orbital motion by Memmesheimer, Gopakumar and Schäfer
- Contributions from the hereditary terms computed exploiting the double periodicity of the PN motion
- Complete expressions for the far-zone energy flux from inspiralling compact binaries moving in eccentric orbits.

Present Work

- Represent GW from a binary evolving negligibly under GRR including precisely upto 3PN order, the effects of eccentricity and periastron precession during epochs of inspiral when the orbital parameters are essentially constant over a few orbital revolutions.
- First step towards the discussion of the *quasi-elliptical* case: the evolution of the binary in an elliptical orbit under GRR

FZ flux - Radiative Multipoles

Following Thorne (1980), the expression for the 3PN accurate far zone energy flux in terms of symmetric trace-free (STF) radiative multipole moments read as

$$\left(\frac{d\mathcal{E}}{dt}\right)_{\rm far-zone} = \frac{G}{c^5}\left\{\frac{1}{5}U^{(1)}_{ij}U^{(1)}_{ij}\right.$$
$$+\frac{1}{c^2}\left[\frac{1}{189}U^{(1)}_{ijk}U^{(1)}_{ijk}+\frac{16}{45}V^{(1)}_{ij}V^{(1)}_{ij}\right]$$
$$+\frac{1}{c^4}\left[\frac{1}{9072}U^{(1)}_{ijkm}U^{(1)}_{ijkm}+\frac{1}{84}V^{(1)}_{ijk}V^{(1)}_{ijk}\right]$$
$$+\frac{1}{c^6}\left[\frac{1}{594000}U^{(1)}_{ijkmn}U^{(1)}_{ijkmn}+\frac{4}{14175}V^{(1)}_{ijkm}V^{(1)}_{ijkm}\right]$$
$$\left.+\mathcal{O}(8)\right\}.$$

PN order of Multipoles

- For a given PN order only a finite number of Multipoles contribute
- At a given PN order the mass l-multipole is accompanied by the current $l-1$-multipole (Recall EM)
- To go to a higher PN order Flux requires new higher order l-multipoles and more importantly higher PN accuracy in the known multipoles.
- 3PN Energy flux requires 3PN accurate Mass Quadrupole, 2PN accurate Mass Octupole, 2PN accurate Current Quadrupole,........ N Mass 2^5-pole, Current 2^4-pole

Radiative moments - Source moments

The relations connecting the different radiative moments U_L and V_L to the corresponding source moments I_L and J_L are given below. For the mass type moments we have (Blanchet 92.. 98)

$$
\begin{aligned}
U_{ij}(U) = \; & I^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{2}\right] I^{(4)}_{ij}(U-\tau) \\
& + \frac{G}{c^5}\left\{ -\frac{2}{7}\int_0^{+\infty} d\tau I^{(3)}_{a<i}(U-\tau) I^{(3)}_{j>a}(U-\tau) \right. \\
& \quad + \frac{1}{7} I^{(5)}_{a<i} I_{j>a} - \frac{5}{7} I^{(4)}_{a<i} I^{(1)}_{j>a} - \frac{2}{7} I^{(3)}_{a<i} I^{(2)}_{j>a} + \frac{1}{3}\varepsilon_{ab<i} I^{(4)}_{j>a} J_b \\
& \quad \left. + 4\left[W^{(2)} I_{ij} - W^{(1)} I^{(1)}_{ij}\right] \right\} \\
& + 2\left(\frac{GM}{c^3}\right)^2 \int_0^{+\infty} d\tau I^{(5)}_{ij}(U-\tau) \\
& \left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right] \\
& + \mathcal{O}(7),
\end{aligned}
$$

Radiative moments - Source moments

$$
\begin{aligned}
U_{ijk}(U) &= I^{(3)}_{ijk}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{97}{60}\right] I^{(5)}_{ijk}(U-\tau) \\
&\quad + \mathcal{O}(5)\,, \\
U_{ijkm}(U) &= I^{(4)}_{ijkm}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{59}{30}\right] I^{(6)}_{ijkm}(U-\tau)\right. \\
&\qquad + \frac{2}{5}\int_0^{+\infty} d\tau I^{(3)}_{<ij}(U-\tau) I^{(3)}_{km>}(U-\tau) \\
&\qquad \left. - \frac{21}{5} I^{(5)}_{<ij} I_{km>} - \frac{63}{5} I^{(4)}_{<ij} I^{(1)}_{km>} - \frac{102}{5} I^{(3)}_{<ij} I^{(2)}_{km>}\right\} + \mathcal{O}(4)\,,
\end{aligned}
$$

Requires one to control the Reln of the Radiative Mass Quadrupole to Source Mass Quadrupole to 3PN accuracy. Hence involves *Tail-of-Tails* for Mass Quadrupole. Other multipoles to lower PN accuracy involving only *Tails*

Current-type moments

$$
\begin{aligned}
V_{ij}(U) &= J^{(2)}_{ij}(U) + \frac{2GM}{c^3}\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{7}{6}\right] J^{(4)}_{ij}(U-\tau) \\
&\quad + \mathcal{O}(5)\,, \\
V_{ijk}(U) &= J^{(3)}_{ijk}(U) + \frac{G}{c^3}\left\{2M\int_0^{+\infty} d\tau \left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{5}{3}\right] J^{(5)}_{ijk}(U-\tau)\right. \\
&\qquad \left. + \frac{1}{10}\varepsilon_{ab<i}I^{(5)}_{j\underline{a}}I_{k>b} - \frac{1}{2}\varepsilon_{ab<i}I^{(4)}_{j\underline{a}}I^{(1)}_{k>b} - 2J_{<i}I^{(4)}_{jk>}\right\} \\
&\quad + \mathcal{O}(4)\,.
\end{aligned}
$$

$$
\begin{aligned}
U_L(U) &= I^{(l)}_L(U) + \mathcal{O}(3)\,, \\
V_L(U) &= J^{(l)}_L(U) + \mathcal{O}(3)\,.
\end{aligned}
$$

$$
U = t - \frac{\rho}{c} - \frac{2\,G\,M}{c^3}\ln\left(\frac{\rho}{c\,r_0}\right)\,.
$$

3PN EOM for ICB

$$a^i = \frac{dv^i}{dt} = -\frac{m}{r^2}\left[(1+\mathcal{A}_E)\,n^i + \mathcal{B}_E\,v^i\right] + \mathcal{O}\left(\frac{1}{c^7}\right)\ ,$$

$$\begin{aligned}
\mathcal{A}_E &= \frac{1}{c^2}\left\{-\frac{3\,\dot{r}^2\,\nu}{2} + v^2 + 3\,\nu\,v^2 - \frac{m}{r}\,(4+2\,\nu)\right\} \\
&+ \frac{1}{c^4}(\cdots) + \frac{1}{c^5}(\cdots) + + \frac{1}{c^6}(\cdots) \\
\mathcal{B}_E &= \frac{1}{c^2}\left\{-4\,\dot{r} + 2\,\dot{r}\,\nu\right\} \\
&+ \frac{1}{c^4}(\cdots) + \frac{1}{c^5}(\cdots) + \frac{1}{c^6}(\cdots)
\end{aligned}$$

3PN Mass Quadrupole for ICB

$$I_{ij} = \nu\, m \left\{ \left[\mathcal{A} - \frac{24}{7}\,\frac{\nu}{c^5}\,\frac{G^2 m^2}{r^2}\,\dot{r} \right] x_{\langle i} x_{j\rangle} + \mathcal{B}\,\frac{r^2}{c^2}\, v_{\langle i} v_{j\rangle} + 2 \left[\mathcal{C}\,\frac{r\,\dot{r}}{c^2} + \frac{24}{7}\,\frac{\nu}{c^5}\,\frac{G^2 m^2}{r} \right] x_{\langle i} v_{j\rangle} \right\},$$

where

$$\mathcal{A} = 1 + \frac{1}{c^2}\left[v^2 \left(\frac{29}{42} - \frac{29\,\nu}{14} \right) + \frac{G\,m}{r}\left(-\frac{5}{7} + \frac{8}{7}\,\nu \right) \right] + \frac{1}{c^4}(\cdots) + \frac{1}{c^6}(\cdots)$$

$$\mathcal{B} = \frac{11}{21} - \frac{11}{7}\,\nu + \frac{1}{c^2}\left[\frac{Gm}{r}\left(\frac{106}{27} - \frac{335}{189}\,\nu - \frac{985}{189}\,\nu^2 \right) + v^2 \left(\frac{41}{126} - \frac{337}{126}\,\nu + \frac{733}{126}\,\nu^2 \right) + \dot{r}^2 \left(\frac{5}{63} - \frac{25}{63}\,\nu + \frac{25}{63}\,\nu^2 \right) \right] + \frac{1}{c^4}(\cdots) + \frac{1}{c^6}(\cdots)$$

$$\mathcal{C} = -\frac{2}{7} + \frac{6}{7}\,\nu + \frac{1}{c^2}\left[v^2 \left(-\frac{13}{63} + \frac{101}{63}\,\nu - \frac{209}{63}\,\nu^2 \right) + \frac{G\,m}{r}\left(-\frac{155}{108} + \frac{4057}{756}\,\nu + \frac{209}{108}\,\nu^2 \right) \right] + \frac{1}{c^4}(\cdots)$$

Instantaneous Terms

$$\left(\frac{d\mathcal{E}}{dt}\right) = \left(\frac{d\mathcal{E}}{dt}\right)_{\rm inst} + \left(\frac{d\mathcal{E}}{dt}\right)_{\rm hered}.$$

$$\begin{aligned}\left(\frac{d\mathcal{E}}{dt}\right)_{\rm inst} = & \frac{G}{c^5}\Bigg\{\frac{1}{5}I^{(3)}_{ij}I^{(3)}_{ij} \\ & +\frac{1}{c^2}\left[\frac{1}{189}I^{(4)}_{ijk}I^{(4)}_{ijk}+\frac{16}{45}J^{(3)}_{ij}J^{(3)}_{ij}\right]+\frac{1}{c^4}\left[\frac{1}{9072}I^{(5)}_{ijkm}I^{(5)}_{ijkm}+\frac{1}{84}J^{(4)}_{ijk}J^{(4)}_{ijk}\right] \\ & +\frac{8G}{5c^5}\left\{I^{(3)}_{ij}\left[I_{ij}W^{(5)}+2I^{(1)}_{ij}W^{(4)}-2I^{(3)}_{ij}W^{(2)}-I^{(4)}_{ij}W^{(1)}\right]\right\} \\ & +\frac{2G}{5c^5}I^{(3)}_{ij}\Bigg\{-\frac{4}{7}I^{(5)}_{ai}I^{(1)}_{aj}-I^{(4)}_{ai}I^{(2)}_{aj}-\frac{4}{7}I^{(3)}_{ai}I^{(3)}_{aj}+\frac{1}{7}I^{(6)}_{ai}I_{aj}+ \\ & \frac{1}{3}\epsilon_{abi}\left(I^{(4)}_{aj}J^{(1)}_{b}+I^{(5)}_{aj}J_{b}\right)\Bigg\} \\ & +\frac{1}{c^6}\left[\frac{1}{594000}I^{(6)}_{ijkmn}I^{(6)}_{ijkmn}+\frac{4}{14175}J^{(5)}_{ijkm}J^{(5)}_{ijkm}\right]+\mathcal{O}(8)\Bigg\}.\end{aligned}$$

3PN Instantaneous Terms

$$\left(\frac{d\mathcal{E}}{dt}\right) = \left[\left(\frac{d\mathcal{E}}{dt}\right)^{0\mathrm{PN}} + \left(\frac{d\mathcal{E}}{dt}\right)^{1\mathrm{PN}} + \left(\frac{d\mathcal{E}}{dt}\right)^{2\mathrm{PN}} + \left(\frac{d\mathcal{E}}{dt}\right)^{2.5\mathrm{PN}} + \left(\frac{d\mathcal{E}}{dt}\right)^{3\mathrm{PN}}\right]_{\mathrm{inst}} + \left(\frac{d\mathcal{E}}{dt}\right)_{\mathrm{her}} + \mathcal{O}(7)\,,$$

$$\left(\frac{d\mathcal{E}}{dt}\right)^{0\mathrm{PN}} = \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^5\,r^4}\left\{v^2 - \frac{11}{12}\dot{r}^2\right\},$$

$$\left(\frac{d\mathcal{E}}{dt}\right)^{1\mathrm{PN}} = \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^7\,r^4}\left\{v^4\left(\frac{785}{336} - \frac{71}{28}\nu\right) + \dot{r}^2\,v^2\left(-\frac{1487}{168} + \frac{58}{7}\nu\right) + \frac{G\,m}{r}v^2\left(-\frac{170}{21} + \frac{10}{21}\nu\right) + \dot{r}^4\left(\frac{687}{112} - \frac{155}{28}\nu\right) + \frac{G\,m}{r}\dot{r}^2\left(\frac{367}{42} - \frac{5}{14}\nu\right) + \frac{G^2\,m^2}{r^2}\left(\frac{1}{21} - \frac{4}{21}\nu\right)\right\},$$

$$\left(\frac{d\mathcal{E}}{dt}\right)^{2.5\mathrm{PN}} = \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^{10}\,r^4}\left\{\dot{r}\,\nu\left(-\frac{12349}{210}\frac{G\,m}{r}v^4 + \frac{4524}{35}\frac{G\,m}{r}v^2\dot{r}^2 - \frac{2753}{126}\frac{G^2\,m^2}{r^2}v^2 - \frac{985}{14}\frac{G\,m}{r}\dot{r}^4 + \frac{13981}{630}\frac{G^2\,m^2}{r^2}\dot{r}^2 - \frac{1}{315}\frac{G^3\,m^3}{r^3}\right)\right\},$$

3PN Instantaneous Terms

$$
\begin{aligned}
\left(\frac{d\mathcal{E}}{dt}\right)^{\mathrm{2PN}} = \; & \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^9\,r^4}\left\{v^6\left(\frac{47}{14}-\frac{5497}{504}\nu+\frac{2215}{252}\nu^2\right)\right. \\
& +\dot{r}^2v^4\left(-\frac{573}{56}+\frac{1713}{28}\nu-\frac{1573}{42}\nu^2\right) \\
& +\frac{G\,m}{r}v^4\left(-\frac{247}{14}+\frac{5237}{252}\nu-\frac{199}{36}\nu^2\right) \\
& +\dot{r}^4v^2\left(\frac{1009}{84}-\frac{5069}{56}\nu+\frac{631}{14}\nu^2\right) \\
& +\frac{G\,m}{r}\dot{r}^2v^2\left(\frac{4987}{84}-\frac{8513}{84}\nu+\frac{2165}{84}\nu^2\right) \\
& +\frac{G^2\,m^2}{r^2}v^2\left(\frac{281473}{9072}+\frac{2273}{252}\nu+\frac{13}{27}\nu^2\right) \\
& +\dot{r}^6\left(-\frac{2501}{504}+\frac{10117}{252}\nu-\frac{2101}{126}\nu^2\right) \\
& +\frac{G\,m}{r}\dot{r}^4\left(-\frac{5585}{126}+\frac{60971}{756}\nu-\frac{7145}{378}\nu^2\right) \\
& +\frac{G^2\,m^2}{r^2}\dot{r}^2\left(-\frac{106319}{3024}-\frac{1633}{504}\nu-\frac{16}{9}\nu^2\right) \\
& \left.+\frac{G^3\,m^3}{r^3}\left(-\frac{253}{378}+\frac{19}{7}\nu-\frac{4}{27}\nu^2\right)\right\},
\end{aligned}
$$

3PN Instantaneous Terms

$$\left(\frac{d\mathcal{E}}{dt}\right)^{3\mathrm{PN}} = \frac{32}{5}\frac{G^3\,m^4\,\nu^2}{c^{11}\,r^4}\left\{v^8\ldots\ldots\right\}$$

The result for 3PN terms involvesi two types of log terms
Gauge dependent Log terms $(\log r_0')$ and
Log terms arising from the regularisation of the moments at infinity $(\log r_0)$

Transfn of World lines

- Having obtained the energy flux in GW we next wish to average this expression over an orbit
- This is required to compute the evolution of the elliptical orbit undr Grav Radn reaction (GRR)
- A technical obstacle is that the standard harmonic coords in which the energy flux is computed involves log terms in its description of the motion (accn) and radiation (GW flux)
- It is not possible to extend the 2PN GQKR to 3PN if these terms are present and one needs to transform to other gauges like Modified harmonic coors or ADM coords which do not contain logs and that is what we do
- This is most conveniently implemented by a transformation of world lines which we employ

Energy Flux - Modified Harmonic Coords

Logs can be removed by the following shift on the particle world-lines:

$$\xi_1^i = \frac{22}{3}\frac{m_1^2 m_2}{c^6\, r^2}\, n^i \,\ln\left(\frac{r}{r_1'}\right),$$

$$\xi_2^i = -\frac{22}{3}\frac{m_1 m_2^2}{c^6\, r^2}\, n^i \,\ln\left(\frac{r}{r_2'}\right).$$

Under shift ξ, Accn of First particle shifts by

$$\delta_\xi a_1^i = \ddot{\xi}_1^i - \left(\xi_1^j - \xi_2^j\right)\partial_j a_1^i .$$

Rel Accn shifts by

$$\delta_\xi a^i = -\frac{m^3\nu}{r^4}\left\{\left[\left(110\dot{r}^2 - 22v^2\right)n^i - 44\dot{r}v^i\right]\ln\left(\frac{r}{r_0'}\right) + \left(-\frac{176}{3}\dot{r}^2 + \frac{22}{3}v^2 - \frac{22}{3}\frac{m}{r}\right)n^i + \frac{44}{3}\dot{r}v^i\right\}$$

Log dependence in the above transformation exactly cancels the log dependence of the acceleration in standard harmonic coordinates. Some 3PN coefficients in the EOM are also modified and the final result for Accn in Modified Harm Coords. agrees with that displayed in e.g. Mora Will

Energy Flux - Modified Harmonic Coords

The only other modification vis a vis the calculation of the energy flux in standard harmonic coordinates is the part related to the mass quadrupole which must be computed to 3PN accuracy.
Under the above shift formula the mass quadruple I_{ij} is shifted by

$$\delta_\xi I_{ij} = STF_{ij}\left(\frac{44}{3}\frac{m^4\,\nu^2}{r^3}\ln\left(\frac{r}{r_0'}\right)x_{ij}\right),$$

Exactly cancels the $\ln r_0'$ dependence of the mass quadrupole in standard harmonic coordinates.
Finally,

$$\left(\dot{\mathcal{E}}\right)_{\rm Mhar} = \left(\dot{\mathcal{E}}\right)_{\rm Shar\to Mhar} + \frac{G^6 m^7 \nu^3}{c^{11} r^7}\left\{\frac{352}{45}\dot{r}^2 + \left(-\frac{1408}{15}v^2 + \frac{2816}{45}\dot{r}^2\right)\log\left(\frac{r}{r_0'}\right)\right.$$

- $\ln r_0'$ above exactly cancels the $\ln r_0'$ dependence of the Energy Flux in standard harmonic coordinates. E Flux in MHar coords independent of $\ln r_0'$.
- The Variables are Mhar variables

The Keplerian representation

The Keplerian parametrisation of a particle moving in a general orbit with $0 \le e \le 1$) is given by:

$$
\begin{aligned}
r &= a\,(1 - e\,\cos u)\ , \\
l \equiv n\,(t - t_0) &= u - e\,\sin u\,, \\
(\phi - \phi_0) &= V\,, \\
\text{where,}\ \ V &= 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\right].
\end{aligned}
$$

- The three angles V, u and l (measured from the perhelion) are called the true anomaly, the eccentric anomaly and the mean anomaly respectively.
- The orbit has semi-major axis a, eccentricity e and mean motion n.

3PN generalised Quasi-Keplerian reprn

- Quasi-Keplerian representation at 1PN was introduced by Damour and Deruelle 1985 to discuss the problem of binary pulsar timing.
- At 1PN relativistic periastron precession first appears and complicates the simpler Newtonian picture.
- This elegant formulation will play a crucial role in the our computation of the hereditary terms
- 2PN extension in ADM coordinates was next given by Damour, Schäfer (1988) and Wex (1993, 1995) (Generalized QKR).
- 3PN parametrization of the orbital motion of the binary was constructed by Memmeshiemer, Gopakumar and Schäfer (2004) in both ADM and modified harmonic coordinates.

3PN generalised Quasi-Keplerian reprn

$$
\begin{aligned}
r &= a_r \left(1 - e_r \cos u\right) \,, \\
l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(V - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right) \sin V + \frac{i_{6t}}{c^6} \sin 2\,V + \frac{h_{6t}}{c^6} \sin 3\,V \,, \\
\frac{2\,\pi}{\Phi}(\phi - \phi_0) &= V + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right) \sin 2V + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right) \sin 3V \\
&\quad + \frac{i_{6\phi}}{c^6} \sin 4V + \frac{h_{6\phi}}{c^6} \sin 5V \,, \\
\text{where,}\ \ V &= 2 \arctan\left[\left(\frac{1 + e_\phi}{1 - e_\phi}\right)^{1/2} \tan \frac{u}{2}\right].
\end{aligned}
$$

Details: Schäfer's Lectures

3PN generalised Quasi-Keplerian reprn

- V is the 3PN generalisation of the Keplerian true anomaly.
- a_r, e_r, l, u, n, e_t, e_ϕ and $2\pi/\Phi$ are some 3PN accurate semi-major axis, radial eccentricity, mean anomaly, eccentric anomaly, mean motion, 'time' eccentricity, angular eccentricity and angle of advance of periastron per orbital revolution respectively.
- Eqns contain three kinds of 'eccentricities' e_t, e_r and e_ϕ labelled after the coordinates t, r, and ϕ respectively. Differ from each other starting at the 1PN order.
- $\Phi/2\pi \equiv K = 1 + k$
- Presense of log terms in Std Harmonic coords obstructs the construction of GQKR which crucially exploits the fact that at order 3PN the radial equation is a fourth order *polynomial* in $1/r$.
- MGS04 thus construct the GQKR for Modified Harmonic coords.
- GQKR in Modified Harmonic coords is of the same form as for ADM but the corresponding eqns for the orbital elements are different.

3PN GQKR - Mhar

$$
\begin{aligned}
a_r \quad = \quad & \frac{1}{(-2E)}\Bigg\{1+\frac{(-2E)}{4\,c^2}\,(-7+\nu)+\frac{(-2E)^2}{16c^4}\Bigg[1+\nu^2 \\
& +\frac{16}{(-2Eh^2)}(-4+7\,\nu)\Bigg]+\frac{(-2E)^3}{6720\,c^6}\Bigg[105-105\,\nu \\
& +105\,\nu^3+\frac{1}{(-2Eh^2)}\Bigg(26880+4305\,\pi^2\nu-215408\,\nu \\
& +47040\,\nu^2\Bigg)-\frac{4}{(-2Eh^2)^2}\Bigg(53760-176024\,\nu+4305\,\pi^2\nu \\
& +15120\,\nu^2\Bigg)\Bigg]\Bigg\},
\end{aligned}
$$

3PN GQKR - Mhar

$$
\begin{aligned}
n &= (-2E)^{3/2}\left\{1+\frac{(-2E)}{8\,c^2}\,(-15+\nu)+\frac{(-2E)^2}{128\,c^4}\,\left[555+30\,\nu\right.\right.\\
&\left.+11\,\nu^2+\frac{192}{\sqrt{(-2Eh^2)}}(-5+2\,\nu)\right]+\frac{(-2E)^3}{3072\,c^6}\,\left[-29385\right.\\
&-4995\,\nu-315\,\nu^2+135\,\nu^3+\frac{5760}{\sqrt{(-2Eh^2)}}(17-9\,\nu+2\,\nu^2)\\
&\left.\left.-\frac{16}{(-2Eh^2)^{3/2}}\left(10080-13952\,\nu+123\,\pi^2\nu+1440\,\nu^2\right)\right]\right\},
\end{aligned}
$$

3PN GQKR - Mhar

$$
\begin{aligned}
{e_t}^2 \quad = \quad & 1 + 2Eh^2 + \frac{(-2E)}{4\,c^2}\left\{ -8 + 8\,\nu - (-2Eh^2)(-17 + 7\,\nu) \right\} \\
& + \frac{(-2E)^2}{8\,c^4}\left\{ 12 + 72\,\nu + 20\,\nu^2 - 24\sqrt{(-2Eh^2)}\,(-5 + 2\,\nu) \right. \\
& - (-2Eh^2)(112 - 47\,\nu + 16\,\nu^2) - \frac{16}{(-2Eh^2)}(-4 + 7\,\nu) \\
& \left. + \frac{24}{\sqrt{(-2Eh^2)}}(-5 + 2\,\nu) \right\} + \frac{(-2E)^3}{6720\,c^6}\left\{ 23520 - 464800\,\nu \right. \\
& + 179760\,\nu^2 + 16800\,\nu^3 - 2520\sqrt{(-2Eh^2)}(265 - 193\,\nu \\
& + 46\,\nu^2) - 525(-2Eh^2)\left(-528 + 200\,\nu - 77\,\nu^2 + 24\,\nu^3 \right) \\
& - \frac{6}{(-2Eh^2)}\left(73920 - 260272\,\nu + 4305\,\pi^2\nu + 61040\,\nu^2 \right)
\end{aligned}
$$

3PN GQKR - Mhar

$$
\begin{aligned}
&+\frac{70}{\sqrt{(-2Eh^2)}}\left(16380-19964\,\nu+123\,\pi^2\nu+3240\,\nu^2\right)\\
&+\frac{8}{(-2Eh^2)^2}\left(53760-176024\,\nu+4305\,\pi^2\nu+15120\,\nu^2\right)\\
&-\frac{70}{(-2Eh^2)^{3/2}}\left(10080-13952\,\nu+123\,\pi^2\nu+1440\,\nu^2\right)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\Phi \quad = \quad & 2\,\pi\Bigg\{1+\frac{3}{h^2c^2}+\frac{(-2E)^2}{4\,c^4}\left[\frac{3}{(-2Eh^2)}(-5+2\,\nu)\right.\\
&\left.-\frac{15}{(-2Eh^2)^2}(-7+2\,\nu)\right]+\frac{(-2E)^3}{128\,c^6}\left[\frac{5}{(-2Eh^2)^3}\Big(7392\right.\\
&-8000\,\nu+336\,\nu^2+123\,\pi^2\nu\Big)+\frac{24}{(-2Eh^2)}(5-5\,\nu+4\,\nu^2)\\
&\left.-\frac{1}{(-2Eh^2)^2}\left(10080-13952\,\nu+123\,\pi^2\nu+1440\,\nu^2\right)\right]\Bigg\},
\end{aligned}
$$

Gauge Invariant Variables

- Memmesheimer, Gopakumar and Schäfer (2004) stress use of gauge invariant variables in the elliptical orbit case
- Damour and Schäfer (1988) showed that the functional form of n and Φ as functions of gauge invariant variables like E and h is the same in different coordinate systems (gauges).
- From the explicit expressions for n and Φ in the ADM and modified harmonic coordinates the gauge invariance of these two parameters is explicit at 3PN.
- MGS04 suggest the use of variables $x_{\rm MGS} = (Gmn/c^3)^{2/3}$ and $k' = (\Phi - 2\pi)/6\pi$ as gauge invariant variables in the general orbit case.
- We propose a variant of the former: $x = (Gmn\,\Phi/2\pi c^3)^{2/3} = (Gmn\,Kc^3)^{2/3} = (Gmn\,(1+k)c^3)^{2/3}$.
- Our choice is the obvious generalisation of gauge invariant variable x in the circular orbit case and thus facilitates the straightforward reading out of the circular orbit limit.

Orbital average - Energy flux -MHar

- To average the energy flux over an orbit requires use of 3PN GQKR $\rightarrow$ Modified Harmonic coords
- Involves evaluation of the the integral,

$$\langle\dot{\mathcal{E}}\rangle = \frac{1}{P}\int_0^P \dot{\mathcal{E}}(t)\,dt = \frac{1}{2\,\pi}\int_0^{2\,\pi}\frac{dl}{du}\,\dot{\mathcal{E}}(u)du\,.$$

- Using GQKR, transform the expression for the energy flux $\dot{\mathcal{E}}\,(r, \dot{r}^2, v^2)$ or more exactly $(dl/du \times \dot{\mathcal{E}})(r, \dot{r}^2, v^2)$ to $(dl/du \times \dot{\mathcal{E}})(x, e_t, u)$.
- Choices: GI97 uses Gm/a_r and e_r. DGI04 employs Gmn/c^3 and e_t. ABIQ06 uses e_t and $x = (Gmn\Phi/2\pi c^3)^{2/3}$
- Recall: 3PN flux contains log terms; convenient to rewrite the expression as

$$\frac{dl}{du}\,\dot{\mathcal{E}} = \sum_{N=3}^{11}\left\{\alpha_N(e_t)\frac{1}{(1-e_t\cos u)^N} + \beta_N(e_t)\frac{\sin u}{(1-e_t\cos u)^N} + \gamma_N(e_t)\frac{\ln(1-e_t\cos u)}{(1-e_t\cos u)^N}\right.$$

- Non-vanishing α_N's, β_N's and γ_N's are too long to be listed.
- β_N's correspond to all the 2.5PN terms.
  γ_N represent the log terms at order 3PN.

Orbit Averaged Energy Flux - MHar

$$
\begin{aligned}
< \dot{\mathcal{E}} >_{\rm MHar} \quad = \quad & \frac{32\nu^2 x^5}{5} \frac{1}{(1-e_t^2)^{7/2}} \Big(< \dot{\mathcal{E}}_{\rm N} >_{\rm MHar} + x < \dot{\mathcal{E}}_{\rm 1PN} >_{\rm MHar} \\
& + x^2 < \dot{\mathcal{E}}_{\rm 2PN} >_{\rm MHar} + x^3 < \dot{\mathcal{E}}_{\rm 3PN} >_{\rm MHar} \Big) \,.
\end{aligned}
$$

$$
\begin{aligned}
< \dot{\mathcal{E}}_N >_{\rm Mhar} \quad &= \quad 1 + e_t^2 \, \frac{73}{24} + e_t^4 \, \frac{37}{96} \,, \\
< \dot{\mathcal{E}}_{\rm 1PN} >_{\rm Mhar} \quad &= \quad \frac{1}{(1-e_t^2)} \\
& \left\{ \left(-\frac{1247}{336} - \frac{35}{12}\nu \right) + e_t^2 \left(\frac{10475}{672} - \frac{1081}{36}\nu \right) \right. \\
& \left. + e_t^4 \left(\frac{10043}{384} - \frac{311}{12}\nu \right) + e_t^6 \left(\frac{2179}{1792} - \frac{851}{576}\nu \right) \right\} ,
\end{aligned}
$$

Orbit Averaged Energy Flux - MHar

$$
\begin{aligned}
< \dot{\mathcal{E}}_{\rm 2PN} >_{\rm Mhar} \quad = \quad & \frac{1}{\left(1-e_t^2\right)^2} \\
& \left\{ -\frac{203471}{9072} + \frac{12799}{504}\nu + \frac{65}{18}\nu^2 + e_t^2 \left(-\frac{3807197}{18144} + \frac{116789}{2016}\nu + \frac{5935}{54}\nu^2 \right. \right. \\
& + e_t^4 \left(-\frac{268447}{24192} - \frac{2465027}{8064}\nu + \frac{247805}{864}\nu^2 \right) \\
& + e_t^6 \left(\frac{1307105}{16128} - \frac{416945}{2688}\nu + \frac{185305}{1728}\nu^2 \right) \\
& + e_t^8 \left(\frac{86567}{64512} - \frac{9769}{4608}\nu + \frac{21275}{6912}\nu^2 \right) \\
& + \sqrt{1-e_t^2} \left[\left(\frac{35}{2} - 7\nu \right) + e_t^2 \left(\frac{6425}{48} - \frac{1285}{24}\nu \right) \right. \\
& \left. \left. + e_t^4 \left(\frac{5065}{64} - \frac{1013}{32}\nu \right) + e_t^6 \left(\frac{185}{96} - \frac{37}{48}\nu \right) \right] \right\},
\end{aligned}
$$

Orbit Averaged Energy Flux - MHar

$$< \dot{\mathcal{E}}_{\rm 3PN} >_{\rm Mhar} \quad = \quad \frac{1}{\left(1 - e_t^2\right)^3} \left(\cdots\cdots\right)$$

Comments

- Note: *No* term at 2.5PN. 2.5PN contribution is proportional to $\dot{r}$ and vanishes after averaging since it always includes only 'odd' terms.
- e_t represents eccentricity in Modified harmonic coordinates e_t^{MHar}.
- x is gauge invariant. No such label is required on it.
- Important to keep track when comparing formulas in different gauges.
- Circular orbit limit - setting $e_t = 0$,

$$
\begin{aligned}
< \dot{\mathcal{E}} > |_{\odot} \quad = \quad & \frac{32}{5} x^5 \nu^2 \Big\{ 1 + x \left(-\frac{1247}{336} - \frac{35}{12}\nu \right) + x^2 \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2 \right. \\
& x^3 \left(\frac{1266161801}{9979200} - \frac{1712}{105} \ln\left(\frac{Gm}{c^2 x r_0} \right) + \left[-\frac{14930989}{272160} + \frac{41}{48}\pi^2 - \frac{8}{3} \right. \right. \\
& \left. -\frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \right) \Big\} .
\end{aligned}
$$

- Exact agreement with BIJ02 after converting the $\gamma = Gm/c^2 r_{\mathrm{SHar}}$ to the gauge invariant variable x.
- This is *only* instantaneous contribution

Comments

- No 2.5PN term in the energy flux after averaging.
- Circular orbit limit as expected is in agreement with BIJ
- Newtonian and 1PN orders have the same form in Mhar coords and ADM coordinates because two coordinates differ starting only at 2PN.
- e_t in the above expression represents e_t^{MHar}, the time eccentricity in Modified harmonic coordinates.

Comments

- Useful internal consistency check of the algebraic correctness of different representations of the energy flux, the coordinate transformations linking the various gauges, and the work of MGS04 on the construction of the 3PN GQKR is verification of the equality Mhar and ADM results using the following transformation between the time eccenticities $e_t^{\rm MHar}$ and $e_t^{\rm ADM}$:

$$
\begin{aligned}
e_t^{\rm MHar} \;=\;& e_t^{\rm ADM}\left\{1-\frac{x^2}{(1-e_t^2)}\left(\frac{1}{4}+\frac{17}{4}\nu\right)-\right. \\
& \frac{x^3}{(1-e_t^2)^2}\left(\frac{1}{2}+\frac{1}{2}e_t^2+\left(\frac{16739}{1680}-\frac{21\pi^2}{16}+\frac{249}{16}e_t^2\right)\nu\right. \\
& \left.\left.-\left(\frac{83}{24}+\frac{241}{24}e_t^2\right)\nu^2\right)\right\}
\end{aligned}
$$

- Relation derives from using Mhar and ADM Eqns and rewriting the E and h^2 dependence in terms of x and e_t.
- No ambiguity in not having a label on the e_t in the 2PN and 3PN terms above.

Energy Flux - Gauge Invariant Variables

- Energy flux represented using x a gauge invariant variable and e_t which however is coordinate dependent.
- e_t is useful in extracting the circular limit for which it has value zero.
- Rewrite the flux in terms of *two* gauge invariant observables defined earlier: x and k'.
- Start from average energy flux in terms of variables x and e_t. Rewrite e_t in terms of x and k'
- Alternatively work from the beginning with the expression for the flux in terms of x and k'.
- Both lead to the same results. The computation can be done independently both in Mhar and ADM coords
- Final result is identical proving the gauge invariance of the energy flux and providing a gauge invariant expression of the energy flux.

Gauge Invariant Variables

$$< \dot{\mathcal{E}} > \;=\; \frac{32\nu^2 x^5}{5}\left(\frac{x}{k'}\right)^{-13/2}\left(< \dot{\mathcal{E}}_{\rm N} > +x < \dot{\mathcal{E}}_{\rm 1PN} > +x^2 < \dot{\mathcal{E}}_{\rm 2PN} > \right.$$
$$\left. +x^3 < \dot{\mathcal{E}}_{\rm 3PN} > \right).$$

$$< \dot{\mathcal{E}}_N > \;=\; \left(\frac{x}{k'}\right)^3 \frac{425}{96} + \left(\frac{x}{k'}\right)^4 \left(-\frac{61}{16}\right) + \left(\frac{x}{k'}\right)^5 \frac{37}{96},$$
$$< \dot{\mathcal{E}}_{\rm 1PN} > \;=\; \left\{\left(\frac{x}{k'}\right)^2\left(-\frac{289}{3}+\frac{3605}{384}\nu\right) + \left(\frac{x}{k'}\right)^3\left(\frac{1865}{24}+\frac{3775}{384}\nu\right)\right.$$
$$\left. + \left(\frac{x}{k'}\right)^4\left(-\frac{5297}{336}-\frac{2725}{384}\nu\right) + \left(\frac{x}{k'}\right)^5\left(\frac{139}{112}+\frac{259}{1152}\nu\right)\right\},$$

Gauge Invariant Variables

$$
\begin{aligned}
< \dot{\mathcal{E}}_{2\mathrm{PN}} > \quad = \quad & \left\{ \frac{x}{k'} \left(\frac{267725837}{258048} + \left[\frac{1440583}{2304} - \frac{609875}{24576} \pi^2 \right] \nu + \frac{24395}{1024} \nu^2 \right) \right. \\
& + \left(\frac{x}{k'} \right)^2 \left(- \frac{51894953}{82944} + \left[- \frac{583921}{512} + \frac{497125}{24576} \pi^2 \right] \nu + \frac{1625}{48} \nu^2 \right) \\
& + \left(\frac{x}{k'} \right)^3 \left(\frac{49183667}{387072} + \left[\frac{14718145}{32256} - \frac{32595}{8192} \pi^2 \right] \nu + \frac{37145}{4608} \nu^2 \right) \\
& + \left(\frac{x}{k'} \right)^{7/2} \left(- \frac{305}{16} + \frac{61}{8} \nu \right) + \left(\frac{x}{k'} \right)^4 \left(- \frac{2145781}{64512} \right. \\
& \left. + \left[- \frac{505639}{10752} + \frac{1517}{8192} \pi^2 \right] \nu - \frac{105}{16} \nu^2 \right) \\
& \left. + \left(\frac{x}{k'} \right)^{9/2} \left(\frac{185}{48} - \frac{37}{24} \nu \right) + \left(\frac{x}{k'} \right)^5 \left(\frac{744545}{258048} + \frac{19073}{32256} \nu + \frac{2849}{27648} \nu^2 \right) \right\} ,
\end{aligned}
$$

Gauge Invariant Variables

$$< \dot{\mathcal{E}}_{\rm 3PN} > \quad = \quad \left\{ \frac{149899221067}{7741440} + \left[-\frac{186950547065}{3096576} + \frac{46739713}{32768}\pi^2 \right] \nu \cdots \right.$$

Hereditary Contributions

- Multipole moments describing GW emitted by an isolated system cannot evolve independently. They couple to each other and with themselves, giving rise to non-linear physical effects.
- Instantaneous terms in the flux must be supplemented by the contributions arising from these non-linear multipole interactions.
- We set up a general theoretical framework to compute the hereditary contributions for binaries moving in elliptical orbits and apply it to evaluate all the tail contributions contained in the 3PN accurate GW energy flux. (TALK AT WORKSHOP)

Hereditary Contributions

$$\mathcal{F}^{\rm 3PN}_{\rm tail} = \frac{32}{5}\,\nu^2\,x^5\Bigg\{4\pi\,x^{3/2}\,\varphi(e_t) + \pi\,x^{5/2}\left[-\frac{8191}{672}\,\psi(e_t) - \frac{583}{24}\nu\,\theta'(e_t)\right]$$
$$+x^3\left[-\frac{116761}{3675}\,\kappa(e_t) + \left[\frac{16}{3}\pi^2 - \frac{1712}{105}C - \frac{1712}{105}\ln\left(\frac{4\omega r_0}{c}\right)\right]F(e_t)\right.$$

- All the enhancement functions are defined in such a way that they reduce to one in the circular case, $e_t = 0$, so that the circular-limit of the formula is immediately seen from inspection and seen to be in complete agreement with Blanchet (98), Blanchet, Iyer Joguet (02)
- There are four enhancement functions which probably do not admit any analytic closed-form expressions: these are $\varphi(e_t)$, $\psi(e_t)$, $\theta(e_t)$ and $\kappa(e_t)$.

Log terms in total energy flux

- Result finally depends on the constant $r_0 = \tau_0/2$ at 3PN. Next discuss in detail structure of the Log term in the complete energy flux, the cancellation of the $\ln r_0$ term and the circular orbit limit of this term for one last final check of this complicated calculation.
- *Log terms in the instantaneous contribution* to the average flux is given by

$$\frac{32}{5}\nu^2\, x^5 \left\{ \frac{1712}{105} F(e_t) \ln\left[x \left(\frac{c^2\, r_0}{Gm}\right) \frac{1+\sqrt{1-e_t^2}}{2\,(1-e_t^2)} \right] \right\}.$$

- *Log terms in the tail contribution* to the average flux is

$$\frac{32}{5}\nu^2\, x^5 \left\{ -\frac{1712}{105} F(e_t) \ln\left[4x^{3/2} \left(\frac{c^2\, r_0}{Gm}\right) \right] \right\}.$$

Log terms in total energy flux

- Summing up, the *log terms in the total 3PN energy flux*

$$-\frac{32\nu^2 x^5}{5}\,\frac{1712}{105}\,F(e_t)\ln\left[\frac{8\sqrt{x}\,(1-e_t^2)}{1+\sqrt{1-e_t^2}}\right].$$

- Dependence on r_0 cancels as expected from general considerations providing a check on the algebra. Moreover, in the circular limit, $F(0)=1$ and the net result for the log term in the average flux is $-\frac{856}{105}\,\ln 16\,x$, in perfect agreement with BIJ 02

- To understand in more detail the occurence of this constant remind that the dependence of the radiative-type quadrupole moment at infinity, say U_{ij}, in terms of the constant r_0 arises at 3PN order, exclusively from the tails of tails (*i.e.* the multipole interaction $\propto M^2\times I_{ij}$), and given by

$$U_{ij}(t) = I^{(3)}_{ij}(t) + \cdots + \frac{214}{105}\,M^2\,I^{(4)}_{ij}(t)\,\ln r_0 + \cdots,$$

Log terms in total energy flux

- At the lowest Newtonian order U_{ij} reduces to the second time derivative of I_{ij}, and where the dots indicate all the terms which do not depend on r_0.
- Trivial to deduce that the corresponding dependence of the tail part of the energy flux on r_0 is given by

$$\mathcal{F}_{\rm tail} = \cdots - \frac{428}{525}\, M^2 \,\langle I^{(4)}_{ij} I^{(4)}_{ij} \rangle \,\ln r_0 + \cdots ,$$

  where inside the time average operation $\langle\rangle$ one can freely operate by parts the time derivatives. Hence, $\langle I^{(3)}_{ij} I^{(5)}_{ij} \rangle = -\langle I^{(4)}_{ij} I^{(4)}_{ij} \rangle$
- Thus, the effect looks like a "quadrupole formula" but where the third time derivative of the moment is replaced by the fourth one.

Log terms in total energy flux

- FZ total energy flux is in terms of the radiative moments,and is true for any PN source, and in particular for a binary system moving on eccentric orbit.
- Thus dependence on eccentricity e_t of the coefficient of $\ln r_0$ must necessarily be given by the function

$$F(e_t) = \frac{\omega^8}{128} \langle \hat{I}_{ij}^{(4)} \hat{I}_{ij}^{(4)} \rangle = \frac{1}{64} \sum_{p=1}^{+\infty} p^8 | \underset{(p)}{\hat{\mathcal{I}}}_{ij} |^2,$$

using reduced quadrupole moment

- The result is thus perfectly in agreement with our finding of the function $F(e)$. The dependence of the tail part of the averaged energy flux on the constant r_0 is such that it cancels out, for any value of the eccentricity, with a similar term coming from the instantaneous part of the flux. Of course such cancellation must be true for any source, and can be shown based on general arguments in Blanchet, but gives an interesting check of our calculations.

Complete 3PN energy flux - Mhar

- At long last, one can write down the *complete* 3PN GW energy flux averaged over an orbit for an ICB moving in an elliptical orbit by summing up the averaged instantaneous contribution and the tail contribution

$$
\begin{aligned}
< \dot{\mathcal{E}} >_{\rm MHar} \;=\; & \frac{32\nu^2 x^5}{5}\frac{1}{(1-e_t^2)^{7/2}}\Big(< \dot{\mathcal{E}}_N >_{\rm MHar} + x < \dot{\mathcal{E}}_{1PN} >_{\rm MHar} \\
& + x^{3/2} < \dot{\mathcal{E}}_{3/2PN} >_{\rm MHar} + x^2 < \dot{\mathcal{E}}_{2PN} >_{\rm MHar} \\
& + x^{5/2} < \dot{\mathcal{E}}_{5/2PN} >_{\rm MHar} + x^3 < \dot{\mathcal{E}}_{3PN} >_{\rm MHar}\Big) \,.
\end{aligned}
$$

Complete 3PN energy flux - Mhar

$$
\begin{aligned}
< \dot{\mathcal{E}}_N >_{\rm Mhar} &= 1 + e_t^2\,\frac{73}{24} + e_t^4\,\frac{37}{96}\,,\\
< \dot{\mathcal{E}}_{\rm 1PN} >_{\rm Mhar} &= \frac{1}{(1-e_t^2)}\\
&\left\{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) + e_t^2\left(\frac{10475}{672} - \frac{1081}{36}\nu\right)\right.\\
&\left.+e_t^4\left(\frac{10043}{384} - \frac{311}{12}\nu\right) + e_t^6\left(\frac{2179}{1792} - \frac{851}{576}\nu\right)\right\},\\
< \dot{\mathcal{E}}_{\rm 1.5PN} >_{\rm Mhar} &= 4\pi\,\varphi(e_t)\,,
\end{aligned}
$$

Complete 3PN energy flux - Mhar

$$
\begin{aligned}
< \dot{\mathcal{E}}_{2\mathrm{PN}} >_{\mathrm{Mhar}} &= \frac{1}{\left(1-e_t^2\right)^2} \\
&\left\{-\frac{203471}{9072}+\frac{12799}{504}\nu+\frac{65}{18}\nu^2+e_t^2\left(-\frac{3807197}{18144}+\frac{11678}{2016}\right.\right. \\
&+e_t^4\left(-\frac{268447}{24192}-\frac{2465027}{8064}\nu+\frac{247805}{864}\nu^2\right) \\
&+e_t^6\left(\frac{1307105}{16128}-\frac{416945}{2688}\nu+\frac{185305}{1728}\nu^2\right) \\
&+e_t^8\left(\frac{86567}{64512}-\frac{9769}{4608}\nu+\frac{21275}{6912}\nu^2\right) \\
&+\sqrt{1-e_t^2}\left[\left(\frac{35}{2}-7\nu\right)+e_t^2\left(\frac{6425}{48}-\frac{1285}{24}\nu\right)\right. \\
&\left.\left.+e_t^4\left(\frac{5065}{64}-\frac{1013}{32}\nu\right)+e_t^6\left(\frac{185}{96}-\frac{37}{48}\nu\right)\right]\right\}, \\
< \dot{\mathcal{E}}_{2.5\mathrm{PN}} >_{\mathrm{Mhar}} &= \pi\left[-\frac{8191}{672}\,\psi(e_t)-\frac{583}{24}\nu\,\theta'(e_t)\right],
\end{aligned}
$$

Complete 3PN energy flux - Mhar

$$
\begin{aligned}
< \dot{\mathcal{E}}_{\rm 3PN} >_{\rm Mhar} \quad &= \quad \frac{1}{\left(1-e_t^2\right)^3} \left\{ \frac{1266161801}{9979200} + \left(\frac{8009293}{54432} - \frac{41}{64}\pi^2 \right) \nu \right. \\
& \left. i - \frac{94403}{3024}\nu^2 - \cdots\cdots \right\}
\end{aligned}
$$

Complete 3PN energy flux - Mhar

- Recall that the e_t above denotes e_t^{Mhar}.
- Circular orbit limit of the above expression is obtained by setting $e_t = 0$ and

$$F(e_t = 0) = \phi(e_t = 0) = \psi'(e_t = 0) = \theta'(e_t = 0) = \kappa(e_t = 0) = 1\,.$$

One obtains,

$$
\begin{aligned}
< \dot{\mathcal{E}} > |_{\odot} \quad = \quad & \frac{32}{5} x^5 \nu^2 \Bigg\{ 1 + x \left(-\frac{1247}{336} - \frac{35}{12}\nu \right) + 4\pi\, x^{3/2} \\
& + x^2 \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2 \right) - \pi\, x^{5/2} \left(\frac{8191}{672} + \frac{583}{24}\nu \right. \\
& + x^3 \left(\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105} C - \frac{856}{105} \ln(16\,x) \right. \\
& \left. + \left[-\frac{14930989}{272160} + \frac{41}{48}\pi^2 - \frac{88}{3}\theta \right] \nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \right) \Bigg\}
\end{aligned}
$$

Present Work

- Extends the circular orbit results at 2.5PN (Blanchet, 1990) and 3PN (Blanchet, Iyer, Joguet, 2002) to the elliptical orbit case. (involve both instantaneous and hereditary terms).
- Extends earlier works on instantaneous contributions for binaries moving in elliptical orbits at 1PN Blanchet Schäfer 89,Junker Schäfer 92) and 2PN (Gopakumar Iyer 97) to 3PN order.
- Extends hereditary contributions at 1.5PN by (Blanchet Schäfer 93) to 2.5PN order and 3PN.
- 3PN hereditary contributions comprise the *tail(tail)* and $tail^2$ and are extensions of (Blanchet 98) for circular orbits to the elliptical case.

Angular Momentum Flux

- For non-circular orbits, in addition to the conserved energy and gravitational wave energy flux, the angular momentum flux needs to be known to determine the phasing of eccentric binaries. A knowledge of the angular momentum flux of the system averaged over an orbit is mandatory to calculate the evolution of the orbital elements of non-circular, in particular, elliptic orbits under GW radiation reaction.
- We compute the angular momentum flux of inspiralling compact binaries moving in non-circular orbits up to 3PN order generalising earlier work at Newtonian order by Peters (1964), at 1PN order by Junker and Schäfer (Junker Schäfer 1992), 1.5PN (tails and spin-orbit) by Schafer and Rieth (1997) and at 2PN order by Gopakumar and Iyer (1997). Unlike at earlier post-Newtonian orders, the 3PN contribution to angular momentum flux comes not only from *instantaneous* terms but also *hereditary* contributions.

Angular Momentum Flux

- Hereditary contributions comprise not only the tails-of-tails and tail-square terms as for the energy flux but also an interesting *memory* contribution at 2.5PN.
- Evolution of orbital elements under gravitational radiation goes back to the classic work of Peters and Mathews (1963). This was progressively extended by Blanchet and Schäfer to 1PN in (1989) and 1.5PN in (BS89, RS97) and finally to 2PN by Gopakumar and Iyer (1997). While JS92, RS97 require the 1PN accurate orbital description of Damour and Deruelle (1985), GI97 crucially employs the 2PN GQK parametrization of the binary's orbital motion in ADM coordinates as given in Damour-Schäfer 88,Schäfer-Wex 93,Wex 95.
- Evolution of orbital elements under GRR relevant to the problem of Binary pulsar timing.. Eg by computing evolution of the orbital period under leading GRR Blanchet and Schäfer showed it contributes to $\dot{P}$ a fractional amount 2.15×10^{-5} for 1913+16 much below the observed accuracy of 1.7×10^{-2}.
- Should be checked for the faster pulsars like the new Double pulsar

Angular Momentum Flux

- We obtain the orbital average of the *instantaneous* part of the angular momentum flux at 3PN using the recently constructed 3PN GQK parametrization of the binary's orbital motion by Memmesheimer, Gopakumar and Schäfer (2004).
- Combining the results for the angular momentum flux obtained with the results for the far-zone flux of energy obtained by earlier, we finally evaluate the evolution of the orbital elements under the instantaneous contribution in the 3PN gravitational under the instantaneous contribution in the 3PN gravitational radiation reaction.

Far Zone Angular Momentum Flux

$$\left(\frac{d\mathcal{J}_i}{dt}\right) = \frac{G}{c^5}\epsilon_{ipq}\left\{\frac{2}{5}U_{pj}U^{(1)}_{qj}\right.$$
$$+\frac{1}{c^2}\left[\frac{1}{63}U_{pjk}U^{(1)}_{qjk}+\frac{32}{45}V_{pj}V^{(1)}_{qj}\right]+\frac{1}{c^4}\left[\frac{1}{2268}U_{pjkl}U^{(1)}_{qjkl}+\frac{1}{28}V_{pjk}V^{(1)}_{qjk}\right]$$
$$\left.+\frac{1}{c^6}\left[\frac{1}{118800}U_{pjklm}U^{(1)}_{qjklm}+\frac{16}{14175}V_{pjkl}V^{(1)}_{qjkl}\right]+\mathcal{O}(8)\right\}.$$

- Using the MPM formalism, the radiative moments can be re-expressed in terms of the source moments to an accuracy sufficient for the computation of the angular momentum flux up to 3PN.
- For the AM flux to be complete up to 3PN approximation, one must compute the mass type radiative quadrupole U_{ij} to 3PN accuracy, mass octupole U_{ijk} and current quadrupole V_{ij} to 2PN accuracy, mass hexadecupole U_{ijkm} and current octupole V_{ijk} to 1PN accuracy and finally U_{ijkmn} and V_{ijkm} to Newtonian accuracy.

Far Zone Angular Momentum Flux

- From the expressions for U_Ls and V_Ls, one can schematically split the total contribution to the angular momentum flux as the sum of the instantaneous and hereditary terms.
- Starting from the expression for the angular momentum flux in terms of the radiative multipole moments and the expressions for the radiative moments in terms of the source multipoles the AMF can be re-written as

$$\left(\frac{d\mathcal{J}_i}{dt}\right) = \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst} + \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm hered} .$$

$$\begin{aligned}
\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm inst(s)} &= \frac{G}{c^5}\varepsilon_{ipq}\left\{\frac{2}{5}I^{(2)}_{pj}I^{(3)}_{qj}\right. \\
&+ \frac{1}{c^2}\left[\frac{1}{63}I^{(3)}_{pjk}I^{(4)}_{qjk}+\frac{32}{45}J^{(2)}_{pj}J^{(3)}_{qj}\right]+\frac{1}{c^4}\left[\frac{1}{2268}I^{(4)}_{pjkl}I^{(5)}_{qjkl}+\frac{1}{28}J^{(3)}_{pjk} \right. \\
&\left. +\frac{1}{c^6}\left[\frac{1}{118800}I^{(5)}_{pjklm}I^{(6)}_{qjklm}+\frac{16}{14175}J^{(4)}_{pjkl}J^{(5)}_{qjkl}\right]\right\},
\end{aligned}$$

Far Zone Angular Momentum Flux

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\mathrm{inst(c)}} = \frac{2G}{5c^5}\varepsilon_{ipq}\left\{\frac{4G}{c^5}\left[W^{(5)}I^{(2)}_{pj}I_{qj} + 2W^{(4)}I^{(2)}_{pj}I^{(1)}_{qj} - 3W^{(2)}I^{(2)}_{pj}I^{(3)}_{qj} - W^{(1)}I\right.\right.$$

$$\left.\left. + W^{(4)}I_{pj}I^{(3)}_{qj} + W^{(3)}I^{(1)}_{pj}I^{(3)}_{qj} - W^{(1)}I^{(3)}_{pj}I^{(3)}_{qj}\right]\right\} ,$$

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\mathrm{inst(r)}} = \frac{G}{c^5}\,\frac{2}{5}\,\frac{G}{c^5}\varepsilon_{ipq}\left\{I^{(3)}_{qj}\left[-\frac{5}{7}I^{(4)}_{a<p}I^{(1)}_{j>a} - \frac{2}{7}I^{(3)}_{a<p}I^{(2)}_{j>a} + \frac{1}{7}I^{(5)}_{a<p}I_{j>a} + \frac{1}{3}\varepsilon_{ab<}\right.\right.$$

$$+I^{(2)}_{pj}\left[-\frac{4}{7}I^{(5)}_{a<q}I^{(1)}_{j>a} - I^{(4)}_{a<q}I^{(2)}_{j>a} - \frac{4}{7}I^{(3)}_{a<q}I^{(3)}_{j>a} + \frac{1}{7}I^{(6)}_{a<q}I_{j>a}\right.$$

$$\left.\left.\left. +\frac{1}{3}\varepsilon_{ab<q}\,I^{(5)}_{j>a}J_b\right)\right]\right\} .$$

3PN AMflux - Shar - Inst Terms

$$\left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm SHar} = \left[\left(\frac{d\mathcal{J}_i}{dt}\right)^{N} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 1PN} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2PN} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 2.5PN} + \left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 3PN}\right]_{\rm inst} + \left(\frac{d\mathcal{J}_i}{dt}\right)_{\rm her} + \mathcal{O}(7).$$

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{N} = \frac{G^2 m^3 \nu^2}{c^5 r^3}\,\tilde{\mathbf{L}}_i\left\{\frac{16}{5}v^2 - \frac{24}{5}\dot{r}^2 - \frac{16}{5}\frac{G\,m}{r}\right\},$$

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{\rm 1PN} = \frac{G^2 m^3 \nu^2}{c^7 r^3}\mathbf{L}_i\left\{v^4\left(\frac{614}{105} - \frac{1096}{105}\nu\right) + v^2\,\dot{r}^2\left(-\frac{296}{35} + \frac{1108}{35}\nu\right) + \frac{G\,m}{r}\,v^2\left(-\frac{464}{105} + \frac{152}{21}\nu\right) + \dot{r}^4\left(\frac{38}{7} - \frac{144}{7}\nu\right) + \frac{G\,m}{r}\,\dot{r}^2\left(\frac{496}{35} + \frac{788}{105}\nu\right) + \frac{G^2\,m^2}{r^2}\left(-\frac{596}{21} + \frac{8}{105}\nu\right)\right\},$$

$$\tilde{\mathbf{L}}_{\mathbf{i}} = \varepsilon_{ijk} x_j\, v_k$$

3PN AMFlux - Shar

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{2\mathrm{PN}} = \frac{G^2m^3\nu^2}{c^9r^3}\left\{v^6\left(\frac{53}{63}-\frac{353}{9}\nu+\frac{614}{15}\nu^2\right)+v^4\,\dot{r}^2\left(-\frac{2246}{105}+\frac{12653}{105}\nu-\frac{1563}{105}\right.\right.$$

$$+\frac{G\,m}{r}\,v^4\left(\frac{11}{21}-\frac{491}{315}\nu+\frac{4022}{315}\nu^2\right)+v^2\,\dot{r}^4\left(\frac{715}{21}-\frac{3361}{21}\nu+\frac{448}{3}\nu^2\right)$$

$$+\frac{G\,m}{r}\,v^2\,\dot{r}^2\left(\frac{21853}{315}-\frac{7201}{105}\nu+\frac{2551}{315}\nu^2\right)$$

$$+\frac{G^2\,m^2}{r^2}\,v^2\left(-\frac{21302}{315}+\frac{2262}{35}\nu-\frac{6856}{315}\nu^2\right)+\dot{r}^6\left(-\frac{52}{3}+\frac{652}{9}\nu-\frac{388}{9}\nu^2\right)$$

$$+\frac{G\,m}{r}\,\dot{r}^4\left(-\frac{22312}{315}+\frac{5914}{45}\nu-\frac{277}{9}\nu^2\right)+\frac{G^2\,m^2}{r^2}\,\dot{r}^2\left(\frac{5624}{105}-\frac{7172}{45}\nu+\frac{30}{10}\right.$$

$$\left.+\frac{G^3\,m^3}{r^3}\left(\frac{340724}{2835}+\frac{15658}{315}\nu+\frac{44}{45}\nu^2\right)\right\}\mathbf{L}_i.$$

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{2.5\mathrm{PN}} = \frac{G^2m^3\nu^2}{5\,c^{10}r^3}\left\{\dot{r}\nu\left[-\frac{27744}{35}\frac{G\,m}{r}\,v^4+\frac{19144}{7}\frac{G\,m}{r}\,v^2\dot{r}^2-\frac{116944}{105}\frac{G^2\,m^2}{r^2}v^2\right.\right.$$

$$\left.\left.+\frac{8976}{7}\left(\frac{Gm}{r}\right)^2\dot{r}^2-1960\frac{G\,m}{r}\,\dot{r}^4-\frac{22864}{105}\left(\frac{Gm}{r}\right)^3\right]\right\}\mathbf{L}_i.$$

3PN AMFlux - Shar

$$\left(\frac{d\mathcal{J}_i}{dt}\right)^{3\mathrm{PN}} = \frac{G^2 m^3\,\nu^2 \mathbf{L}_i}{r^3\,c^{11}}\left\{v^8\left[\frac{145919}{13860} - \frac{110423\,\nu}{1260} + \frac{1079083\,\nu^2}{4620} - \frac{30229\,\nu^3}{165}\right]\cdot\cdot\right.$$

Orbital Averaged AMF - ADM

- Using the QK representation of the orbit in ADM coordinates and the instantaneous angular momentum flux in ADM coordinates, one transforms the expression for the magnitude of the angular momentum flux $d\mathcal{J}/dt\,(r,\dot{r}^2,v^2) \equiv |d\mathcal{J}_i/dt|$ to $d\mathcal{J}/dt\,(E,h,e_r,u)$ where E is the conserved orbital energy and h is related the conserved angular momentum $\mathbf{J}$ as $h = |\mathbf{J}|/Gm$. This expression up to 3PN order is schematically given as

$$\frac{d\mathcal{J}}{dt} = \frac{du}{ndt}\sum_{N=2}^{10}\left[\frac{\alpha_N(e_t)}{(1-e_t\cos u)^N} + \beta_N(e_t)\,\frac{\sin u}{(1-e_t\ \cos u)^N} + \gamma_N(e_t)\,\frac{\ln(1-e_t\cos}{(1-e_t\cos u)}\right.$$

$$\alpha_{\rm N}(E,h) = \frac{\nu^2}{G\,c^5}(-E)^5\beta_{\rm N}(E,h)\,.$$

$\beta_N(E,h)$ can be written down as a PN series but too long to be listed here.

Orbital Averaged AMF - ADM

- Computation of the orbital average involves the evaluation of the integral,

$$\langle\frac{d\mathcal{J}}{dt}\rangle = \frac{1}{P}\int_0^P \frac{d\mathcal{J}}{dt}(t)\,dt \;=\frac{1}{2\,\pi}\int_0^{2\,\pi}\left(\frac{ndt}{du}\right)\frac{d\mathcal{J}}{dt}(u)du\,.$$

- Rewriting AMF using the GQKR, the flux can be averaged over an orbit to order 3PN extending the results at 2PN.

Orbital Averaged AMF - ADM

$$\left\langle\frac{d\mathcal{J}}{dt}\right\rangle_{\rm inst}^{\rm ADM} = \frac{4}{5}\,c^2\,m\,\zeta^{7/3}\,\nu^2\,\frac{1}{(1-e_t{}^2)^{7/2}}\left[\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm Newt} + \left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 1PN} + \left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 2PN} + \left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 2.5PN} + \left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 3PN}\right],$$

where $\zeta = \frac{G\,m\,n}{c^3}$ and the individual terms read as:

$$\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm Newt} = \frac{8+7e_t^2}{(1-e_t^2)^2}\,,$$

$$\left\langle\frac{d\mathcal{J}}{dt}\right\rangle^{\rm 1PN} = \zeta^{2/3}\frac{1}{(1-e_t)^3}\left\{\frac{1105}{42}-\frac{70\nu}{3}+e_t^2\left[\frac{5077}{42}-\frac{335\nu}{3}\right] + e_t^4\left[\frac{8399}{336}-\frac{275\nu}{12}\right]\right\},$$

Orbital Averaged AMF - ADM

$$
\begin{aligned}
\langle \frac{d\mathcal{J}}{dt} \rangle^{\rm 2PN} \quad = \quad & \zeta^{4/3}\, \frac{1}{(1-e_t^2)^4} \left\{ \left[\frac{7238}{81} - \frac{10175\,\nu}{63} + \frac{260\,\nu^2}{9} \right] \right. \\
& + e_t^2 \left[\frac{376751}{756} - \frac{37047\,\nu}{28} + \frac{1546\,\nu^2}{3} \right] \\
& + e_t^4 \left[\frac{377845}{756} - \frac{168863\,\nu}{168} + 569\,\nu^2 \right] \\
& + e_t^6 \left[\frac{30505}{2016} - \frac{2201\,\nu}{56} + \frac{1519\,\nu^2}{36} \right] \\
& \left. + \sqrt{1-e_t^2} \left[80 - 32\,\nu + e_t^2 (335 - 134\,\nu) + e_t^4 (35 - 14\,\nu) \right] \right\} \,,
\end{aligned}
$$

Orbital Averaged AMF - ADM

$$\langle\frac{d\mathcal{J}}{dt}\rangle^{3\mathrm{PN}} = \zeta^2\,\frac{1}{(1-e_t^2)^5}\Bigg\{\Bigg[\frac{265845199}{138600}-\frac{20318135\,\nu}{6804}+\frac{287\pi^2\,\nu}{4}+$$

$$\frac{187249\,\nu^2}{378}-\frac{1550\,\nu^3}{81}\Bigg]\cdots$$

Checks

- Circular orbit limit ($e_t = 0$) As an algebraic check, we take the circular orbit limit of the orbital average of angular momentum flux and the energy flux in ADM coordinates expressed in terms of ζ and e_t. For circular orbit binaries the angular momentum flux and the energy flux must be simply related as

$$\frac{d\mathcal{E}}{dt} = \omega \, \frac{d\mathcal{J}}{dt}$$

in any coordinate system. Here $\frac{d\mathcal{J}}{dt}$ is the magnitude of the angular momentum flux.

- The circular orbit limit of our calculation agrees with the above expression with ω and is given by

$$\begin{aligned} \omega &= \left(\frac{c^3 \, \zeta}{G \, m} \right) \left\{ 1 + 3 \, \zeta^{2/3} + \zeta^{4/3} \left[\frac{39}{2} - 7\nu \right] \right. \\ &\left. + \zeta^2 \left[\frac{315}{2} + \frac{1}{32} \left(-6536 + 123\pi^2 \right) \nu + 7\nu^2 \right] \right\} , \end{aligned}$$

where $\zeta = \frac{G \, m \, n}{c^3}$.

Evoln of orbital elements under GRR

- Most important application of the 3PN angular momentum flux obtained here and the energy flux obtained is to calculate how the orbital elements of the binary evolve with time under GRR. By 3PN evolution of orbital elements under GRR we mean its evolution under 5.5PN terms beyond leading newtonian order in the EOM.
- We compute the rate of change of n, e_t and a_r averaged over an orbit, due to GRR.
- We start with the 3PN accurate expressions for n and e_t in terms of the 3PN conserved energy (E) and angular momentum (J). Differentiating them w.r.t time and using *heuristic balance equations* for energy and angular momentum up to 3PN order, we compute the rate of change of the orbital elements.
- Extends the earlier analyses at Newtonian order by Peters (64), 1PN computation of Blanchet Schäfer 89,Junker Schäfer 92 and at 2PN order by Gopakumar Iyer 97,Damour Gopakumar Iyer 04. The 1.5PN hereditary effects also have been accounted in the orbital element evolution in Blanchet Schäfer 93, Rieth Schäfer 97.

Evoln of orbital elements under GRR

- 3PN accurate expressions for the mean motion n, eccentricity e_t and semi-major axis a_r read are listed. Let us use the example of n to outline the procedure adopted for the computation of orbital elements in more detail. The expression for n is symbolically written as

$$n = n(E, J).$$

Differentiating with respect to t one obtains

$$\frac{dn}{dt} = \gamma_1(e_t, \zeta, \nu)\,\frac{dE}{dt} + \gamma_2(e_t, \zeta, \nu)\,\frac{d|\mathbf{J}|}{dt},$$

where γ_1 and γ_2 are PN expansions in powers of ζ. Now we use the balance equations,

$$\frac{dE}{dt} = -\frac{d\mathcal{E}}{dt},$$
$$\frac{d|\mathbf{J}|}{dt} = -\frac{d\mathcal{J}}{dt}.$$

Evoln of orbital element n under GRR

- Replace the time derivatives of the conserved energy and angular momentum (on the right side of the expression for $\frac{dn}{dt}$) with the energy and angular momentum fluxes and compute the final expression for the orbital average by using the orbital averages of the energy and angular momentum fluxes up to 3PN. It may be noted that, the angular momentum flux is needed only up to 1PN accuracy for the computation of $\langle\frac{dn}{dt}\rangle$ where as the energy flux is needed up to 3PN. The structure of the evolution equations is similar for the other orbital elements also and the same procedure can be employed. The final expression for the 3PN evolution of n reads

$$\langle\frac{dn}{dt}\rangle^{\rm ADM}_{\rm inst} = \frac{c^6}{G^2\,m^2}\,\zeta^{11/3}\left[\langle\frac{dn}{dt}\rangle_{\rm Newt} + \langle\frac{dn}{dt}\rangle_{\rm 1PN} + \langle\frac{dn}{dt}\rangle_{\rm 2PN} + \langle\frac{dn}{dt}\rangle_{\rm 3PN}\right]$$

Evoln of orbital element n under GRR

$$
\begin{aligned}
\langle \frac{dn}{dt} \rangle_{\rm Newt} &= \frac{1}{\left(1-e_t^2\right)^{7/2}} \left\{ \frac{96}{5} + \frac{292 e_t^2}{5} + \frac{37 e_t^4}{5} \right\} , \\
\langle \frac{dn}{dt} \rangle_{\rm 1PN} &= \frac{\zeta^{2/3}}{\left(1-e_t^2\right)^{9/2}} \left\{ \frac{2546}{35} - \frac{264\,\nu}{5} + e_t^2 \left[\frac{5497}{7} - 570\,\nu \right] \right. \\
&+ \left. e_t^4 \left[\frac{14073}{20} - \frac{5061\,\nu}{10} \right] + e_t^6 \left[\frac{11717}{280} - \frac{148\,\nu}{5} \right] \right\} , \\
\langle \frac{dn}{dt} \rangle_{\rm 2PN} &= \frac{\zeta^{4/3}}{\left(1-e_t^2\right)^{11/2}} \left\{ \frac{393527}{945} + e_t^2 \left[\frac{4098457}{945} - \frac{108047\nu}{15} + \frac{182387\nu^2}{90} \right] \right. \\
&+ e_t^4 \left[\frac{1678961}{180} - \frac{2098263\nu}{140} + \frac{396443\nu^2}{72} \right] + e_t^6 \left[\frac{1249229}{336} - \frac{76689\nu}{16} + \frac{19294}{90} \right. \\
&+ \sqrt{1-e_t^2} \left[48 - \frac{47491\nu}{105} + \frac{944\nu^2}{15} + e_t^2 \left[2134 - \frac{4268\nu}{5} \right] + e_t^4 \left[2193 - \frac{4386}{5} \right. \right. \\
&+ \left. e_t^6 \left[\frac{175}{2} - 35\nu \right] - \frac{96\nu}{5} \right] + e_t^8 \left[\frac{391457}{3360} - \frac{6037\nu}{56} + \frac{2923\nu^2}{45} \right] \Bigg\} ,
\end{aligned}
$$

Evoln of orbital element n under GRR

$$\langle \frac{dn}{dt} \rangle_{\rm 3PN} = \frac{\zeta^2}{\left(1 - e_t^2\right)^{13/2}} \left\{ \left[\frac{6687854333}{1039500} - \frac{113898769\,\nu}{11340} + \frac{2337\pi^2\,\nu}{10} \right.\right.$$

$$+ \quad + \frac{564197\,\nu^2}{420} - \frac{1121\,\nu^3}{27} \cdots$$

Evoln of orbital element e_t under GRR

- Let us next consider the orbital average of $\frac{de_t}{dt}$. Both energy and angular momentum fluxes are now required up to 3PN in order to compute the 3PN evolution of e_t.

$$\langle\frac{de_t}{dt}\rangle^{\rm ADM}_{\rm inst} = \frac{c^3\, e_t}{G\, m}\left[\langle\frac{de_t}{dt}\rangle_{\rm Newt} + \langle\frac{de_t}{dt}\rangle_{\rm 1PN} + \langle\frac{de_t}{dt}\rangle_{\rm 2PN} + \langle\frac{de_t}{dt}\rangle_{\rm 3PN}\right],$$

$$\begin{aligned}
\langle\frac{de_t}{dt}\rangle_{\rm Newt} &= \frac{\zeta^{8/3}}{\left(1-e_t^2\right)^{5/2}}\left\{\frac{304}{15}+\frac{121e_t^2}{15}\right\},\\
\langle\frac{de_t}{dt}\rangle_{\rm 1PN} &= \frac{\zeta^{10/3}}{\left(1-e_t^2\right)^{7/2}}\left\{\frac{14207}{105}-\frac{4084\nu}{45}+e_t^2\left[\frac{12231}{35}-\frac{7753\nu}{30}\right]\right.\\
&\quad\left.+e_t^4\left[\frac{13929}{280}-\frac{1664\nu}{45}\right]\right\},
\end{aligned}$$

Evoln of orbital element e_t under GRR

$$
\begin{aligned}
\langle \frac{de_t}{dt} \rangle_{\rm 2PN} \quad = \quad & \frac{\zeta^4}{\left(1-e_t^2\right)^{9/2}} \Bigg\{ \frac{257771}{378} - \frac{13271\,\nu}{14} + \frac{752\,\nu^2}{5} \\
& + e_t^2 \left[\frac{7199837}{2520} - \frac{4133467\,\nu}{840} + \frac{64433\,\nu^2}{40} \right] \\
& + e_t^4 \left[\frac{34890643}{15120} - \frac{15971227\,\nu}{5040} + \frac{127411\,\nu^2}{90} \right] \\
& + e_t^6 \left[\frac{420727}{3360} - \frac{362071\,\nu}{2520} + \frac{821\,\nu^2}{9} \right] \\
& + \sqrt{1-e_t^2} \left[\frac{1336}{3} - \frac{2672\,\nu}{15} + e_t^2 \left[\frac{2321}{2} - \frac{2321\,\nu}{5} \right] \right. \\
& \left. + e_t^4 \left[\frac{565}{6} - \frac{113\,\nu}{3} \right] \right] \Bigg\} \,,
\end{aligned}
$$

Evoln of orbital element e_t under GRR

$$\langle\frac{de_t}{dt}\rangle_{\rm 3PN} = \frac{\zeta^{14/3}}{\left(1-e_t^2\right)^{11/2}}\left\{\frac{81933388819}{6237000} - \left(\frac{378365677}{22680} - \frac{10081\pi^2}{30}\right)\nu \cdots\right.$$

Evoln of orbital element a_r under GRR

- Finally we compute the orbital average of the time derivative of semi-major axis a_r. Similar to the case of n, one requires a 3PN energy flux expression for its evaluation but only 1PN angular momentum flux. The final result reads

$$\langle \frac{da_r}{dt} \rangle^{\rm ADM}_{\rm inst} = \nu\, c\, \zeta^2 \left[\langle \frac{da_r}{dt} \rangle_{\rm Newt} + \langle \frac{da_r}{dt} \rangle_{\rm 1PN} + \langle \frac{da_r}{dt} \rangle_{\rm 2PN} + \langle \frac{da_r}{dt} \rangle_{\rm 3PN} \right]$$

$$\langle \frac{da_r}{dt} \rangle_{\rm Newt} = \frac{1}{\left(1 - e_t^2\right)^{9/2}} \left\{ -\frac{64}{5} - \frac{392 e_t^2}{15} + 34 e_t^4 + \frac{74 e_t^6}{15} \right\} ,$$

$$\langle \frac{da_r}{dt} \rangle_{\rm 1PN} = \frac{\zeta^{2/3}}{\left(1 - e_t^2\right)^{11/2}} \left\{ -\frac{5092}{105} + \frac{176\nu}{5} + e_t^2 \left[-\frac{16626}{35} + \frac{1724\nu}{5} \right] \right.$$
$$+ e_t^4 \left[\frac{11429}{210} - \frac{213\nu}{5} \right] + e_t^6 \left[\frac{37061}{84} - \frac{953\nu}{3} \right]$$
$$\left. + e_t^8 \left[\frac{11717}{420} - \frac{296\nu}{15} \right] \right\} ,$$

Evoln of orbital element a_r under GRR

$$
\begin{aligned}
\langle \frac{da_r}{dt} \rangle_{\rm 2PN} &= \frac{\zeta^{4/3}}{(1-e_t^2)^{13/2}} \Bigg\{ -\frac{180998}{567} + \frac{22054\,\nu}{63} - \frac{608\,\nu^2}{15} \\
&+ e_t^2 \left[-\frac{7080622}{2835} + \frac{154921\,\nu}{35} - 1309\,\nu^2 \right] \\
&+ e_t^4 \left[-\frac{19396577}{5670} + \frac{2153051\,\nu}{420} - \frac{27935\,\nu^2}{12} \right] \\
&+ e_t^6 \left[\frac{28278521}{7560} - \frac{5582839\,\nu}{840} + \frac{81053\,\nu^2}{36} \right] \\
&+ e_t^8 \left[\frac{814607}{336} - \frac{8012201\,\nu}{2520} + \frac{12449\,\nu^2}{9} \right] \\
&+ e_t^{10} \left[\frac{366593}{5040} - \frac{9703\,\nu}{126} + \frac{1924\,\nu^2}{45} \right] \\
&+ \sqrt{1-e_t^2} \Bigg[-96 + \frac{192\,\nu}{5} + e_t^2 \left[-1356 + \frac{2712\,\nu}{5} \right] \\
&+ e_t^4 \left[99 - \frac{198\,\nu}{5} \right] + e_t^6 \left[1279 - \frac{2558\,\nu}{5} \right] \\
&+ e_t^8 \left[74 - \frac{148\,\nu}{5} \right] \Bigg] \Bigg\} \,,
\end{aligned}
$$

Evoln of orbital element a_r under GRR

$$\langle \frac{da_r}{dt} \rangle_{3\mathrm{PN}} = \frac{\zeta^2}{\left(1-e_t^2\right)^{15/2}} \Bigg\{ -\frac{7894936583}{1559250} - \frac{\left(-72118997 + 1600641\pi^2\right)\,\nu}{8505}$$

$$- \quad -\frac{412199\,\nu^2}{378} + \frac{122\,\nu^3}{5}$$

Evoln of orbital elements under GRR

- The three expressions obtained here are the 3PN generalizations of the expressions given in Peters which are at the lowest quadrupolar order. They could be used to provide 3PN extensions of $n(e)$ and $a(e)$ relations in the future.
- The above results have to be supplemented by the computation of *hereditary* terms at 2.5PN and 3PN for completion. These hereditary terms include the tails at 2.5PN and tail of tails and tail-square terms at 3PN.
- Formally one can analytically solve the coupled evolution system by successive approximations, reducing it to simple quadratures. Eg, at the leading order $\mathcal{O}(c^{-5})$ one can first eliminate t by dividing $d\bar{n}/dt$ by $d\bar{e}_t/dt$, thereby obtaining an equation of the form $d\ln\bar{n} = f_0(\bar{e}_t)d\bar{e}_t$. Integration of this equation yields

$$\bar{n}(\bar{e}_t) = n_i \, \frac{e_i^{18/19}\,(304 + 121\,e_i^2)^{1305/2299}}{(1-e_i^2)^{3/2}} \, \frac{(1-e_t^2)^{3/2}}{e_t^{18/19}\,(304 + 121\,e_t^2)^{1305/2299}}\,,$$

e_i is the value of e_t when $n = n_i$. First obtained by Peters 64.

PART II

Based on

Phasing of Gravitational waves fluxes from inspiralling eccentric binaries 2.5PN/3.5PN

T. Damour, A. Gopakumar and B. R. Iyer

Phys.Rev. D70 (2004) 064028

C. Koenigsdoerffer, A. Gopakumar

Phys.Rev. D73 (2006) 124012

Beyond Orbital Averages

- GW obsvns of ICB, are analogous to the high precision Radio wave obsvns of binary pulsars. Uses accurate relativistic `timing formula' (Damour Deruelle 85, Damour Taylor 92).. Requires soln to rel EOM for CB moving in elliptical orbit
- GW obsvns demand accurate `phasing', i.e. an accurate mathematical modeling of the continuous time evolution of the gravitational waveform.
- GW emitted from inspiralling *circular* orbits, contain only two different time scales: orbital motion and radiation reaction
- *Inspiralling eccentric* orbits involve *three different time scales*: orbital period, periastron precession and radiation-reaction time scales.
- By using an improved `method of variation of constants', one can combine these three time scales, without making the usual approximation of treating the radiative time scale as an adiabatic process. Relies on techniques from (Damour 83, 85) to implement PN `phasing' for elliptical orbits.

Beyond Orbital Averages

- Going beyond the average evolution of the orbit under Grav Radn reaction the method allows one to deal with both a 'slow' (radiation-reaction time-scale) secular drift and 'fast' (orbital time-scale) periodic oscillations.
- Method implemented at the 2.5PN (Damour, Iyer, Gopakumar) and 3.5PN (Königsdörffer, Gopakumar)
- Results compute new 'post-adiabatic' short period contributions to the orbital phasing, or equivalently, new short-period contributions to GW polarizations, $h_{+,\times}$, to be explicitly added to PN expn for $h_{+,\times}$, if one treats radiative effects on the orbital phasing in the usual adiabatic approximation.
- Should be of importance both for the LIGO/VIRGO/GEO network of ground based interferometric GW detectors and for space-based interferometer LISA.

Phasing of GWF

- Theoretical templates for compact binaries, required to analyze the noisy data from the detectors consist of h_+ and $h_\times$: two independent GW polarization states, expressed in terms of the binary's intrinsic dynamical variables and location.

$$h_+ = \frac{1}{2}\left(p_i\, p_j - q_i\, q_j\right) h_{ij}^{TT}\,,$$
$$h_\times = \frac{1}{2}\left(p_i\, q_j + p_j\, q_i\right) h_{ij}^{TT}\,,$$

h_{ij}^{TT} = (TT) part of Radn field expressible in terms of PN expn in (v/c). $\mathbf{p}$ and $\mathbf{q}$ are two orthogonal unit vectors in the plane of the sky *i.e.* in the plane transverse to the radial direction linking the source to the observer.

Phasing of GWF

- TT radn field is given by wave generation formalisms, as a PN expansion of the form

$$h_{ij}^{\rm TT} = \frac{1}{c^4}\left[h_{ij}^0 + \frac{1}{c}h_{ij}^1 + \frac{1}{c^2}h_{ij}^2 + \frac{1}{c^3}h_{ij}^3 + \frac{1}{c^4}h_{ij}^4 + \frac{1}{c^5}h_{ij}^5 + \frac{1}{c^6}h_{ij}^6 + \cdots\right]$$

- Leading (`quadrupolar') approximation is given in terms of the relative separation vector $\mathbf{x}$ and relative velocity vector $\mathbf{v}$ as

$$\frac{1}{c^4}(h_{km}^0) = \frac{4\,G\,\mu}{c^4\,R'}\mathcal{P}_{ijkm}(\mathbf{N})\left(v_{ij} - \frac{G\,m}{r}\,n_{ij}\right),$$

$\mathcal{P}_{ijkm}(\mathbf{N})$ TT projection operator projecting normal to $\mathbf{N}$, $\mathbf{N} = \mathbf{R}'/R'$, R' radial distance to the binary.

- When inserting the explicit expression of h_{ij}^0, and its higher-PN analogues h_{ij}^1, $h_{ij}^2 \cdots$ which are currently known up to h_{ij}^4 one ends up with a corresponding expression for the two independent polarization amplitudes, as functions of the relative separation r and the `true anomaly' ϕ, *i.e.* the polar angle of $\mathbf{x}$, and their time derivatives,

Phasing of GWF

$$h_{+,\times}(r,\phi,\dot r,\dot\phi) = \frac{1}{c^4}\Big[h^0_{+,\times}(r,\phi,\dot r,\dot\phi) + \frac{1}{c}\,h^1_{+,\times}(r,\phi,\dot r,\dot\phi) + \frac{1}{c^2}\,h^2_{+,\times}(r,\phi,\dot r,\dot\phi) + \frac{1}{c^3}\,h^3_{+,\times}(r,\phi,\dot r,\dot\phi) + \frac{1}{c^4}\,h^4_{+,\times}(r,\phi,\dot r,\dot\phi) + \cdots\Big]\,.$$

- Choose convention: $\mathbf{N}$ from the source to the observer and $\mathbf{p}$ toward the correspondingly defined 'ascending' node

$$\mathbf{x} = \mathbf{p}\, r\, \cos\phi + (\mathbf{q}\,\cos i + \mathbf{N}\,\sin i) r\, \sin\phi,$$

$i =$ inclination of orbital plane wrt plane of sky

$$\frac{1}{c^4} h^0_{+}(r,\phi,\dot r,\dot\phi) = -\frac{G\,m\,\eta}{c^4\,R'}\Big\{(1+C^2)\Big[\Big(\frac{G\,m}{r} + r^2\,\dot\phi^2 - \dot r^2\Big)\cos 2\,\phi + 2\,\dot r\, r\,\dot\phi\,\sin 2\,\phi \Big] + S^2\Big[\frac{G\,m}{r} - r^2\,\dot\phi^2 - \dot r^2\Big]\Big\}\,,$$

$$\frac{1}{c^4} h^0_{\times}(r,\phi,\dot r,\dot\phi) = -2\frac{G\,m\,\eta\,C}{c^4\,R'}\Big\{\Big(\frac{G\,m}{r} + r^2\,\dot\phi^2 - \dot r^2\Big)\sin 2\phi - 2\dot r\, r\,\dot\phi\,\cos 2\phi\Big\}\,,$$

$C = \cos i$ and $S = \sin i$

Phasing of GWF

- Orbital phase $= \phi$, $\dot{\phi} = d\phi/dt$ and $\dot{r} = dr/dt = \mathbf{n}\cdot\mathbf{v}$, where $\mathbf{v} = \mathbf{p}\,(\dot{r}\,\cos\phi - r\,\dot{\phi}\,\sin\phi) + (\mathbf{q}\,\cos i + \mathbf{N}\,\sin i)\,(\dot{r}\,\sin\phi + r\,\dot{\phi}\,\cos\phi)$.
- Must be supplemented by explicit expressions describing the temporal evolution of the relative motion, *i.e.* describing the explicit time dependences $r(t)$, $\phi(t)$, $\dot{r}(t)$, and $\dot{\phi}(t)$.
- Refer to as *phasing*, any explicit way to define the latter time-dependences, because it is the crucial input needed beyond the 'amplitude' expansions, given by to derive some ready to use waveforms $h_{+,\times}(t)$.
- Structure for GW polarization amplitudes has only the relative motion $\mathbf{x}$, $\mathbf{v}$, because one go to a suitable center-of-mass frame .. Validity of a CM theorem .. $\mathcal{O}(c^{-7})$ 'recoil' of the center-of-mass is expected to influence the waveform only at the $\mathcal{O}(c^{-8})$ level.
- Time-dependent recoil of the latter rest frame will introduce both a $\mathbf{N}\cdot\mathbf{v}_{\mathrm{CM}}/c$ Doppler shift of the phasing and a corresponding modification of the amplitudes $h_{+,\times}$.

Phasing of GWF

- $h_{+,\times}$ expressed only in terms of r, ϕ and their time derivatives because restricted to non-spinning objects. In the presence of spin interactions, the orbital plane is no longer fixed in space and one needs to introduce further variables, notably a (slowly varying) 'longitude of the node' Ω. Correspondingly, the polarization direction $\mathbf{p}$ cannot be defined anymore as the line of nodes.
- Such a situation dealt with in the problem of the timing of binary pulsars (Damour Taylor 92) and might be advantageous to use the conventions used there to define $\mathbf{p}$ and $\mathbf{q}$. Namely, in terms of (DT92), $\mathbf{p} = \mathbf{I}_0$, $\mathbf{q} = \mathbf{J}_0$. Note that the binary pulsar convention uses as the third vector $\mathbf{I}_0 \times \mathbf{J}_0$, the direction from the observer to the source.
- Explicit functional forms for $h_+(r, \phi, \dot{r}, \dot{\phi})$, $h_\times(r, \phi, \dot{r}, \dot{\phi})$ and phasing relations $r(t), \phi(t), \dot{r}(t)$ and $\dot{\phi}(t)$ depend on the coordinate system used, though the final results $h_+(t)$ and $h_\times(t)$ do not
- h_{ij}^{TT} and therefore $h_+(t)$ and $h_\times(t)$ are *coordinate independent* asymptotic quantities.

Method of variation of constants

- A version of the general Lagrange method of variation of arbitrary constants, which was employed to compute within GR the orbital evolution of the Hulse-Taylor binary pulsar (Damour 83, 85).
- Begins by splitting the relative acceleration of the compact binary $\mathcal{A}$ into two parts: an integrable leading part $\mathcal{A}_0$ and a perturbation part, $\mathcal{A}'$

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}' \, .$$

- Eg to work at 2.5PN accuracy choose $\mathcal{A}_0$ to be the acceleration at 2PN order and $\mathcal{A}'$ to be the c^{-5} RR; for 3.5PN-accurate calculation $\mathcal{A}_0$ would be the conservative part of the 3PN dynamics, and $\mathcal{A}'$ the $\mathcal{O}(c^{-5}) + \mathcal{O}(c^{-7})$ RR

Method of variation of constants

- First construct soln to the 'unperturbed' system, defined by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v}\,,\\ \dot{\mathbf{v}} &= \mathcal{A}_0(\mathbf{x},\mathbf{v}).\end{aligned}$$

The solution to the exact system

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v}\,,\\ \dot{\mathbf{v}} &= \mathcal{A}(\mathbf{x},\mathbf{v})\,,\end{aligned}$$

obtained by *varying the constants* in the generic solutions of the unperturbed system. The method assumes (true for $\mathcal{A}_{2PN}^{\rm conservative}$ or $\mathcal{A}_{3PN}^{\rm conservative}$) that the unperturbed system admits sufficiently many integrals of motion to be integrable. E.g. if $\mathcal{A}_0 = \mathcal{A}_{2PN}$, we have four first integrals: the 2PN accurate energy and 2PN accurate angular momentum of the binary written in the 2PN accurate center of mass frame as c_1 and c_2^i:

Method of variation of constants

$$c_1 = \mathcal{E}(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{2(3)\mathrm{PN\ CM}} ,$$
$$c_2^i = \mathcal{J}_i(\mathbf{x_1}, \mathbf{x_2}, \mathbf{v_1}, \mathbf{v_2})|_{2(3)\mathrm{PN\ CM}} ,$$

- Vectorial structure of c_2^i, indicates that the unperturbed motion takes place in a plane. Problem is restricted to a plane even in the presence of radiation reaction. Introduce polar coordinates in the plane of the orbit r and ϕ such that $\mathbf{x} = \mathbf{i}\, r \cos\phi + \mathbf{j}\, r \sin\phi$ with, $\mathbf{i} = \mathbf{p}$, $\mathbf{j} = \mathbf{q} \cos i + \mathbf{N} \sin i$.
- Functional form for the solution to the unperturbed equations of motion may be expressed as

$$r = S(l; c_1, c_2) \quad ; \quad \dot{r} = n \frac{\partial S}{\partial l}(l; c_1, c_2) ,$$
$$\phi = \lambda + W(l; c_1, c_2) \quad ; \quad \dot{\phi} = (1 + k)n + n \frac{\partial W}{\partial l}(l; c_1, c_2) ,$$

where λ and l are two basic angles, which are 2π periodic and $c_2 = |c_2^i|$.

Method of variation of constants

- Functions $S(l)$ and $W(l)$ and therefore $\frac{\partial S}{\partial l}(l)$ and $\frac{\partial W}{\partial l}(l)$ are periodic in l with a period of 2π. n denotes the unperturbed 'mean motion', given by $n = \frac{2\pi}{P}$, P being the radial (periastron to periastron) period, while $k = \Delta\Phi/2\pi$, $\Delta\Phi$ being the advance of the periastron in the time interval P.
- Angles l and λ satisfy, for the unperturbed system, $\dot{l} = n$ and $\dot{\lambda} = (1+k)n$, which integrate to

$$\begin{aligned} l &= n(t - t_0) + c_l \,, \\ \lambda &= (1+k)n(t - t_0) + c_\lambda \,, \end{aligned}$$

t_0 some initial instant and the constants c_l and c_λ, corresponding values for l and λ. Unperturbed solution depends on four integration constants: c_1, c_2, c_l and c_λ.

Method of variation of constants

- At 2PN, one can write down explicit expressions for the functions $S(l)$ and $W(l)$. GQKR yields:

$$\begin{aligned} S(l;c_1,c_2) &= a_r(1-e_r\cos u)\,, \\ W(l;c_1,c_2) &= (1+k)(v-l)+\frac{f_\phi}{c^4}\sin 2v+\frac{g_\phi}{c^4}\sin 3v\,, \end{aligned}$$

where v and u are some 2PN accurate true and eccentric anomalies, which must be expressed as functions of l, c_1, and c_2, say $v=\mathcal{V}(l;c_1,c_2)=V(\mathcal{U}(l;c_1,c_2))$ and $u=\mathcal{U}(l;c_1,c_2)$.

- a_r and e_r are some 2PN accurate semi-major axis and radial eccentricity, while f_ϕ and g_ϕ are certain functions, given in terms of c_1 and c_2.

$$v=V(u)\equiv 2\arctan\left(\left(\frac{1+e_\phi}{1-e_\phi}\right)^{1/2}\tan\frac{u}{2}\right).$$

Fn $u=\mathcal{U}(l)$ defined by inverting the 'Kepler equation' $l=l(u)$

$$l=u-e_t\sin u+\frac{f_t}{c^4}\sin V(u)+\frac{g_t}{c^4}(V(u)-u)\,.$$

Method of variation of constants

Fn $v = \mathcal{V}(l)$ obtained by inserting $u = \mathcal{U}(l)$ in $v = V(u)$, *i.e.* $\mathcal{V}(l) \equiv V(\mathcal{U}(l))$. e_t and e_ϕ are some time and angular eccentricity and f_t and g_t are certain functions of c_1 and c_2, appearing at the 2PN order.

- Use the following exact relation for $v - u$, which is also periodic in u, given by

$$v - u = 2\,\tan^{-1}\left(\frac{\beta_\phi\,\sin u}{1 - \beta_\phi\,\cos u}\right),$$

$\beta_\phi = \frac{1-\sqrt{1-e_\phi^2}}{e_\phi}$.

- Use explicit unperturbed solution, for the construction of the general solution of the perturbed system. Keep exactly the same functional form for r, $\dot{r}$, ϕ and $\dot{\phi}$, as functions of l and λ, *i.e.* by writing

$$r = S(l; c_1, c_2) \quad ; \quad \dot{r} = n\frac{\partial S}{\partial l}(l; c_1, c_2),$$

$$\phi = \lambda + W(l; c_1, c_2) \quad ; \quad \dot{\phi} = (1+k)n + n\frac{\partial W}{\partial l}(l; c_1, c_2),$$

but allowing temporal variation in $c_1 = c_1(t)$ and $c_2 = c_2(t)$

Method of variation of constants

- And, with corresponding temporal variation in $n = n(c_1, c_2)$ and $k = k(c_1, c_2)$, *and*, by modifying the unperturbed expressions, for the temporal variation of the basic angles l and λ entering Eqs. into the new expressions:

$$
\begin{aligned}
l &\equiv \int_{t_0}^{t} n\, dt \,+\, c_l(t)\,, \\
\lambda &\equiv \int_{t_0}^{t} (1+k)\, n\, dt \,+\, c_\lambda(t)\,,
\end{aligned}
$$

  involving two new evolving quantities $c_l(t)$, and $c_\lambda(t)$.

- Seek for solutions of the exact system, in the 'unperturbed' form given with *four* 'varying constants' $c_1(t)$, $c_2(t)$, $c_l(t)$ and $c_\lambda(t)$. Four variables $\{c_1, c_2, c_l, c_\lambda\}$ replace the original four dynamical variables r, $\dot{r}$, ϕ and $\dot{\phi}$.

- The alternate set $\{c_1, c_2, c_l, c_\lambda\}$ satisfies, like the original phase-space variables, first order evolution equations (Damour 83,85). These evolution equations have a rather simple functional form,

Method of variation of constants

$$\frac{dc_\alpha}{dt} = F_\alpha(l; c_\beta) \; ; \; \alpha, \beta = 1, 2, l, \lambda \, ,$$

where RHS is linear in the perturbing acceleration, $\mathcal{A}'$. Note the presence of the sole angle l (apart from the implicit time dependence of c_β) on the RHS.

$$\begin{aligned}
\frac{dc_1}{dt} &= \frac{\partial c_1(\mathbf{x}, \mathbf{v})}{\partial v^i} \mathcal{A}'^i \, , \\
\frac{dc_2}{dt} &= \frac{\partial c_2(\mathbf{x}, \mathbf{v})}{\partial v^j} \mathcal{A}'^j \, , \\
\frac{dc_l}{dt} &= -\left(\frac{\partial S}{\partial l}\right)^{-1} \left(\frac{\partial S}{\partial c_1} \frac{dc_1}{dt} + \frac{\partial S}{\partial c_2} \frac{dc_2}{dt}\right) , \\
\frac{dc_\lambda}{dt} &= -\frac{\partial W}{\partial l} \frac{dc_l}{dt} - \frac{\partial W}{\partial c_1} \frac{dc_1}{dt} - \frac{\partial W}{\partial c_2} \frac{dc_2}{dt} \, .
\end{aligned}$$

- Evolution eqns for c_1 and c_2 clearly arise from the fact that c_1 and c_2 were defined as some first integrals in phase-space.

Method of variation of constants

Alternative expression for $\frac{dc_l}{dt}$ reads

$$\frac{dc_l}{dt} = \left(\frac{\partial Q}{\partial l}\right)^{-1} \left(\mathcal{A}' \cdot \mathbf{n} - \frac{\partial Q}{\partial c_1}\,\frac{dc_1}{dt} - \frac{\partial Q}{\partial c_2}\,\frac{dc_2}{dt}\right),$$

$Q^2(l, c_1, c_2) = \dot{r}^2(S(l, c_1, c_2), c_1, c_2)$ and $\frac{\partial Q}{\partial l}$ defined by

$$\frac{\partial Q}{\partial l} = \frac{P}{4\,\pi}\,\frac{\partial Q^2}{\partial r}\,.$$

- Both expressions, involve formal delicate limits of the $0/0$ form, for some (different) values of l. Taken together, they prove that these limits are well-defined and yield for dc_l/dt, an everywhere regular function of l.
- Definition of l given by $l = \int_{t_0}^{t} n(c_a(t))\,dt \,+\, c_l(t)$, is equivalent to the differential form, $\frac{dl}{dt} = n + \frac{dc_l}{dt} = n + F_l(l, c_a); a = 1\,,\,2$, which allow to define a set of differential equations for c_α as functions of l similar to c_α as functions of t. The exact form of the differential equations for $c_\alpha(l)$ reads

Method of variation of constants

$$\frac{dc_\alpha}{dl} = \frac{F_\alpha(l; c_a)}{n(c_a) + F_l(l; c_a)},$$

c_a, $a = 1, 2$, stands for c_1 and c_2.

- Neglecting terms quadratic in F_α, *i.e.* quadratic in the perturbation $\mathcal{A}'$ (e.g. neglecting $\mathcal{O}(c^{-10})$ terms in our application), simplify the system above to

$$\frac{dc_\alpha}{dl} \simeq \frac{1}{n(c_a)} F_\alpha(l; c_a) \equiv G_\alpha(l; c_a)\ ;\ \alpha = 1, 2, l, \lambda\,; a = 1\ ,\ 2\,.$$

  Simplified system adequate.

- Crucial to note, that the RHS is a function of c_1, c_2 and the sole angle l (and not of λ), and that it is a *periodic* function of l. Periodicity, together with the slow ($G_\alpha \propto F_\alpha \propto \mathcal{A}' = \mathcal{O}(c^{-5})$) evolution of the c_α's, implies that the evolution of $c_\alpha(l)$ contains both a 'slow' (radiation-reaction time-scale) secular drift and 'fast' (orbital time-scale) periodic oscillations.

Method of variation of constants

- For the purpose of phasing, to model the combination of slow drift and the fast oscillations present in c_α, introduce a two-scale decomposition for $c_\alpha(l)$ in the following manner

$$c_\alpha(l) = \bar{c}_\alpha(l) + \tilde{c}_\alpha(l)\,,$$

  where the first term $\bar{c}_\alpha(l)$ represents a slow drift (which can ultimately lead to *large* changes in the 'constants' c_α) and $\tilde{c}_\alpha(l)$ represents fast oscillations (which will stay always *small, i.e.* of order $\mathcal{O}(G_\alpha) = \mathcal{O}(c^{-5})$).

- Proved by first decomposing the periodic functions $G_\alpha(l)$ (considered for fixed values of the other arguments c_a) into its *average* part and its *oscillatory* part:

$$\bar{G}_\alpha(c_a) \equiv \frac{1}{2\pi}\int_0^{2\pi} dl\, G(l, c_a)\,,$$
$$\tilde{G}_\alpha(l; c_a) \equiv G_\alpha(l; c_a) - \bar{G}_\alpha(c_a).$$

Method of variation of constants

- *By definition*, the oscillatory part $\tilde{G}_\alpha(l)$ is a periodic function with *zero average* over l. Assuming that $\tilde{c}_\alpha$ is always small ($\tilde{c}_\alpha = \mathcal{O}(G_\alpha) = \mathcal{O}(c^{-5})$), one can expand the RHS of the exact evolution system, as

$$\begin{aligned} \frac{d\bar{c}_\alpha}{dl} + \frac{d\tilde{c}_\alpha}{dl} &= G_\alpha(l; \bar{c}_a + \tilde{c}_a) = G_\alpha(l; \bar{c}_a) + \mathcal{O}(G_\alpha^2)\,, \\ &= \bar{G}_\alpha(l; \bar{c}_a) + \tilde{G}_\alpha(l; \bar{c}_a) + \mathcal{O}(G_\alpha^2)\,. \end{aligned}$$

- Solve, modulo $\mathcal{O}(G_\alpha^2)$, the evolution equation by defining $\bar{c}_\alpha(l)$ as a solution of the 'averaged system'

$$\frac{d\bar{c}_\alpha}{dl} = \bar{G}_\alpha(\bar{c}_a)\,,$$

and by defining $\tilde{c}_\alpha(l)$ as a solution of the 'oscillatory part' of the system

$$\frac{d\tilde{c}_\alpha}{dl} = \tilde{G}_\alpha(l, \bar{c}_a)\,.$$

Method of variation of constants

- During one orbital period ($0 \leq l \leq 2\pi$) the quantities $\bar{c}_a$ on the RHS change only by $\mathcal{O}(G_\alpha)$. Therefore, by neglecting again terms of order $\mathcal{O}(G_\alpha^2) \sim \mathcal{O}(c^{-10})$ in the evolution of $\tilde{c}_\alpha$, further define $\tilde{c}_\alpha(l)$ as the *unique zero-average* solution of Eq. (**??**), considered for fixed values of $\bar{c}_a$, *i.e.*

$$\tilde{c}_\alpha(l) = \int dl \, \tilde{G}_\alpha(l; \bar{c}_a)|_{\bar{c}_a = \bar{c}_a(l)} = \int \frac{dl}{n} \, \tilde{F}_\alpha(l; \bar{c}_a) \, .$$

  zero-average periodic primitive of the zero-average (periodic) function $\tilde{G}_\alpha(l)$. During that integration, the arguments $\bar{c}_a$ are kept fixed, and, after the integration, they are replaced by the slowly drifting solution of the averaged system. Since one gets $\tilde{c}_\alpha = \mathcal{O}(G_\alpha)$, which was assumed above, one verifies the consistency of the (approximate) two-scale integration method used here.

- To apply the above described method of variation of arbitrary constants, which gave us the evolution equations for $\bar{c}_\alpha$ and $\tilde{c}_\alpha$, to GW phasing, we use 2PN accurate expressions for the dynamical variables r, $\dot{r}$, ϕ and $\dot{\phi}$ entering the expressions for $h_\times$ and h_+.

Method of variation of constants

- To do the phasing, solve the evolution equations for $\{c_1, c_2, c_l, c_\lambda\}$, on the 2PN accurate orbital dynamics. This leads to an evolution system, in which the RHS contains terms of order $\mathcal{O}(c^{-5}) \times \left[1 + \mathcal{O}(c^{-2}) + \mathcal{O}(c^{-4})\right] = \mathcal{O}(c^{-5}) + \mathcal{O}(c^{-7}) + \mathcal{O}(c^{-9})$.
- As a first step, restrict attention to the leading order contributions to $\bar{G}_\alpha$ and $\tilde{G}_\alpha$, which define the evolution of $\{\bar{c}_\alpha, \tilde{c}_\alpha\}$ under GRR to $\mathcal{O}(c^{-5})$ order. Impose these variations, on to $h_\times$ and h_+. This will allow to obtain GW polarizations, which are Newtonian accurate in their amplitudes and 2.5PN accurate in orbital dynamics. 2.5PN accurate phasing of GW. Since $\tilde{G}_\alpha$'s create only *periodic 2.5PN corrections* to the dynamics, Later look at the consequences of considering PN corrections to $\bar{G}_\alpha$ by computing $\mathcal{O}(c^{-9})$ contributions to relevant $\frac{d\bar{c}_\alpha}{dt}$. This is required as $\bar{G}_\alpha$ directly contribute to the highly important adiabatic evolution of $h_\times$ and h_+.
- Two constants c_1 and c_2 assumed to be energy and the angular momentum. Any functions of these conserved quantities can do as well. Convenient to use as c_1 the mean motion n, and as c_2 the time-eccentricity e_t. Will require to express 2PN accurate orbital dynamics in terms of l, n and e_t. Evolution equations for $\frac{dn}{dt}$, $\frac{de_t}{dt}$, $\frac{dc_l}{dt}$ and $\frac{dc_\lambda}{dt}$ in terms of l, n and e_t.

Implementation

- Compute 3PN accurate parametric expressions for the dynamical variables r, $\dot{r}$, ϕ, and $\dot{\phi}$ entering the expressions for $h_+|_{\rm Q}$ and $h_\times|_{\rm Q}$
- Solve the evolution equations for c_1, c_2, c_l, and c_λ, on the 3PN accurate orbital dynamics. This leads to an evolution system, where the RHS contains dominant $\mathcal{O}(c^{-5})$ terms and their first corrections, i.e., $\mathcal{O}(c^{-7})$ terms.
- Later impose these variations on the 3PN accurate expressions for the dynamical variables r, $\dot{r}$, ϕ, and $\dot{\phi}$, appearing in $h_+|_{\rm Q}$ and $h_\times|_{\rm Q}$. This allows us to obtain GW polarizations, which are Newtonian accurate in their amplitudes and 3.5PN accurate in the orbital dynamics. Called 3.5PN accurate phasing of GW.
- Quasi-periodic oscillations in c_α, governed by $\tilde{G}_\alpha$, are restricted to the 1PN reactive order. Therefore, not explore higher-PN corrections to the above $\tilde{G}_\alpha$.

Implementation

- Consider higher-PN corrections to $\bar{G}_\alpha$ by computing $\mathcal{O}(c^{-9})$ contributions to relevant $d\bar{c}_\alpha/dt$. This is desirable as $\bar{G}_\alpha$ directly contributes to the highly important adiabatic evolution of h_+ and $h_\times$.
- Employ as c_1 the mean motion n, and as c_2 the time eccentricity e_t. Employ 3PN accurate expressions for n and e_t in terms of E and L derived MGS.
- Have to express the 3PN accurate orbital dynamics in terms of l, n, and e_t. Derive the evolution equations for dn/dt, de_t/dt, dc_l/dt, and dc_λ/dt in terms of l, n, and e_t.
- Using these expressions, the evolution equations, for $\bar{n}$, $\bar{e}_t$, $\bar{c}_l$, $\bar{c}_\lambda$, $\tilde{n}$, $\tilde{e}_t$, $\tilde{c}_l$, and $\tilde{c}_\lambda$ will be obtained in terms of l, $\bar{n}$, and $\bar{e}_t$.
- 2PN - Damour, Gopakumar, Iyer - ADM
  3PN - Königsdörffer and Gopakumar - Modified Harmonic

3PN accurate conservative dynamics

- 3PN accurate orbital dynamics, namely, r, $\dot{r}$, ϕ, and $\dot{\phi}$, explicitly in terms of u, n, and e_t using explicit expressions for the PN orbital elements a_r, e_r, e_ϕ, k, $f_{4\phi}$, $f_{6\phi}$, $g_{4\phi}$, $g_{6\phi}$, $i_{6\phi}$, $h_{6\phi}$, g_{4t}, g_{6t}, f_{4t}, f_{6t}, i_{6t}, and h_{6t} of the GQKR (MGS).
- PN orbital elements, given in terms of E and L, can easily be expressed in terms of n and e_t with the help of the following 3PN accurate relations for $-2E$ and $-2EL^2$:

$$
\begin{aligned}
-2E &= (GMn)^{2/3}\Bigg\{1+\frac{\xi^{2/3}}{12}(15-\eta)+\frac{\xi^{4/3}}{24}\left[15-15\eta-\eta^2+\frac{24}{\sqrt{1-e_t^2}}(5-2\eta)\right]+ \\
&\quad \frac{\xi^2}{5184}\Bigg[-4995-6075\eta-450\eta^2-35\eta^3+\frac{864}{\sqrt{1-e_t^2}}(15+23\eta-20\eta^2)+ \\
&\quad \frac{18}{(1-e_t^2)^{3/2}}\left(11520-15968\eta+123\pi^2\eta+2016\eta^2\right)\Bigg]\Bigg\},
\end{aligned}
$$

3PN accurate conservative dynamics

$$
\begin{aligned}
-2EL^2 \quad = \quad & \Bigg(1 + \frac{\xi^{2/3}}{4(1-e_t^2)} \left[9 + \eta - (17 - 7\eta) e_t^2 \right] + \frac{\xi^{4/3}}{24(1-e_t^2)^2} \Bigg[189 - 45\eta + \eta^2 - \\
& 2(111 + 7\eta + 15\eta^2) e_t^2 + (225 - 277\eta + 29\eta^2) e_t^4 - (360 - 144\eta) e_t^2 \sqrt{1 - e_t^2} \Bigg] \\
& \frac{\xi^2}{6720(1-e_t^2)^3} \Bigg\{ 35(5535 - 9061\eta + 246\pi^2\eta + 142\eta^2 - \eta^3) + \\
& (299145 - 1197667\eta + 25830\pi^2\eta + 173250\eta^2 + 2345\eta^3) e_t^2 + \\
& 35(3549 - 12783\eta + 6154\eta^2 - 131\eta^3) e_t^4 - 35(2271 - 7381\eta + 2414\eta^2 - 65\eta \\
& 70 \Big[24(45 - 13\eta - 2\eta^2) - (17880 - 20120\eta \ + 123\pi^2\eta + 2256\eta^2) e_t^2 + \\
& 96(55 - 40\eta + 3\eta^2) e_t^4 \Big] \sqrt{1 - e_t^2} \Bigg\} \Bigg) (1 - e_t^2) \,,
\end{aligned}
$$

$\xi \equiv GMn/c^3$

3PN accurate conservative dynamics

- To compute expressions for $\dot{r}$ and $\dot{\phi}$, need the following relations:

$$\begin{aligned}
\frac{\partial S}{\partial l} &= a_r e_r \sin u \frac{\partial u}{\partial l} \,, \\
\frac{\partial W}{\partial l} &= \left[1 + k + 2\left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6} \right) \cos 2v + 3\left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6} \right) \cos 3v + 4\frac{i_{6\phi}}{c^6} \cos 4v + \right. \\
&\quad \left. 5\frac{h_{6\phi}}{c^6} \cos 5v \right] \frac{\partial v}{\partial u} \frac{\partial u}{\partial l} - (1+k) \,, \\
\frac{\partial u}{\partial l} &= \left\{ 1 - e_t \cos u - \frac{g_{4t}}{c^4} - \frac{g_{6t}}{c^6} + \left[\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \cos v + 2\frac{i_{6t}}{c^6} \cos 2v \right. \right. \\
&\quad \left. \left. 3\frac{h_{6t}}{c^6} \cos 3v \right] \frac{\partial v}{\partial u} \right\}^{-1} \,, \\
\frac{\partial v}{\partial u} &= \frac{(1 - e_\phi^2)^{1/2}}{1 - e_\phi \cos u} \,.
\end{aligned}$$

3PN accurate conservative dynamics

- Radial motion, defined by $r(l,n,e_t)$ and $\dot{r}(l,n,e_t)$, reads (both in the compact and in the 3PN expanded form) 5

$$r = S(l,n,e_t) = a_r(n,e_t)[1 - e_r(n,e_t)\cos u] = r_{\rm N} + r_{\rm 1PN} + r_{\rm 2PN} + r_{\rm 3PN}\,,$$

$$r_{\rm N} = \left(\frac{GM}{n^2}\right)^{1/3}(1 - e_t\cos u)\,,$$

$$r_{\rm 1PN} = r_{\rm N}\times\frac{\xi^{2/3}}{6(1-e_t\cos u)}[-18 + 2\eta - (6 - 7\eta)e_t\cos u]\,,$$

$$r_{\rm 2PN} = r_{\rm N}\times\frac{\xi^{4/3}}{72(1-e_t^2)(1-e_t\cos u)}\Big\{-72(4-7\eta) + \Big[72 + 30\eta + 8\eta^2 - i$$

$$(72 - 231\eta + 35\eta^2)e_t\cos u\Big](1-e_t^2) - 36(5-2\eta)(2 + e_t\cos u)\sqrt{1-e_t^2}\Big\}$$

$$r_{\rm 3PN} = r_{\rm N}\times\frac{\xi^2}{(1-e_t^2)^2(1-e_t\cos u)}\Bigg(-\frac{70}{3} + \frac{56221}{840}\eta - \frac{123}{64}\pi^2\eta - \frac{151}{36}\eta^2 + \frac{2}{81}\eta^3 +$$

$$\left(-\frac{2}{3} + \frac{87}{16}\eta - \frac{437}{144}\eta^2 + \frac{49}{1296}\eta^3\right)\times e_t\cos u$$

$$\ldots\ldots\ldots\ldots\Bigg)$$

3PN accurate conservative dynamics

$$\dot{r} = n\frac{\partial S}{\partial l}(l, n, e_t) = \dot{r}_{\rm N} + \dot{r}_{\rm 1PN} + \dot{r}_{\rm 2PN} + \dot{r}_{\rm 3PN}\,,$$

where

$$\dot{r}_{\rm N} = \frac{(GMn)^{1/3}}{(1 - e_t\cos u)} e_t \sin u\,,$$

$$\dot{r}_{\rm 1PN} = \dot{r}_{\rm N} \times \frac{\xi^{2/3}}{6}(6 - 7\eta)\,,$$

$$\dot{r}_{\rm 2PN} = \dot{r}_{\rm N} \times \frac{\xi^{4/3}}{72(1 - e_t\cos u)^3}\Big[-468 - 15\eta + 35\eta^2 + (135\eta - 9\eta^2)e_t^2 + $$
$$+(324 + 342\eta - 96\eta^2)e_t\cos u + (216 - 693\eta + 105\eta^2)(e_t\cos u)^2$$
$$-(72 - 231\eta + 35\eta^2)(e_t\cos u)^3 + \frac{36}{\sqrt{1 - e_t^2}}(1 - e_t\cos u)^2(4 - e_t\cos u)(5 - 2\eta$$

$$\dot{r}_{\rm 3PN} = \dot{r}_{\rm N} \times \frac{\xi^2}{(1 - e_t^2)^{3/2}(1 - e_t\cos u)^5}\left(75 - \frac{2071}{18}\eta + \frac{41}{48}\pi^2\eta + \frac{41}{3}\eta^2 \cdots\cdots\cdots\cdots\right)$$

3PN accurate conservative dynamics

- The angular motion, described in terms of ϕ and $\dot{\phi}$, is given by

$$
\begin{aligned}
\phi(\lambda, l) &= \lambda + W(l)\,, \\
\lambda &= (1+k)l\,, \\
k &= \frac{3\xi^{2/3}}{1-e_t^2} + \frac{\xi^{4/3}}{4(1-e_t^2)^2}\left[78 - 28\eta + (51-26\eta)e_t^2\right] + \\
&\quad \frac{\xi^2}{128(1-e_t^2)^3}\Big\{18240 - 25376\eta + 492\pi^2\eta + 896\eta^2 + \\
&\quad (28128 - 27840\eta + 123\pi^2\eta + 5120\eta^2)e_t^2 + (2496 - 1760\eta + 1040\eta^2)e_t^4 \\
&\quad + \left[1920 - 768\eta + (3840 - 1536\eta)e_t^2\right]\sqrt{1-e_t^2}\Big\}\,, \\
W(l) &= W_{\rm N} + W_{\rm 1PN} + W_{\rm 2PN} + W_{\rm 3PN}\,, \\
W_{\rm N} &= v - u + e_t \sin u\,, \\
W_{\rm 1PN} &= \frac{3\xi^{2/3}}{1-e_t^2}(v - u + e_t \sin u)\,,
\end{aligned}
$$

3PN accurate conservative dynamics

$$
\begin{aligned}
W_{2\mathrm{PN}} &= \frac{\xi^{4/3}}{32(1-e_t^2)^2(1-e_t\cos u)^3} \\
&\quad \times\Bigg(8\left[78-28\eta+(51-26\eta)e_t^2-6(5-2\eta)(1-e_t^2)^{3/2}\right](v-u)(1-e_t\cos u)^3 \\
&+ \Big\{624-284\eta+4\eta^2+(408-88\eta-8\eta^2)e_t^2-(60\eta-4\eta^2)e_t^4+ \\
&- \left[1872+792\eta-8\eta^2-(1224-384\eta-16\eta^2)e_t^2+(120\eta-8\eta^2)e_t^4\right]e_t\cos u \\
&+ \left[1872-732\eta+4\eta^2+(1224-504\eta-8\eta^2)e_t^2-(60\eta-4\eta^2)e_t^4\right](e_t\cos u)^2 \\
&+ \left[-624+224\eta-(408-208\eta)e_t^2\right](e_t\cos u)^3\Big\}e_t\sin u+ \\
&- \Big\{(8+153\eta-27\eta^2)e_t^2+(4\eta-12\eta^2)e_t^4+\Big[8+152\eta-24\eta^2+(8+146\eta-6\eta^2) \\
&\quad \times e_t\cos u+\left[-8-148\eta+12\eta^2-(\eta-3\eta^2)e_t^2\right]\times(e_t\cos u)^2\Big\}e_t\sin u\sqrt{1-e_t^2} \\
W_{3\mathrm{PN}} &= \frac{\xi^2}{(1-e_t^2)^3(1-e_t\cos u)^5}\Bigg(\;\cdots\cdots\;\Bigg),
\end{aligned}
$$

3PN accurate conservative dynamics

$$
\begin{aligned}
\dot{\phi} &= \dot{\phi}_{\rm N} + \dot{\phi}_{\rm 1PN} + \dot{\phi}_{\rm 2PN} + \dot{\phi}_{\rm 3PN} \,, \\
\dot{\phi}_{\rm N} &= \frac{n\sqrt{1-e_t^2}}{(1-e_t\cos u)^2} \,, \\
\dot{\phi}_{\rm 1PN} &= \dot{\phi}_{\rm N} \times \frac{\xi^{2/3}}{(1-e_t^2)(1-e_t\cos u)} \Big[3-(4-\eta)e_t^2+(1-\eta)e_t\cos u \Big] \,, \\
\dot{\phi}_{\rm 2PN} &= \dot{\phi}_{\rm N} \times \frac{\xi^{4/3}}{12(1-e_t^2)^2(1-e_t\cos u)^3} \Big\{ \cdots\cdots \Big\} \\
\dot{\phi}_{\rm 3PN} &= \dot{\phi}_{\rm N} \times \frac{\xi^{2}}{(1-e_t^2)^3(1-e_t\cos u)^5} \Big\{ \cdots\cdots \Big\}
\end{aligned}
$$

3PN accurate conservative dynamics

- Eccentric anomaly $u = \mathcal{U}(l, n, e_t)$ given by inverting the 3PN accurate Kepler equation, connecting l and u.

$$
\begin{aligned}
l \quad = \quad & u - e_t \sin u + \\
& \frac{\xi^{4/3}}{8\sqrt{1-e_t^2}(1-e_t\cos u)} \Big[(15\eta - \eta^2) e_t \sin u \sqrt{1-e_t^2} + 12(5-2\eta)(v-u)(1-e_t \cos
\\
+ \quad & \frac{\xi^2}{6720(1-e_t^2)^{3/2}(1-e_t\cos u)^3} \Bigg(\Big\{ 67200 + 143868\eta - 4305\pi^2\eta - 62160\eta^2 - 280
\\
& (134400 + 139896\eta - 8610\pi^2\eta - 67200\eta^2 - 3920\eta^3) e_t \cos u \\
& +(67200 - 752\eta - 4305\pi^2\eta - 15260\eta^2 - 1820\eta^3)(e_t \cos u)^2 \\
& + \Big[-148960\eta + 45500\eta^2 - 1540\eta^3 + (143640\eta - 13440\eta^2 - 3920\eta^3) e_t \cos u \\
& -(1120\eta + 11620\eta^2 - 1820\eta^3) \times (e_t \cos u)^2\Big] e_t^2 + (3220\eta - 10220\eta^2 + 1820\eta^3) e_t^4
\\
& \times e_t \sin u \sqrt{1-e_t^2} + \Big[302400 - 461440\eta + 4305\pi^2\eta + 33600\eta^2 + \\
& i\ (100800 - 97440\eta + 36960\eta^2) e_t^2\Big] (v-u)(1-e_t\cos u)^3 \Bigg) .
\end{aligned}
$$

- No 1PN contribution, in ADM or modified harmonic coords. Expected differences in the Kepler equation at higher-PN orders between the harmonic and the ADM gauge.

3.5PN accurate reactive dynamics

- In addition to the above explicit expressions for r, $\dot{r}$, ϕ, and $\dot{\phi}$, need to evaluate RHS.
- To study the effects of the 2.5PN and 3.5PN contributions to radiation reaction on the 3PN accurate conservative motion, truncate away all effects that would correspond to higher-PN orders.
- For nonspinning point masses, highly difficult to go beyond 3.5PN accuracy for the oscillatory effects associated with the two-scale decomposition
- Secular effects can be computed to higher-PN orders,
- Available are required inputs for the computation of the 3.5PN accurate evolution equations for the sets $\{\bar{c}_\alpha\}$ and $\{\tilde{c}_\alpha\}$, where $\alpha = n, e_t, c_l, c_\lambda$.
- Need $\boldsymbol{\mathcal{A}}'$ to the 3.5PN order for this purpose. in the appropriate gauge as the conservative 3PN dynamics
- Pati Will 2002, Nissanke Blanchet 2005, Königsdörffer Faye Schäfer 2003,

3.5PN accurate reactive dynamics

$$\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}}'_{2.5\mathrm{PN}} + \boldsymbol{\mathcal{A}}'_{3.5\mathrm{PN}} \,,$$

$$\begin{aligned}
\boldsymbol{\mathcal{A}}'_{2.5\mathrm{PN}} &= \frac{8}{15}\frac{G^2M^2\eta}{c^5r^3}\left[\left(9v^2+17\frac{GM}{r}\right)\dot{r}\boldsymbol{n} - \left(3v^2+9\frac{GM}{r}\right)\boldsymbol{v}\right] \,, \\
\boldsymbol{\mathcal{A}}'_{3.5\mathrm{PN}} &= \frac{2}{105}\frac{G^2M^2\eta}{c^7r^3}\Bigg(\Bigg\{-(549+630\eta)v^4+(5985+630\eta)v^2\dot{r}^2-5880\dot{r}^4 \\
&- \frac{GM}{r}\left[(1038-2534\eta)v^2+(3087+3948\eta)\dot{r}^2\right] - \frac{G^2M^2}{r^2}(5934+1932\eta)\Bigg\}\dot{r}\boldsymbol{n} \\
&\quad + \Bigg\{(939+126\eta)v^4-(7119+126\eta)v^2\dot{r}^2+6300\dot{r}^4 - \\
&\quad \frac{GM}{r}\left[(410+1554\eta)v^2-(1435+2968\eta)\dot{r}^2\right] + \frac{G^2M^2}{r^2}(2650+1092\eta)\Bigg\}\boldsymbol{v}\Bigg) \,, \\
v^2 &= \boldsymbol{v}\cdot\boldsymbol{v} = \dot{r}^2 + r^2\dot{\phi}^2 .
\end{aligned}$$

3.5PN accurate reactive dynamics

- Explicit computations of RHS require only 1PN accurate expressions for the orbital elements because we are trying to obtain the phasing to the 3.5PN order and the reactive dynamics only involves 2.5PN and 3.5PN contributions.
- BUT, this does not mean that the orbital dynamics is only 1PN accurate. Phasing formalism allows us to impose the fully 1PN accurate reactive dynamics on the 3PN accurate conservative dynamics to provide the 3.5PN accurate phasing

3.5PN accurate reactive dynamics

- Finally, the evolution equations for dn/dl, de_t/dl, dc_l/dl, and dc_λ/dl in terms of $u(l, n, e_t)$, n, and e_t,

$$
\begin{aligned}
\frac{dn}{dl} \quad = \quad & -\frac{8\xi^{5/3}n\eta}{5}\left\{\frac{6}{\chi^3}-\frac{32}{\chi^4}+\frac{49-9e_t^2}{\chi^5}-\frac{35(1-e_t^2)}{\chi^6}\right\}- \\
& \frac{\xi^{7/3}n\eta}{35}\left\{\frac{-360+1176\eta}{\chi^3}+\frac{2680-11704\eta}{\chi^4}+\right. \\
& \Big[-4012+34356\eta+(36-756\eta)e_t^2\Big]\frac{1}{\chi^5}+ \\
& \Big[1470-47880\eta-(350-17080\eta)e_t^2\Big]\frac{1}{\chi^6}+ \\
& \Big[13510+31780\eta-(24220+30520\eta)e_t^2+(10710-1260\eta)e_t^4\Big]\frac{1}{\chi^7}- \\
& \left.\frac{(27594+5880\eta)(1-e_t^2)^2}{\chi^8}+\frac{11760(1-e_t^2)^3}{\chi^9}\right\},
\end{aligned}
$$

$\chi \equiv 1 - e_t \cos u$ and $u = u(l, n, e_t)$.

3.5PN accurate reactive dynamics

$$
\begin{aligned}
\frac{de_t}{dl} \quad = \quad & \frac{8\xi^{5/3}\eta(1-e_t^2)}{15e_t}\left\{\frac{3}{\chi^3}-\frac{17}{\chi^4}+\frac{49-9e_t^2}{\chi^5}-\frac{35(1-e_t^2)}{\chi^6}\right\}+\\
& \frac{\xi^{7/3}\eta}{315e_t}\left\{\left[-4320+6636\eta+(5328-7644\eta)e_t^2\right]\frac{1}{\chi^3}\right.\\
& +\left[20430-36176\eta-(25470-41216\eta)e_t^2\right]\frac{1}{\chi^4}+\\
& \left[-73650+106120\eta+(112692-116200\eta)e_t^2-(39042-10080\eta)e_t^4\right]\frac{1}{\chi^5}+\\
& \left[102312-154280\eta-(201264-216160\eta)e_t^2+(98952-61880\eta)e_t^4\right]\frac{1}{\chi^6}+\\
& \left[2730+95340\eta+(210-186900\eta)e_t^2-(8610-87780\eta)e_t^4+(5670+3780\eta)e_t^6\right]\frac{}{\chi}\\
& \left.\frac{(82782+17640\eta)(1-e_t^2)^3}{\chi^8}+\frac{35280(1-e_t^2)^4}{\chi^9}\right\},
\end{aligned}
$$

3.5PN accurate reactive dynamics

$$
\begin{aligned}
\frac{dc_l}{dl} &= \frac{8\xi^{5/3}\eta\sin u}{15e_t}\left\{\frac{12e_t^2}{\chi^3}+\frac{3-43e_t^2}{\chi^4}+\frac{-14+23e_t^2-9e_t^4}{\chi^5}+\frac{35(1-e_t^2)^2}{\chi^6}\right\} \\
&+ \frac{\xi^{7/3}\eta\sin u}{315e_t}\Bigg\{\frac{(-4176+9408\eta)e_t^2}{\chi^3} \\
&+ \Big[-4320+6636\eta+(23808-71596\eta)e_t^2\Big]\frac{1}{\chi^4} \\
&+ \Big[16110-29540\eta-(23862-128240\eta)e_t^2+(4392-6300\eta)e_t^4\Big]\frac{1}{\chi^5} \\
&+ \Big[-57540+76580\eta+(103320-159880\eta)e_t^2-(45780-83300\eta)e_t^4\Big]\frac{1}{\chi^6} \\
&+ \frac{\Big[44772-77700\eta-(59934-151620\eta)e_t^2-(14448+70140\eta)e_t^4+(29610-3780\eta}{\chi^7} \\
&+ \frac{(47502+17640\eta)(1-e_t^2)^3}{\chi^8}-\frac{35280(1-e_t^2)^4}{\chi^9}\Bigg\},
\end{aligned}
$$

3.5PN accurate reactive dynamics

$$
\begin{aligned}
\frac{dc_\lambda}{dl} \quad = \quad & -\frac{8\xi^{5/3}\eta\sin u}{15e_t}\Biggl\{\left[\frac{3}{\chi^4}-\frac{14-9e_t^2}{\chi^5}+\frac{35(1-e_t^2)}{\chi^6}\right]\sqrt{1-e_t^2}-\frac{12e_t^2}{\chi^3}-\frac{3-43e_t^2}{\chi^4} \\
& +\frac{14-23e_t^2+9e_t^4}{\chi^5}-\frac{35(1-e_t^2)^2}{\chi^6}\Biggr\} \\
& +\frac{2\xi^{7/3}\eta\sin u}{315e_t(1-e_t^2)}\Biggl\{\Biggl[\biggl[1404-3318\eta+(360+3066\eta)e_t^2\biggr]\frac{1}{\chi^4} \\
& +\biggl[-4527+14770\eta-(4029+15232\eta)e_t^2+(576+882\eta)e_t^4\biggr]\frac{1}{\chi^5} \\
& +\biggl[19950-38290\eta-(38640-58520\eta)e_t^2+(18690-20230\eta)e_t^4\biggr]\frac{1}{\chi^6} \\
& +\biggl[-22386+38850\eta+(27447-75810\eta)e_t^2+(12264+35070\eta)e_t^4-(17325-1890\eta)e_t^6\biggr]\frac{1}{\chi^7} \\
& -\frac{(23751+8820\eta)(1-e_t^2)^3}{\chi^8}+\frac{17640(1-e_t^2)^4}{\chi^9}\Biggr]\sqrt{1-e_t^2}+\biggl[(2448+4704\eta)e_t^2+(2088-4704\eta)e \\
& +\biggl[-1404+3318\eta-(4332+39116\eta)e_t^2-(11904-35798\eta)e_t^4\biggr]\frac{1}{\chi^4}+
\end{aligned}
$$

3.5PN accurate reactive dynamics

$$+\left[4527-14770\eta-(14190-78890\eta)e_t^2+(11859-67270\eta)e_t^4-(2196-3150\eta)e_t^6\right]\frac{1}{\chi^5}$$

$$+\left[-19950+38290\eta+(62790-118230\eta)e_t^2-(65730-121590\eta)e_t^4+(22890-41650\eta)e_t^6\right]\frac{1}{\chi^6}$$

$$+\frac{\left[22386-38850\eta-(52353-114660\eta)e_t^2+(22743-110880\eta)e_t^4+(22029+33180\eta)e_t^6-(14805-1890\right.}{\chi^7}$$

$$\left.+\frac{(23751+8820\eta)(1-e_t^2)^4}{\chi^8}-\frac{17640(1-e_t^2)^5}{\chi^9}\right\}$$

$$+\frac{48\xi^{7/3}\eta(v-u)}{5(1-e_t^2)}\left\{\frac{1}{\chi^3}-\frac{5}{\chi^4}\right\},$$

These are inputs to explore the secular and periodic variations of c_α to the 3.5PN order

Secular variations

- One first extracts secular variations of c_α. Secular evolution of c_α can be obtained by orbital averaging using l.
- It is preferable to perform the orbital averaging in terms of u rather than l, as RHS are explicit functions of u, and to this accuracy one can use $dl \simeq (1 - e_t \cos u) du$, due to the fact that Eq. for l does not have a 1PN contribution.
- Required integration over u is done by the following definite integral,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e_t \cos u)^{N+1}} = \frac{1}{(1 - e_t^2)^{(N+1)/2}} P_N \left(\frac{1}{\sqrt{1 - e_t^2}} \right) ,$$

where P_N is the Legendre polynomial.
- To express these differential equations in terms of the original time variable t rather than l, use $dl = \bar{n} dt$.

Secular variations

$$\frac{d\bar{n}}{dt} = \frac{\xi^{5/3} n^2 \eta}{5(1-e_t^2)^{7/2}} \left\{ 96 + 292 e_t^2 + 37 e_t^4 \right\}$$
$$+ \frac{\xi^{7/3} n^2 \eta}{280(1-e_t^2)^{9/2}} \left\{ 20368 - 14784\eta + (219880 - 159600\eta) e_t^2 \right.$$
$$\left. + (197022 - 141708\eta) e_t^4 + (11717 - 8288\eta) e_t^6 \right\},$$
$$\frac{d\bar{e}_t}{dt} = -\frac{\xi^{5/3} n \eta e_t}{15(1-e_t^2)^{5/2}} \left\{ 304 + 121 e_t^2 \right\}$$
$$- \frac{\xi^{7/3} n \eta e_t}{2520(1-e_t^2)^{7/2}} \left\{ 340968 - 228704\eta + (880632 - 651252\eta) e_t^2 \right.$$
$$\left. + (125361 - 93184\eta) e_t^4 \right\}$$

n and e_t, on RHS of these eqns, stand for $\bar{n}$ and $\bar{e}_t$, respectively.

- In agreement with equivalent expressions, in Blanchet Schäfer 89, Junker Schäfer 92 computed using balance arguments involving local radiation damping and far-zone fluxes.
- These secular evolutions of n and e_t, namely, $\bar{n}$ and $\bar{e}_t$, are crucial for the phasing,

Secular variations

- Regarding secular variation of c_l and c_λ, namely, $\bar{c}_l$ and $\bar{c}_\lambda$, we find that there are no secular evolutions for both c_l and c_λ to the 1PN order of radiation reaction. $\bar{G}_l = 0 = \bar{G}_\lambda$, where $G_l = F_l/n$ and $G_\lambda = F_\lambda/n$, respectively,
- In this case RHS are functions of the form $\sin u \times f(\cos u)$ and $(v - u) \times f(\cos u)$, respectively, and hence they are odd under $u \to -u$. Therefore, their average over $dl \simeq (1 - e_t \cos u)du$ exactly vanishes, leading to $\bar{G}_l = 0 = \bar{G}_\lambda$ to the 3.5PN order.
- Related to time-odd character of the perturbing force $\boldsymbol{\mathcal{A}}'$, $\partial c_1/\partial v^i$, and $\partial c_2/\partial v^j$, respectively, ending up with the conclusion that dc_l/dt and dc_λ/dt are time odd.

$$
\begin{aligned}
\frac{d\bar{c}_l}{dt} &= 0\,; \\
\bar{c}_l(t) &= \bar{c}_l(t_0)\,, \\
\frac{d\bar{c}_\lambda}{dt} &= 0\,; \\
\bar{c}_\lambda(t) &= \bar{c}_\lambda(t_0)\,.
\end{aligned}
$$

Periodic variations

- To complete this study look at the difl eqns for $\tilde{n}$, $\tilde{e}_t$, $\tilde{c}_l$, and $\tilde{c}_\lambda$, which give orbital period oscillations to dynamical variables at $\mathcal{O}(c^{-5})$ and $\mathcal{O}(c^{-7})$.
- The oscillatory part are zero-average oscillatory functions of l.

$$\begin{aligned}\frac{d\tilde{n}}{dl} \quad = \quad & -\frac{8\xi^{5/3}n\eta}{5}\left\{\frac{6}{\chi^3}-\frac{32}{\chi^4}+\frac{49-9e_t^2}{\chi^5}-\frac{35(1-e_t^2)}{\chi^6}\right\} \\ & -\frac{\xi^{5/3}n\eta}{5(1-e_t^2)^{7/2}}\left\{96+292e_t^2+37e_t^4\right\} \\ & -\frac{\xi^{7/3}n\eta}{35}\left\{\frac{-360+1176\eta}{\chi^3}+\frac{2680-11704\eta}{\chi^4}\cdots\cdots\right\} \\ & -\frac{\xi^{7/3}n\eta}{280(1-e_t^2)^{9/2}}\left\{20368-14784\eta+(219880-159600\eta)e_t^2\cdots\cdots\right\},\end{aligned}$$

- n and e_t, on RHS stand for $\bar{n}$ and $\bar{e}_t$.

Periodic variations

$$
\begin{aligned}
\frac{d\tilde{e}_t}{dl} &= \frac{8\xi^{5/3}\eta(1-e_t^2)}{15e_t}\left\{\frac{3}{\chi^3}-\frac{17}{\chi^4}+\frac{49-9e_t^2}{\chi^5}-\frac{35(1-e_t^2)}{\chi^6}\right\} \\
&+\frac{\xi^{5/3}\eta e_t}{15(1-e_t^2)^{5/2}}\left\{304+121e_t^2\right\} \\
&+\frac{\xi^{7/3}\eta}{315e_t}\left\{\left[-4320+6636\eta+(5328-7644\eta)e_t^2\right]\frac{1}{\chi^3}\cdots\cdots\right\} \\
&+\frac{\xi^{7/3}\eta e_t}{2520(1-e_t^2)^{7/2}}\left\{340968-228704\eta\cdots\cdots\right\},
\end{aligned}
$$

- Since $\bar{G}_l = 0 = \bar{G}_\lambda$ to the 1PN reactive order, diffl eqns for $\tilde{c}_l$ and $\tilde{c}_\lambda$ are identical to those for c_l and c_λ, but with n and e_t replaced by $\bar{n}$ and $\bar{e}_t$,

$$
\begin{aligned}
\frac{d\tilde{c}_l}{dl} &= \text{RHS of Eq. for } \frac{dc_l}{dl}[n\to\bar{n}, e_t\to\bar{e}_t]\,, \\
\frac{d\tilde{c}_\lambda}{dl} &= \text{RHS of Eq. for } \frac{dc_\lambda}{dl}[n\to\bar{n}, e_t\to\bar{e}_t]\,.
\end{aligned}
$$

Periodic variations

- One can analytically integrate Eqs. to get $\tilde{n}$, $\tilde{e}_t$, $\tilde{c}_l$, and $\tilde{c}_\lambda$ as zero-average oscillatory functions of l. Expressed in terms of u,

$$\tilde{n} = \frac{\xi^{5/3} n \eta e_t \sin u}{15(1-e_t^2)^3}\left\{\frac{602+673e_t^2}{\chi} + \frac{314-203e_t^2-111e_t^4}{\chi^2} + \frac{98-124e_t^2-46e_t^4+72}{\chi^3}\right.$$
$$\left.+\frac{210(1-e_t^2)^3}{\chi^4}\right\}$$
$$+\frac{\xi^{5/3} n\eta}{5(1-e_t^2)^{7/2}}\left\{96+292e_t^2+37e_t^4\right\}\left\{2\tan^{-1}\left(\frac{\beta_t \sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right\}$$
$$+\frac{\xi^{7/3} n\eta e_t \sin u}{4200(1-e_t^2)^4}\left\{\left[827796-601720\eta\cdots\cdots\right\}\right.$$
$$+\frac{\xi^{7/3} n\eta}{280(1-e_t^2)^{9/2}}\left\{20368-14784\eta+(219880-159600\eta)e_t^2\cdots\cdots\right\}$$
$$\times\left\{2\tan^{-1}\left(\frac{\beta_t \sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right\},$$

Periodic variations

$$
\begin{aligned}
\tilde{e}_t &= -\frac{\xi^{5/3}\eta\sin u}{45(1-e_t^2)^2}\left\{\frac{134+1069e_t^2+72e_t^4}{\chi}+\frac{134+157e_t^2-291e_t^4}{\chi^2}\right.\\
&\left.+\frac{98-124e_t^2-46e_t^4+72e_t^6}{\chi^3}+\frac{210(1-e_t^2)^3}{\chi^4}\right\}\\
&-\frac{\xi^{5/3}\eta e_t}{15(1-e_t^2)^{5/2}}\left\{304+121e_t^2\right\}\left\{2\tan^{-1}\left(\frac{\beta_t\sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right\}\\
&-\frac{\xi^{7/3}\eta\sin u}{37800(1-e_t^2)^3}\left\{\left[78768+1960\eta\cdots\cdots\right.\right\}\\
&-\frac{\xi^{7/3}\eta e_t}{2520(1-e_t^2)^{7/2}}\left\{340968-228704\eta\cdots\cdots\right\}\left\{2\tan^{-1}\left(\frac{\beta_t\sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right\}
\end{aligned}
$$

Periodic variations

$$\begin{aligned}\tilde{c}_l &= -\frac{2\xi^{5/3}\eta}{45e_t^2}\left\{\frac{144e_t^2}{\chi}+\frac{18-258e_t^2}{\chi^2}+\frac{-56+92e_t^2-36e_t^4}{\chi^3}+\frac{105(1-e_t^2)^2}{\chi^4}\right.\\ &\quad\left.-\frac{1}{2(1-e_t^2)^{3/2}}\left[134+103e_t^2-252e_t^4\right]\right\}\\ &\quad+\frac{\xi^{7/3}\eta}{4725e_t^2}\left\{\frac{(62640-141120\eta)e_t^2}{\chi}\dots\dots\right\},\end{aligned}$$

Periodic variations

$$
\begin{aligned}
\tilde{c}_\lambda \quad = \quad & \frac{2\xi^{5/3}\eta}{45e_t^2}\Bigg\{\left[\frac{18}{\chi^2}-\frac{56-36e_t^2}{\chi^3}+\frac{105(1-e_t^2)}{\chi^4}\right]\sqrt{1-e_t^2}-\frac{144e_t^2}{\chi}-\frac{18-258e_t^2}{\chi^2} \\
& +\frac{56-92e_t^2+36e_t^4}{\chi^3}-\frac{105(1-e_t^2)^2}{\chi^4} \\
& +\frac{1}{2(1-e_t^2)^2}\left[\left(134+103e_t^2-252e_t^4\right)\sqrt{1-e_t^2}-134-295e_t^2-36e_t^4\right]\Bigg\} \\
& -\frac{\xi^{7/3}\eta}{4725e_t^2(1-e_t^2)}\Bigg(\Bigg\{\Big[21060-49770\eta \\
& +(5400+45990\eta)e_t^2\Big]\frac{1}{\chi^2}\cdots\cdots\Bigg\}\Bigg)+\frac{48\xi^{7/3}\eta}{5(1-e_t^2)}\int(v-u)\left(\frac{1}{\chi^3}-\frac{5}{\chi^4}\right)\chi\,du\,,
\end{aligned}
$$

$\beta_t=(1-\sqrt{1-e_t^2})/e_t$.

- The constant contributions to the time evolution of $\tilde{c}_l$ and $\tilde{c}_\lambda$, appearing in are required to guarantee the zero-average behaviour.
- We note that the remaining integral in Eq. can be numerically evaluated.

Periodic variations

- Above results modify the temporal evolution of the basic angles l and λ, entering the reactive dynamics
- From the definitions of $l(t)$ and $\lambda(t)$, we can also split these angles in secular and oscillatory pieces, denoted by $\bar{l}$, $\bar{\lambda}$, and $\tilde{l}$, $\tilde{\lambda}$, respectively,

$$\begin{aligned}
l(t) &= \bar{l}(t) + \tilde{l}[l; \bar{c}_a(t)] \,, \\
\lambda(t) &= \bar{\lambda}(t) + \tilde{\lambda}[l; \bar{c}_a(t)] \,, \\
\bar{l}(t) &\equiv \int_{t_0}^{t} \bar{n}(t) dt + \bar{c}_l(t) \,, \\
\bar{\lambda}(t) &\equiv \int_{t_0}^{t} [1 + \bar{k}(t)] \bar{n}(t) dt + \bar{c}_\lambda(t) \,.
\end{aligned}$$

- Note that $\bar{c}_l(t) = \bar{c}_l(t_0)$ and $\bar{c}_\lambda(t) = \bar{c}_\lambda(t_0)$ are constants.
- The oscillatory contributions to l and λ are given by

Periodic variations

$$\tilde{l}(l;\bar{c}_a) = \int \frac{\tilde{n}(l)}{n} dl + \tilde{c}_l(l)\,,$$

$$\tilde{\lambda}(l;\bar{c}_a) = \int \left[\frac{\tilde{n}}{n} + \bar{k}\frac{\tilde{n}}{n} + \tilde{k}\right] dl + \tilde{c}_\lambda(l)\,,$$

- $\tilde{k} \equiv (\partial k/\partial n)\tilde{n} + (\partial k/\partial e_t)\tilde{e}_t$ denotes the oscillatory piece in k.
- Finally, to complete our study of the oscillatory contributions associated with the reactive dynamics, we compute the integrals and add them to the previous results for $\tilde{c}_l(l)$ and $\tilde{c}_\lambda(l)$, respectively.

Periodic variations

$$\tilde{l}(l;\bar{c}_a) = \frac{\xi^{5/3}\eta}{15(1-e_t^2)^3}\Big\{(602+673e_t^2)\chi+(314-203e_t^2-111e_t^4)\ln\chi-(602+673e_t^2)$$

$$+\frac{-98+124e_t^2+46e_t^4-72e_t^6}{\chi}-\frac{105(1-e_t^2)^3}{\chi^2}\Big\}$$

$$+\frac{\xi^{5/3}\eta}{5(1-e_t^2)^{7/2}}\Big\{96+292e_t^2+37e_t^4\Big\}\Big\{\int\Big[2\tan^{-1}\Big(\frac{\beta_t\sin u}{1-\beta_t\cos u}\Big)+e_t\sin u\Big]$$

$$+\frac{\xi^{7/3}\eta}{4200(1-e_t^2)^4}\Big\{[827796-601720\eta\cdots\cdots\Big\}$$

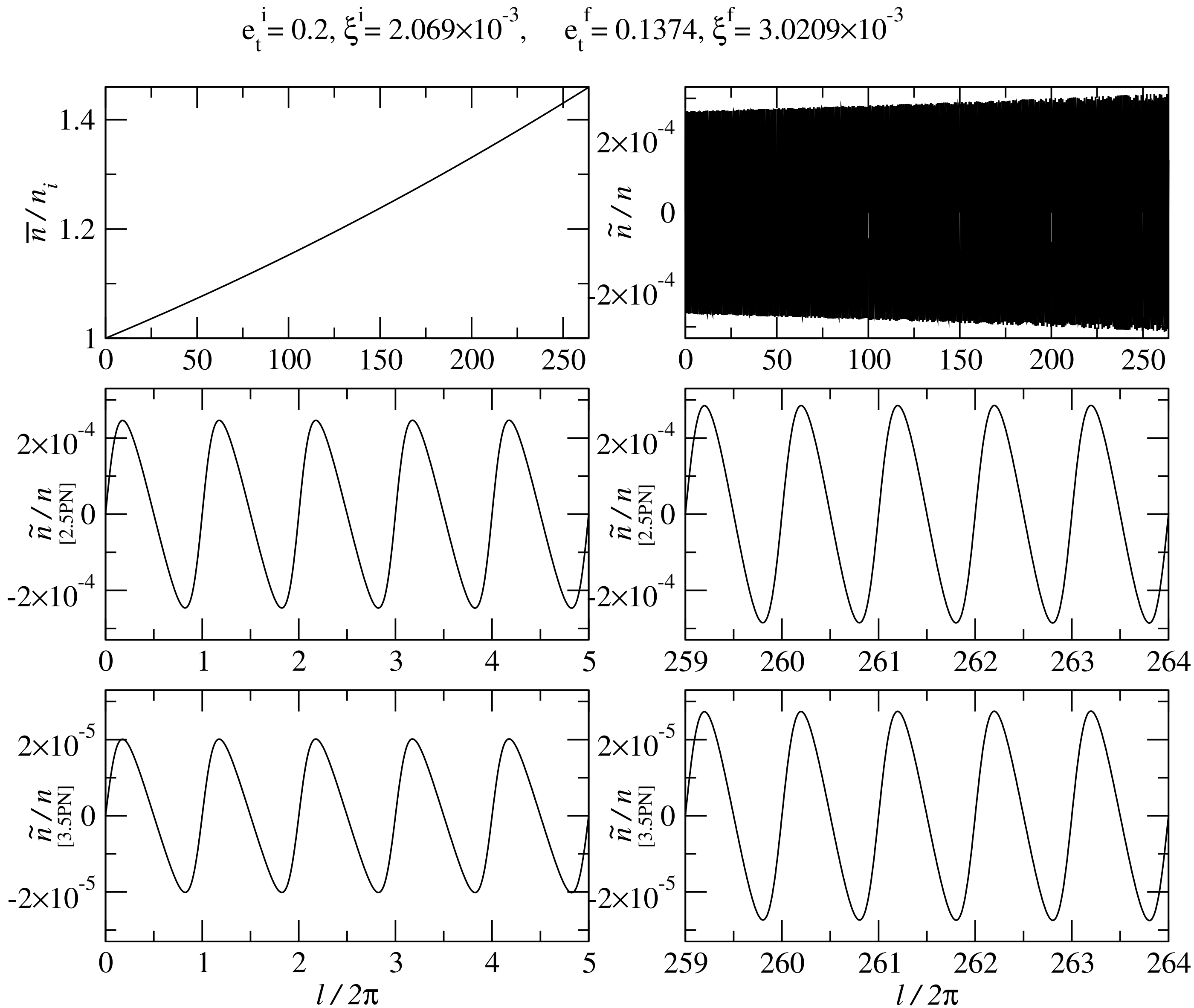
$$+\frac{\xi^{7/3}\eta}{280(1-e_t^2)^{9/2}}\Big\{20368-14784\eta\cdots\cdots\Big\}\Big\{\int\Big[2\tan^{-1}\Big(\frac{\beta_t\sin u}{1-\beta_t\cos u}\Big)+e_t$$

Periodic variations

$$
\begin{aligned}
\tilde{\lambda}(l;\bar{c}_a) \quad = \quad & \frac{\xi^{5/3}\eta}{15(1-e_t^2)^3}\Bigg\{(602+673e_t^2)\chi + (314-203e_t^2-111e_t^4)\ln\chi - (602+673e_t^2) \\
& + \frac{-98+124e_t^2+46e_t^4-72e_t^6}{\chi} - \frac{105(1-e_t^2)^3}{\chi^2}\Bigg\} \\
& + \frac{\xi^{5/3}\eta}{5(1-e_t^2)^{7/2}}\left\{96+292e_t^2+37e_t^4\right\}\Bigg\{\int\left[2\tan^{-1}\left(\frac{\beta_t\sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right] \\
& + \frac{\xi^{7/3}\eta}{4200(1-e_t^2)^4}\Bigg\{\Bigg[1595556-601720\eta\cdots\cdots\Bigg\} \\
& + \frac{\xi^{7/3}\eta}{280(1-e_t^2)^{9/2}}\Big\{47248-14784\eta+(267592-159600\eta)e_t^2+(193830-1417 \\
& +(11717-8288\eta)e_t^6\Big\}\Bigg\{\int\left[2\tan^{-1}\left(\frac{\beta_t\sin u}{1-\beta_t\cos u}\right)+e_t\sin u\right]\chi\,du\Bigg\} + \tilde{c}_\lambda(l)
\end{aligned}
$$

$\tilde{c}_l(l)$ and $\tilde{c}_\lambda(l)$ are given earlier and $\beta_t = (1-\sqrt{1-e_t^2})/e_t$. Note that the contributions to $\tilde{\lambda}(l)$ arising from the periastron advance constant k only appear at $\mathcal{O}(c^{-7})$.

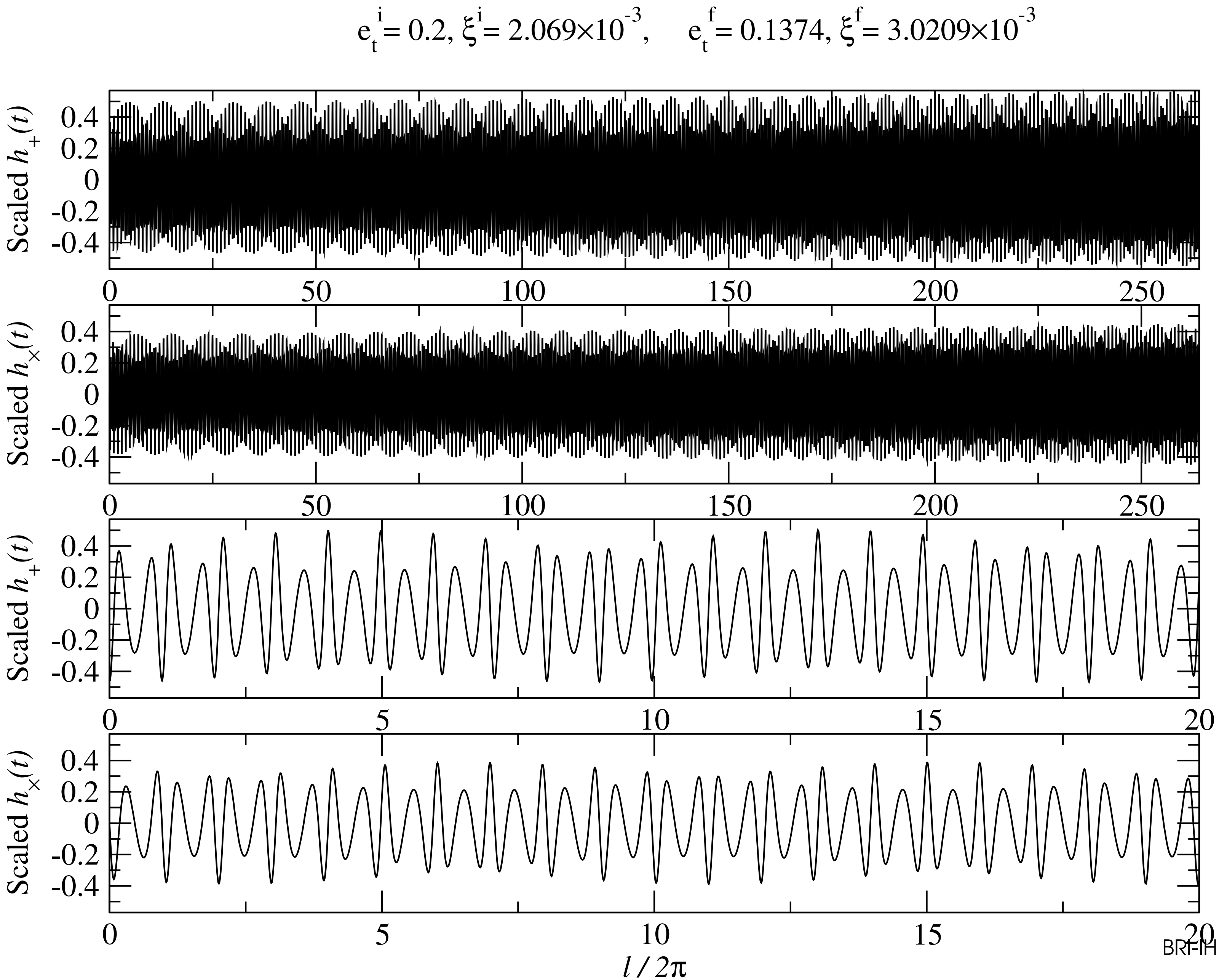
$\bar{n}/n_i$ and $\tilde{n}/n$ versus $l/(2\pi)$



$\bar{n}/n_i$ and $\tilde{n}/n$ versus $l/(2\pi)$

- The plots for $\bar{n}/n_i$ and $\tilde{n}/n$ versus $l/(2\pi)$, which gives the number of orbital revolutions. The adiabatic increase of $\bar{n}$ is clearly visible in panel 1, and the quasi-periodic nature of the variations in $\tilde{n}$ is portrayed in panels 2–6. These variations are governed by the reactive 2.5PN and 3.5PN equations of motion. In the second and third row, these contributions to $\tilde{n}$ are plotted individually and separated for the initial and final stages. The parameters e_t^i and e_t^f denote initial and final values of the time eccentricity e_t, while ξ^i and ξ^f stand for similar values of the adimensional mean motion $\xi = GMn/c^3$. The panels are plotted for $\eta = 0.25$ and the orbital evolution is terminated when $j = \sqrt{48}$.
- Conversion to familiar quantities like orbital frequency f (in hertz) is given by $f \equiv n/(2\pi) = c^3\xi/(2\pi GM) = 3.2312 \times 10^4 \xi(M_\odot/M)$. This implies that for a compact binary with the total mass $M = M_\odot$ and $\xi = 10^{-3}$, the orbital frequency will be ~ 30 Hz.

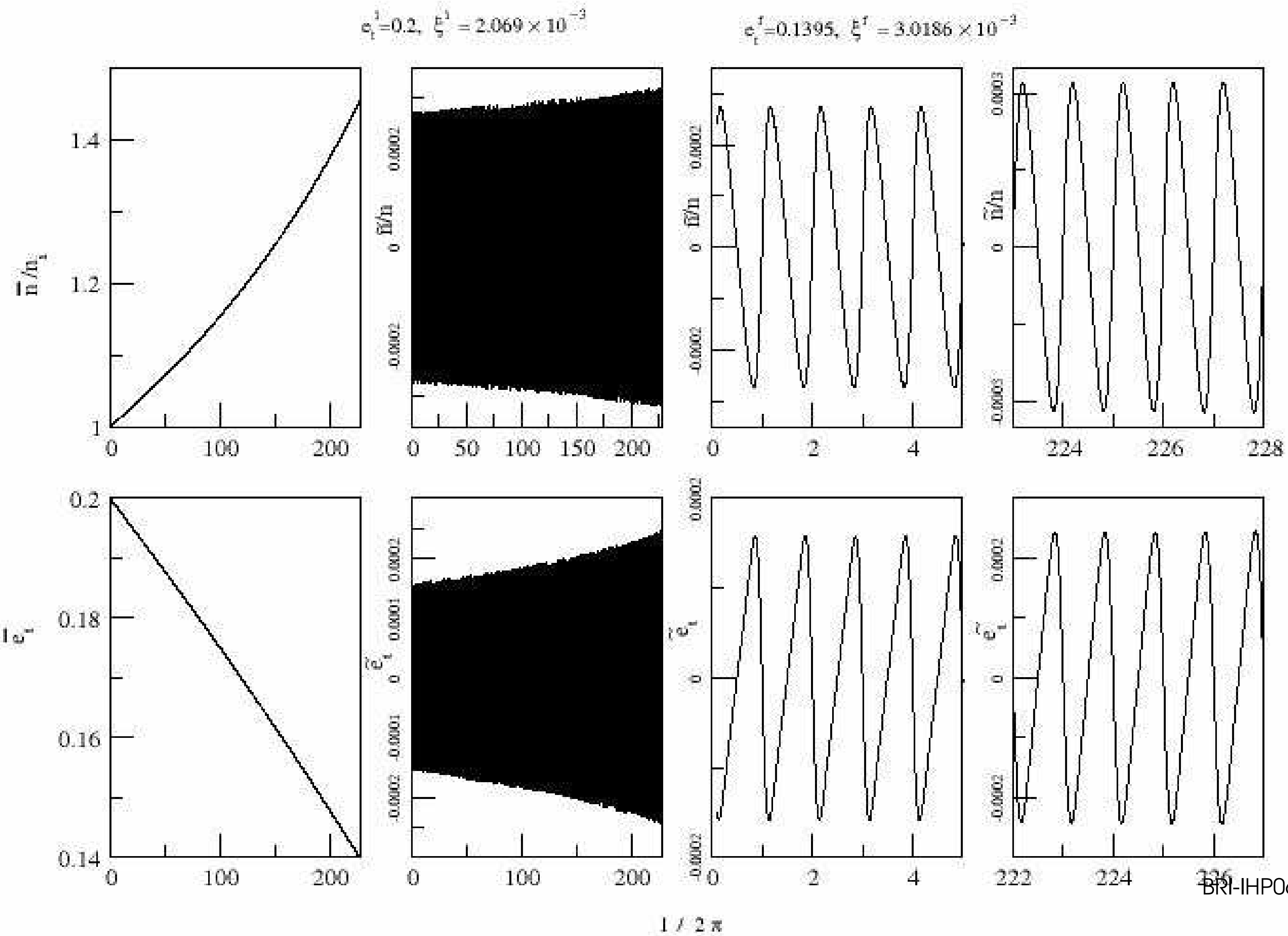
$h_+(t)$ and $h_\times(t)$



$h_+(t)$ and $h_\times(t)$

- The plots for the scaled $h_+(t)$ and $h_\times(t)$ (Newtonian in amplitude and 3.5PN in orbital motion) as functions of $l/(2\pi)$. The slow chirping and the amplitude modulation due to the periastron precession are clearly visible in the two upper panels. In the two bottom panels, we zoom into the initial stages of the orbital evolution in order to show the effect of the periodic orbital motion and the periastron advance on the scaled $h_+(t)$ and $h_\times(t)$. The initial and final values of the relevant orbital elements are marked on top of the plots. The panels are plotted for a binary consisting of equal masses, so that $\eta = 0.25$, and the orbital inclination angle is given by $i = \pi/3$. The orbital evolution is terminated when $j = \sqrt{48}$.

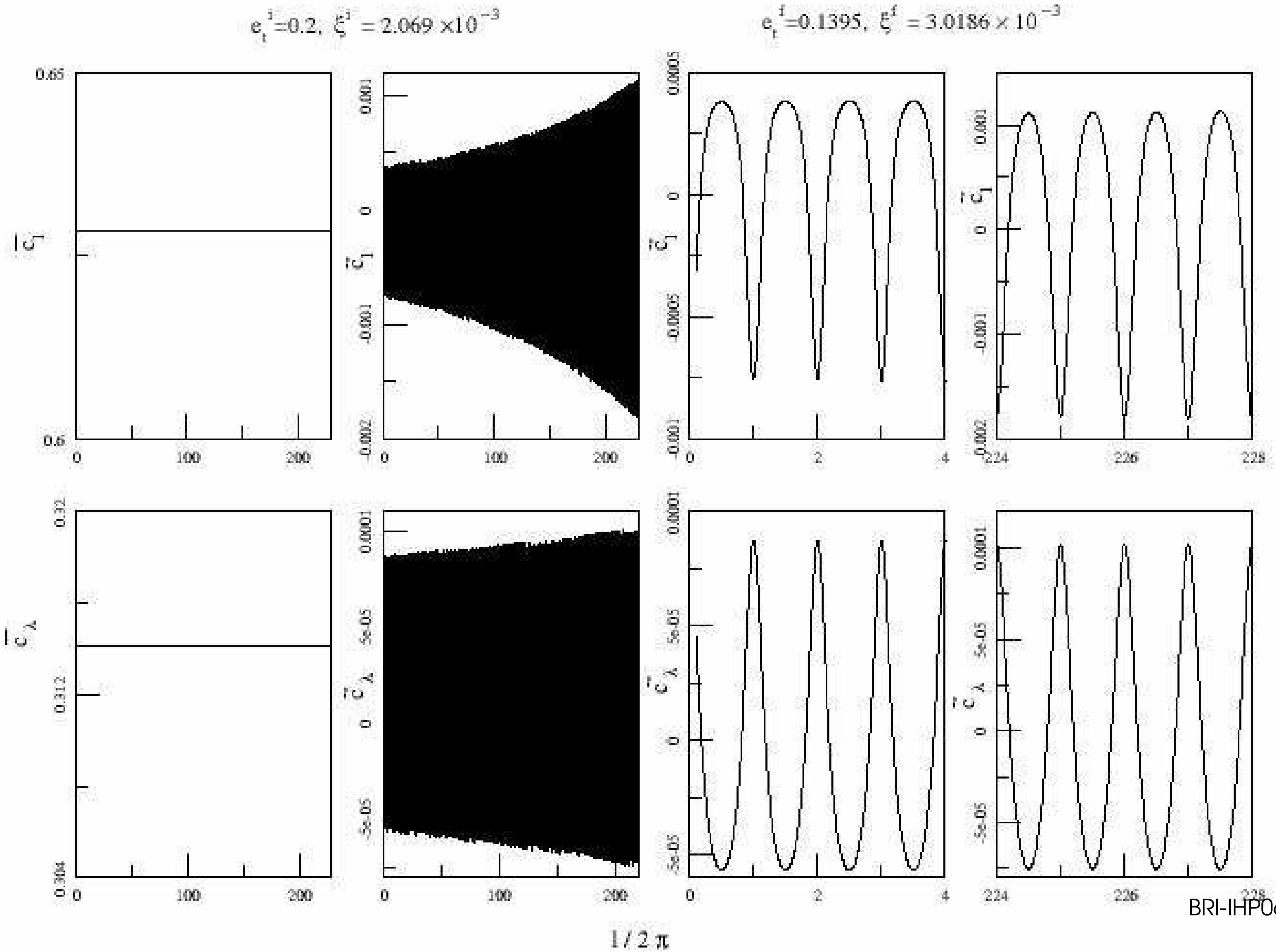
$\bar{n}/n_i$ and $\tilde{n}/n$ $\bar{e}_t$ and $\tilde{e}_t$ versus $l/(2\pi)$



$\bar{n}/n_i$ and $\tilde{n}/n$ $\bar{e}_t$ and $\tilde{e}_t$ versus $l/(2\pi)$

- The plots for $\bar{n}/n_i, \tilde{n}/n, \bar{e}_t$ and $\tilde{e}_t$ versus $\frac{l}{2\,\pi}$, which gives the number of orbital revolutions. These variations are governed by the reactive 2.5PN equations of motion. Periodic nature of the variations in $\tilde{n}$ and $\tilde{e}_t$ are clearly visible. e_t^i and e_t^f denote initial and final values for the time eccentricity e_t, while ξ^i and ξ^f stand for similar values of the adimensional mean motion $\frac{G\,m\,n}{c^3}$. The plots are for $\eta = 0.25$ and the evolution is terminated when $j^2 = 48$.

$\bar{c}_l$ and $\tilde{c}_l$ $\bar{c}_\lambda$ and $\tilde{c}_\lambda$ versus $l/(2\pi)$



$\bar{c}_l$ *and* $\tilde{c}_l$ $\bar{c}_\lambda$ *and* $\tilde{c}_\lambda$ *versus* $l/(2\pi)$

- The plots $\bar{c}_l, \tilde{c}_l, \bar{c}_\lambda$ and $\tilde{c}_\lambda$ against orbital cycles, given by $\frac{l}{2\,\pi}$. These variations are governed by the reactive 2.5PN equations of motion. Periodic nature of the variations in $\tilde{c}_l$ and $\tilde{c}_\lambda$ as well as the constancy of $\bar{c}_l$ and $\bar{c}_\lambda$ are clearly visible.

Validity of Results

- Circular orbits above the LSO, in the presence of GRR, can be described as an adiabatic sequence of circular orbits.
- Approximation breaks down when the binary reaches the LSO, at which point the orbital motion changes into a kind of plunge.
- For inspiralling *eccentric* orbits, evolving under GRR, our treatment can only be valid above an eccentric analog of the LSO, *i.e.* stable eccentric orbits (separated by a potential barrier from any plunge motion)
- To delineate them one can use the non-perturbative effective one body (EOB) formalism, because the transition between bound and plunge orbits has a non-perturbative, strong-field origin.

Validity of Results

- As one approaches the plunge boundary motion deviates from slow precessing QK to zoom-whirl. Hence we need to be be sufficiently away from the plunge boundary for validity of QKR.
- QKR assumes orbits are slowly precessing..Hence one needs to set upper limit on periastron precession
- If $\xi \equiv \frac{G\,m\,n}{c^3}$ $j \equiv c\mathcal{J}/(\mu G m)$, the constraint on parameters ξ and e_t becomes

$$\frac{\xi}{(1-e_t^2)^{3/2}} = \left(\frac{1}{j}\right)^3 < 3.0 \times 10^{-3}$$

- This restriction is due to our use of GQKR. One may be able to go beyond the limit by using instead the exact Schwarzschild-like motion in the EOB metric.

References

1. P. C. Peters, Phys. Rev. **136**, B1224 (1964).
2. P. C. Mathews and J. Mathews, Phys. Rev. B **136**, 435 (1963).
3. T. Damour, Phys. Rev. Lett. **51**, 1019, (1983).
4. T. Damour and G. Schäfer, Nuovo Cimento B **101**, 127 (1988).
5. L. Blanchet and G. Schäfer, Mon. Not. R. Astron. Soc. **239**, 845 (1989);
6. W. Junker and G. Schäfer, Mon. Not. R. Astron. Soc. **254**, 146 (1992).
7. L. Blanchet and G. Schäfer, Class. Quantum Grav. **10**, 2699 (1993).
8. G.. Schäfer and N. Wex, Phys. Lett. **174 A**, 196, (1993); erratum **177**, 461.
9. N. Wex, Class. Quantum Gr. **12**, 983, (1995).
10. A. Gopakumar and B. R. Iyer, Phys. Rev. D **56**, 7708 (1997).
11. K. Martel and E. Poisson, Phys.Rev. **D60**, 124008, (1999);
12. A. Gopakumar and B. R. Iyer, Phys. Rev. D **65**, 84011 (2002).
13. C. Koenigsdoerffer, A. Gopakumar, Phys.Rev. D73 (2006) 124012
14. R.M. Memmesheimer, A. Gopakumar, G. Schäfer Phys.Rev. D70 (2004) 104011
15. T. Damour, A. Gopakumar, B. R. Iyer Phys.Rev. D70 (2004) 064028
16. K G Arun, Luc Blanchet, Bala R Iyer, Moh'd S S Qusailah (2006) To be submitted