Parameter estimation of gravitational wave 'chirps' using the 3.5PN phasing formula

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PART I

Based on

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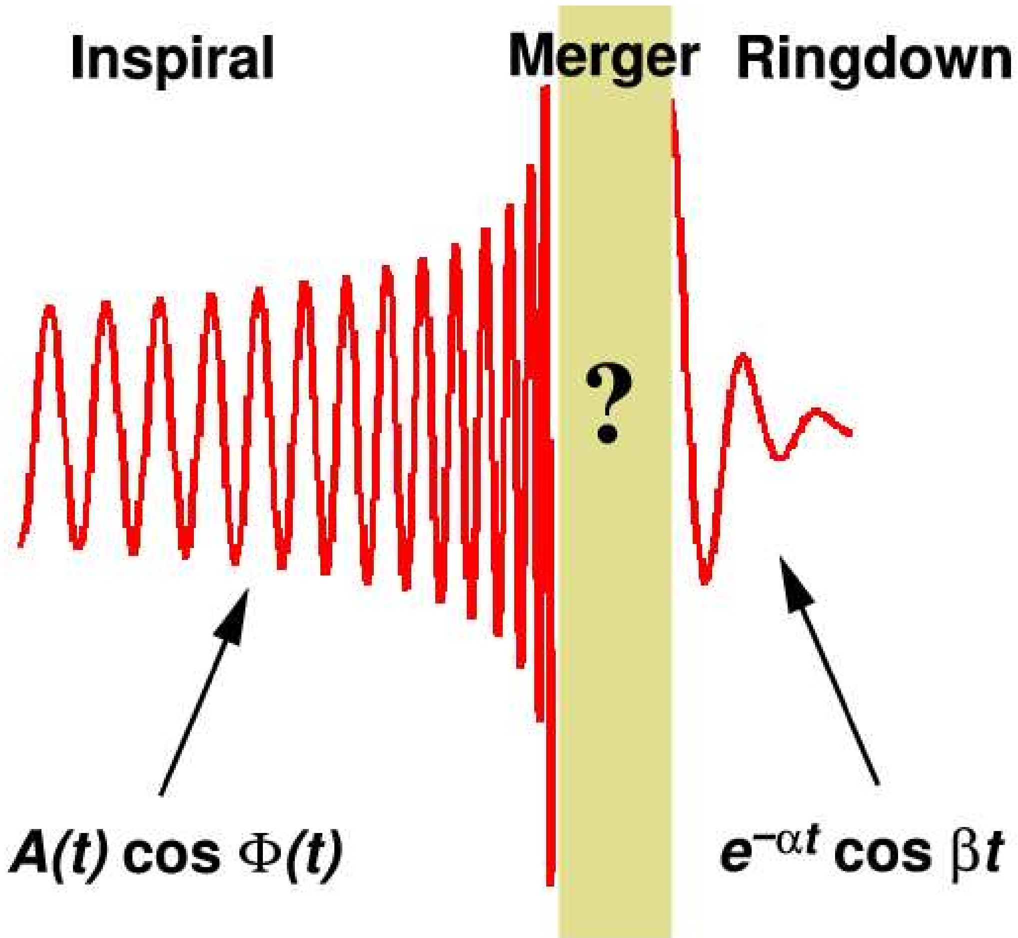
Introduction

- Late Inspiral and Merger Epochs of ICB (NS or BH) are potentially the most important sources for LI GW Detectors like LIGO and VIRGO.
- The waveform is a chirp

Chirp

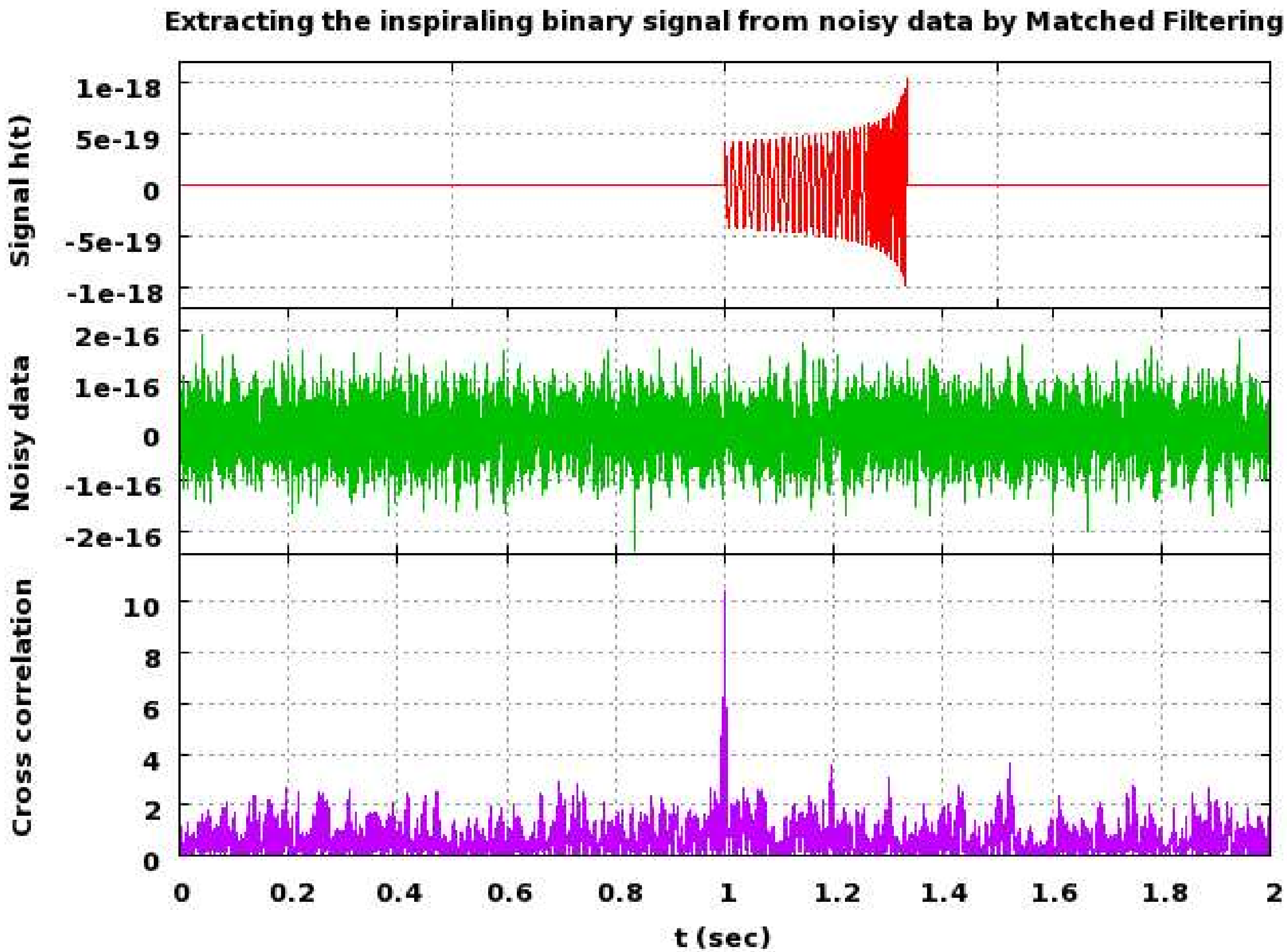


- GWs from ICB $\Rightarrow$ chirp: Amplitude and Freq. increasing with time

Introduction ... Contd

- GW are WEAK SIGNALS buried in NOISE of detector
- Require Matched Filtering (MF) Both for their Detection or Extraction and Parameter Estimation
- Success of MF requires Accurate model of signal using Gen Rel;
- Favours sources like ICB (NS-NS, BH-BH, NS-BH)
- Post detection, a very accurate estimation of parameters (masses, spins, distance,...) of the binary is essential for exploiting GW as a tool to fashion a New Astronomy and open a New Window to the Universe

Matched Filtering



From Anand Sengupta (IUCAA)

Present Work

- Theoretical study of the parameter estimation problem.
  - Implications of higher PN order modelling of the binary in the context of parameter estimation of the chirp signal.
  - Comparison (theoretical) of detector performances of initial LIGO, advanced LIGO and VIRGO interferometers.
  - Effect of Bandwidth and sensitivity on Parameter estimation.
- Present analysis is for the case of nonspinning binaries.

Parameter Estimation

- Matched filtering $\Rightarrow$ detector output is 'filtered' using pre-calculated waveforms with different signal parameters.
- The 'measured' values of the signal parameters correspond to that of the template which has maximum SNR.
- 'Measured' values need not be the 'actual' parameter values of the source. The *errors* could be one of the following:

Statistical Vs Systematic errors

There are two types of errors

1. Systematic errors due to approximate model we use
2. Statistical errors due to the noise present

- Parameter estimation theory aims at calculating the probability distribution for the measured values of a signal and to compute the interval in which the true parameters of the signal lie (at a specified confidence level).
- In this work we are addressing the statistical errors and variation of them with PN orders

The Error estimation

- The error estimates crucially depend on the noise model one employs. In the present work, we assume the noise to be random, stationary Gaussian.
- At high SNR, errors in estimation of parameters $\Delta\theta^a$ obey Gaussian probability distribution of the form

$$p(\Delta\theta^a) \quad = \quad p^{(0)} e^{-\frac{1}{2}\Gamma_{bc}\Delta\theta^b\Delta\theta^c}$$

 where θ^a represents the set of parameters describing the GW signal and Γ^{ab} is the Fisher information matrix.
- Fisher information matrix is given in terms of the Fourier domain signal $\tilde{h}(f)$ and noise Power spectral density $S_h(f)$ as...

Error estimation (contd..)

$$\Gamma_{ab} = 2\int_0^\infty \frac{\tilde{h}_a^*(f)\tilde{h}_b(f) + \tilde{h}_a(f)\tilde{h}_b^*(f)}{S_h(f)}\, df.$$

where $\tilde{h}_a = \frac{\partial \tilde{h}}{\partial \theta^a}$

- Covariance matrix is the inverse of Fisher matrix defined as

$$\Sigma^{ab} \equiv \langle \Delta\theta^a \Delta\theta^b \rangle = (\Gamma^{-1})^{ab}$$

- The root-mean-square error σ_a in the estimation of the parameters θ^a is

$$\sigma_a = \langle (\Delta\theta^a)^2 \rangle^{1/2} = \sqrt{\Sigma^{aa}}\,.$$

Correlation coefficients

- The off-diagonal elements of the covariance matrix are the correlation coefficients

$$c^{ab} = \frac{\langle \Delta\theta^a \Delta\theta^b \rangle}{\sigma_a \sigma_b} = \frac{\Sigma^{ab}}{\sqrt{\Sigma^{aa}\Sigma^{bb}}}.$$

- When the correlation coefficients between two parameters is close to 1(or -1), this indicates that two parameters are perfectly correlated (or anti-correlated) (and therefore redundant) and a value close to zero means the parameters are uncorrelated.
- Covariances close to 1 (or -1) cause large dispersion in their measurement.

Literature Update: Ground-based detectors

- Cutler and Flanagan (1994):
  Parameter estimation using 1.5PN phasing
  Effect of the new 1.5PN tail term and 1.5P spin-orbit term on parameter estimation
- Blanchet and Sathyaprakash (1995):
  Detectability of GW tails at 1.5PN
- Królak, Kokkotas and Schäfer (1995), Poisson and Will (1995):
  Implications of 2PN phasing
  Effect of the new spin-spin coupling term (KKS95, PW95)
  Alternate theories of gravity, effect of eccentricity (KKS95)
- Balasubramanian, Sathyaprakash and Dhurandhar (1995, 1996, 1998)
  Monte-Carlo simulations using 2PN phasing
  Comparison of covariance matrix errors with the simulated ones.
- Sintes and Vecchio (2000), Hellings and Moore (2002) and Van den Broeck and Sengupta (2006):
  Parameter estimation using nonrestricted-waveform.

The Fourier Domain waveform

- The two inputs needed are the Fourier domain gravitational waveform and the noise PSD of the detector.
- The gravitational waveform in the Fourier Domain reads as

$$\tilde{h}(f) = \mathcal{A} f^{-7/6} e^{i\psi(f)},$$

where $\mathcal{A} \propto \mathcal{M}^{5/6} Q(\text{angles})/D$, and to 3.5PN order the phase of the Fourier domain waveform is given by

$$\psi(f) \quad = \quad 2\pi f t_c - \phi_c - \frac{\pi}{4} + \frac{3}{128\,\eta\,v^5} \sum_{k=0}^{N} \alpha_k\, v^k,$$

- $v = (\pi m f)^{1/3}$
- The PN phasing coefficients alphas are given by

The phasing formula: Coefficients

Blanchet, Faye, Iyer and Joguet - 2002
Damour, Iyer and Sathyaprakash - 2002

$$
\begin{aligned}
\alpha_0 &= 1, \\
\alpha_1 &= 0, \\
\alpha_2 &= \frac{20}{9}\left(\frac{743}{336} + \frac{11}{4}\eta\right), \\
\alpha_3 &= -4(4-\beta)\pi, \\
\alpha_4 &= 10\left(\frac{3058673}{1016064} + \frac{5429}{1008}\,\eta + \frac{617}{144}\,\eta^2 - \sigma\right), \\
\alpha_5 &= \pi\left(\frac{38645}{756} + \frac{38645}{252}\,\log\left(\frac{v}{v_{\rm lso}}\right) - \frac{65}{9}\eta\left[1 + 3\log\left(\frac{v}{v_{\rm lso}}\right)\right]\right),
\end{aligned}
$$

$\beta \longrightarrow$ spin-orbit coupling

$\sigma \longrightarrow$ spin-spin coupling

Phasing formula (contd..)

$$\alpha_6 = \left(\frac{11583231236531}{4694215680} - \frac{640\,\pi^2}{3} - \frac{6848\,\gamma}{21}\right)$$
$$+ \quad \eta\left(-\frac{15335597827}{3048192} + \frac{2255\,\pi^2}{12} - \frac{1760\,\theta}{3} + \frac{12320\,\lambda}{9}\right)$$
$$+ \quad \frac{76055}{1728}\eta^2 - \frac{127825}{1296}\eta^3 - \frac{6848}{21}\log(4\,v)\,,$$
$$\alpha_7 = \pi\left(\frac{77096675}{254016} + \frac{1014115}{3024}\,\eta - \frac{36865}{378}\,\eta^2\right).$$
$$\alpha_7 = \pi\left(\frac{77096675}{254016} + \frac{378515}{1512}\,\eta - \frac{74045}{756}\,\eta^2\right).$$

$$\lambda = -\frac{1987}{3080} \simeq -.6451;$$

Damour, Jaranowski, Schäfer - 2001;
Blanchet, Damour, Esposito-Farése - 2004

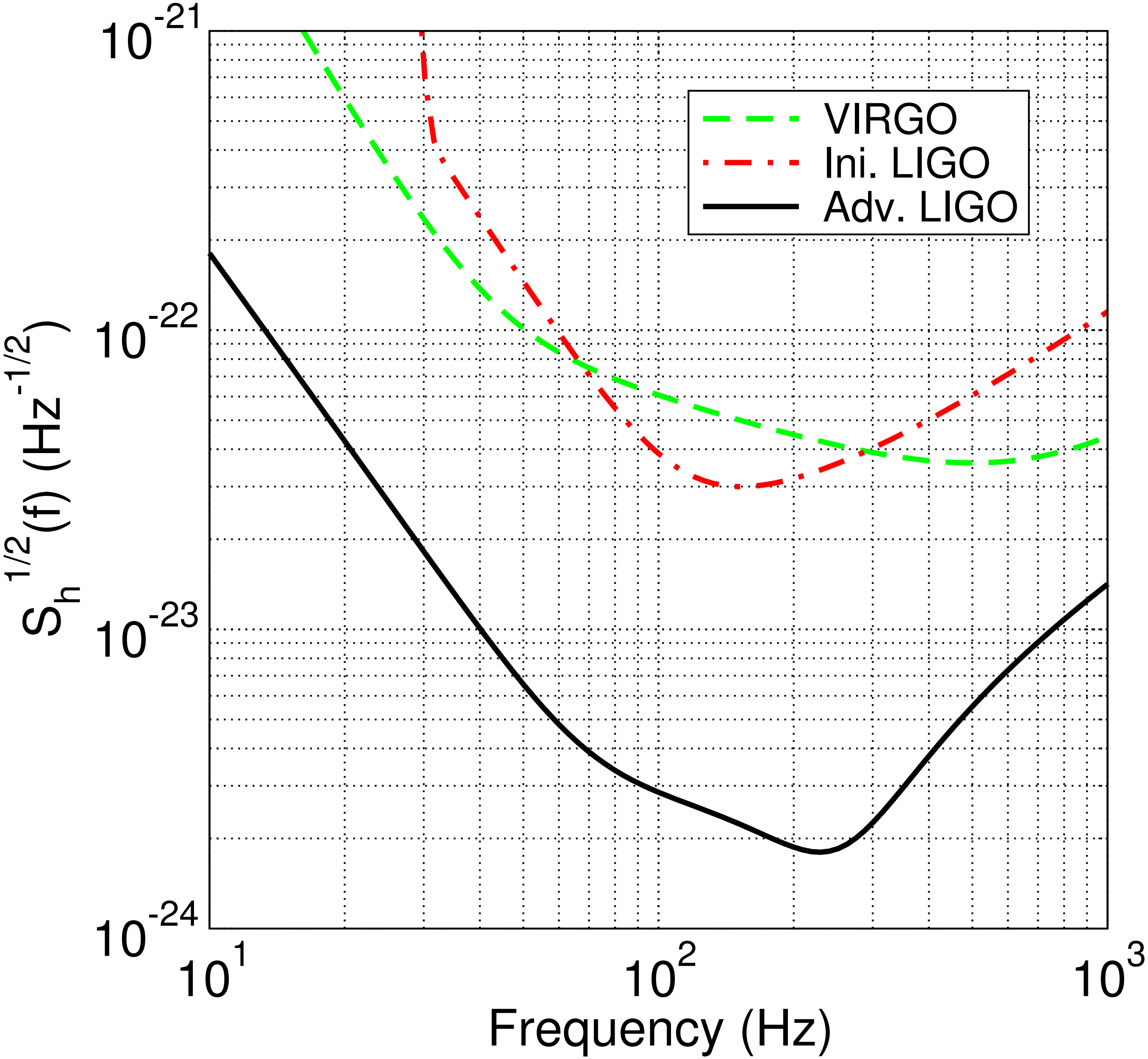
$$\theta = -\frac{11831}{9240} \simeq -1.28$$

Blanchet, Damour, Iyer and Esposito-Farése - 2005

Features of the phasing formula

- Newtonian term depends only on the chirp mass $\mathcal{M} = \eta^{3/5} m$. Not sufficient to estimate the individual masses
- 1PN term breaks the degeneracy and enables estimation of total mass m and mass ratio η or $\sim m_1$ and m_2
- 1.5PN term enables a test of GR since independent estimation of masses is possible for non-spinning binaries (Blanchet and Sathyaprakash)
- Spin-orbit coupling is a 1.5PN effect and spin-spin coupling appears at 2PN
- θ and λ appearing at 3PN are now determined: one of the motivations for this study. Recommend their use to get the best templates
- Extending these results to higher orders in SO is possible due to recent availability of the Energy and Flux functions including spin effects - Blanchet, Buonanno, Faye 2006

The noise curves: Plots



The noise curve: features

- Advanced LIGO has the largest sensitivity (smallest noise amplitude) followed by initial LIGO and VIRGO (which are comparable).
- VIRGO has the best bandwidth amongst the three followed by Advanced LIGO and initial LIGO (in the descending order).
- Lower cut-off frequencies of Adv. LIGO and VIRGO are chosen to be 20 Hz and that of initial LIGO to be 40Hz.
- Upper cut-off for different sources are of the order of their F_{lso}. For 3 prototypical binary systems made of of NS $(1.4 M_\odot)$ and/or BH $(10 M_\odot)$ we have NS-NS: 1570 Hz, NS-BH: 386 Hz, BH-BH: 220 Hz

The Scheme of Analysis

- The set of parameters describing the chirp is

$$\theta^a = \{t_c, \phi_c, \mathcal{M}, \eta\}$$

- We construct the corresponding 4×4 Fisher matrix and invert it to get the associated errors.
- Do it for all PN orders and for different detectors and systems.
- The analysis is done for fixed SNR as well as for sources at fixed distance in order to study the effect of BW and Sensitivity (respectively).
- We consider 3 prototypical systems of NS ($1.4M_\odot$) and BH ($10M_\odot$), viz, NS-NS, NS-BH and BH-BH.

Errors at diff. PN orders: fixed SNR

1

TABLE I: Convergence of measurement errors from 2PN to 3.5PN at a SNR of 10 for the three prototypical binary systems: NS-NS, NS-BH and BH-BH using the phasing formula, in steps of 0.5PN. For each of the three detector noise curves the table presents Δt_c (in msec), $\Delta\phi_c$ (in radians), $\Delta\mathcal{M}/\mathcal{M}$ and $\Delta\eta/\eta$.

	NS-NS				NS-BH				BH-BH			
PN Order	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$
Advanced LIGO												
2PN	0.4623	1.392	0.0143%	1.764%	0.7208	1.848	0.0773%	2.669%	1.404	2.850	0.6240%	10.79%
2.5PN	0.5090	1.354	0.0134%	1.334%	0.9000	1.213	0.0686%	1.515%	1.819	1.555	0.5300%	5.934%
3PN	0.4938	1.326	0.0135%	1.348%	0.8087	1.126	0.0698%	1.571%	1.544	1.559	0.5466%	6.347%
3.5PN	0.5198	1.273	0.0133%	1.319%	0.9980	0.9203	0.0679%	1.456%	2.086	1.137	0.5237%	5.730%
Initial LIGO												
2PN	0.4109	1.816	0.0423%	3.007%	1.148	3.597	0.2903%	6.316%	2.900	7.179	2.851%	32.82%
2.5	0.4605	1.642	0.0384%	2.129%	1.467	1.964	0.2491%	3.305%	3.836	3.070	2.351%	16.48%
3PN	0.4402	1.610	0.0389%	2.170%	1.286	1.787	0.2554%	3.474%	3.159	3.069	2.446%	17.94%
3.5PN	0.4760	1.507	0.0383%	2.098%	1.668	1.311	0.2455%	3.148%	4.531	1.851	2.313%	15.75%
VIRGO												
2PN	0.1562	0.7515	0.0098%	1.085%	0.5918	1.561	0.0611%	2.215%	1.395	2.667	0.5199%	9.625%
2.5PN	0.1743	0.7015	0.0091%	0.7957%	0.7384	1.035	0.0541%	1.263%	1.787	1.527	0.4417%	5.370%
3PN	0.1671	0.6890	0.0092%	0.8083%	0.6632	0.9625	0.0551%	1.309%	1.532	1.528	0.4552%	5.724%
3.5PN	0.1799	0.6527	0.0091%	0.7854%	0.8195	0.7914	0.0536%	1.214%	2.031	1.150	0.4366%	5.193%

Advanced LIGO

PN Order	Δt_c	$\Delta \phi_c$	$\Delta \mathcal{M}/\mathcal{M}$	$\Delta \eta/\eta$
NS-NS				
2PN	0.4623	1.392	0.0143%	1.764%
2.5PN	0.5090	1.354	0.0134%	1.334%
3PN	0.4938	1.326	0.0135%	1.348%
3.5PN	0.5198	1.273	0.0133%	1.319%
NS-BH				
2PN	0.7208	1.848	0.0773%	2.669%
2.5PN	0.9000	1.213	0.0686%	1.515%
3PN	0.8087	1.126	0.0698%	1.571%
3.5PN	0.9980	0.9203	0.0679%	1.456%
BH-BH				
2PN	1.404	2.850	0.6240%	10.79%
2.5PN	1.819	1.555	0.5300%	5.934%
3PN	1.544	1.559	0.5466%	6.347%
3.5PN	2.086	1.137	0.5237%	5.730%

LIGO, VIRGO: NS-NS

PN Order	Δt_c	$\Delta \phi_c$	$\Delta \mathcal{M}/\mathcal{M}$	$\Delta \eta/\eta$
Initial LIGO				
1PN	0.3598	1.238	0.0771%	9.792%
1.5PN	0.4154	1.942	0.0419%	2.768%
2PN	0.4109	1.816	0.0423%	3.007%
2.5	0.4605	1.642	0.0384%	2.129%
3PN	0.4402	1.610	0.0389%	2.170%
3.5PN	0.4760	1.507	0.0383%	2.098%
VIRGO				
1PN	0.1363	0.5134	0.0183%	3.044%
1.5PN	0.1578	0.7981	0.0098%	1.004%
2PN	0.1562	0.7515	0.0098%	1.085%
2.5PN	0.1743	0.7015	0.0091%	0.7957%
3PN	0.1671	0.6890	0.0092%	0.8083%
3.5PN	0.1799	0.6527	0.0091%	0.7854%

LIGO, VIRGO: NS-BH

PN Order	Δt_c	$\Delta \phi_c$	$\Delta \mathcal{M}/\mathcal{M}$	$\Delta \eta/\eta$
Initial LIGO				
1PN	0.9550	2.510	0.5217%	20.06%
1.5PN	1.182	4.135	0.2850%	5.410%
2PN	1.148	3.597	0.2903%	6.316%
2.5	1.467	1.964	0.2491%	3.305%
3PN	1.286	1.787	0.2554%	3.474%
3.5PN	1.668	1.311	0.2455%	3.148%
VIRGO				
1PN	0.4906	1.069	0.1134%	5.782%
1.5PN	0.6069	1.763	0.0603%	1.923%
2PN	0.5918	1.561	0.0611%	2.215%
2.5PN	0.7384	1.035	0.0541%	1.263%
3PN	0.6632	0.9625	0.0551%	1.309%
3.5PN	0.8195	0.7914	0.0536%	1.214%

LIGO, VIRGO: BH-BH

PN Order	Δt_c	$\Delta \phi_c$	$\Delta \mathcal{M}/\mathcal{M}$	$\Delta \eta/\eta$
Initial LIGO				
1PN	2.406	5.038	4.750%	216.2%
1.5PN	2.986	8.143	2.781%	28.81%
2PN	2.900	7.179	2.851%	32.82%
2.5	3.836	3.070	2.351%	16.48%
3PN	3.159	3.069	2.446%	17.94%
3.5PN	4.531	1.851	2.313%	15.75%
VIRGO				
1PN	1.621	1.854	0.8745%	52.12%
1.5PN	1.430	2.972	0.5095%	8.586%
2PN	1.395	2.667	0.5199%	9.625%
2.5PN	1.787	1.527	0.4417%	5.370%
3PN	1.532	1.528	0.4552%	5.724%
3.5PN	2.031	1.150	0.4366%	5.193%

Inferences from the table

- The values listed are for a fixed SNR of 10
- Compared to 2PN phasing, the 3.5PN phasing provides an improved parameter estimation of the mass parameters $\mathcal{M}$ and η.
- Improvement is as high as 19% and 52% for $\mathcal{M}$ and η for a BH-BH binary in the initial LIGO sensitivity band.
- Errors oscillate at every half-a-PN order
- More massive systems have larger errors associated with their parameters.
- At fixed SNR, VIRGO leads to the least errors followed by Adv. LIGO and Initial LIGO configurations
- This is because VIRGO observes the signal over a larger BW compared to the other two.

Percentage improvements: NS-NS

Interferometer	NS-NS			
	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
Adv. LIGO	-10.27	2.083	6.294	23.36
Ini. LIGO	-13.00	8.260	8.274	27.80
VIRGO	-12.42	5.602	6.122	25.43

Improvement: NS-BH

Interferometer	NS-BH			
	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
Adv. LIGO	−32.09	34.85	10.61	41.89
Ini. LIGO	−38.24	45.93	13.43	46.17
VIRGO	−32.07	34.21	10.80	41.63

Improvement:BH-BH

Interferometer	BH-BH			
	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
Adv. LIGO	-40.81	42.04	13.99	42.85
Ini. LIGO	-48.00	54.06	16.49	47.65
VIRGO	-38.21	39.41	13.95	42.07

Variation of correlation coefficients:

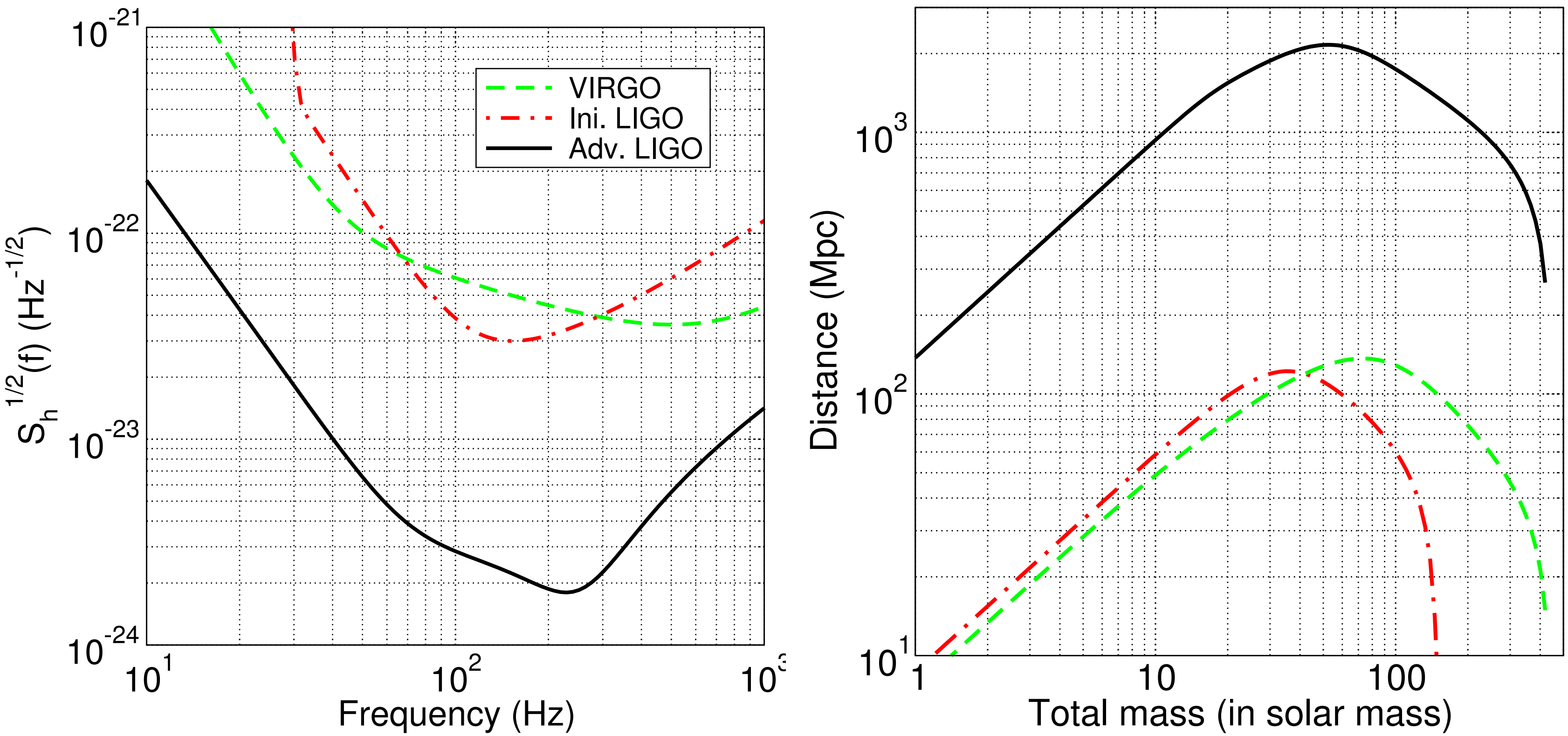
NS-NS, Initial LIGO

Order	$c_{t_c\mathcal{M}}$	$c_{t_c\eta}$	$c_{\mathcal{M}\eta}$	Δt_c (ms)	$\Delta\mathcal{M}/\mathcal{M}$ (%)	$\Delta\eta$
2PN	0.7290	0.8579	0.9571	0.4109	0.0423	3
2.5PN	0.7517	0.8887	0.9477	0.4605	0.0384	2
3PN	0.7393	0.8777	0.9489	0.4402	0.0389	2
3.5PN	0.7610	0.8963	0.9472	0.4760	0.0383	2

Effect of Bandwidth

For the fixed SNR case, where the Sensitivity aspect of the detector is fully suppressed, the detector with the largest BW provides the best estimate.

Sensitivity and Span of detectors



Luminosity distance at which different detectors would produce an SNR of 5.

Errors for fixed distance

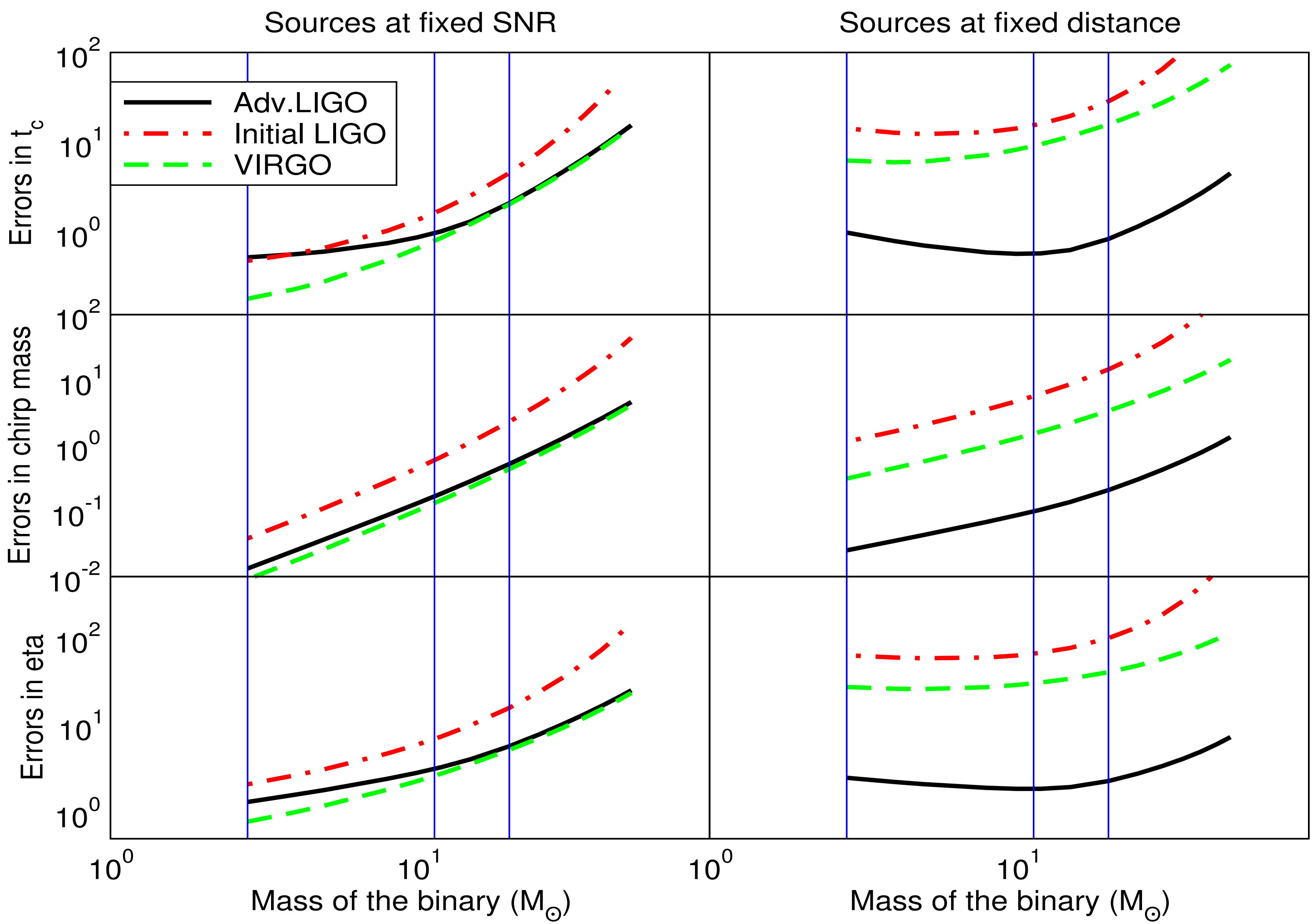
- Errors at fixed SNR cannot be used to gauge detector performance as by keeping SNR constant one ie effectively suppressing the sensitivity of a better detector. A more sensitive detector has larger SNR for a given source and hence lesser errors.
- Errors $\sigma \propto 1/\rho$ (ρ is SNR).
- SNR corresponding to optimal filter is

$$\rho^2 = 4 \int_0^\infty df \, \frac{|\tilde{h}(f)|^2}{S_h(f)}.$$

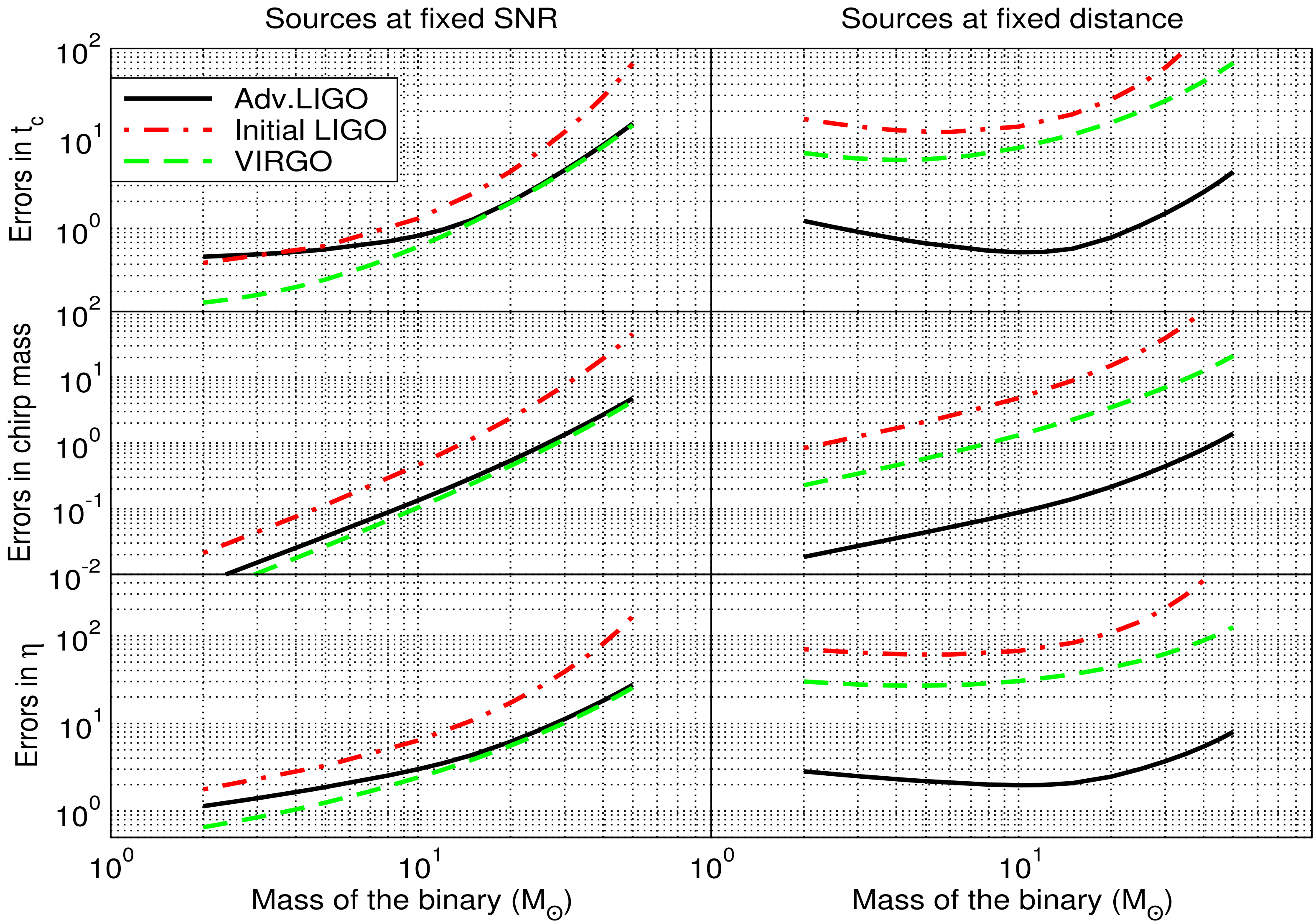
Errors for fixed distance

- We can re-tabulate the errors for sources at fixed distance by rescaling the fixed SNR results by $10/\rho_a$, where ρ_a is the SNR at the fixed distance of 300 Mpc.

Fixed-SNR vs Fixed-Distance



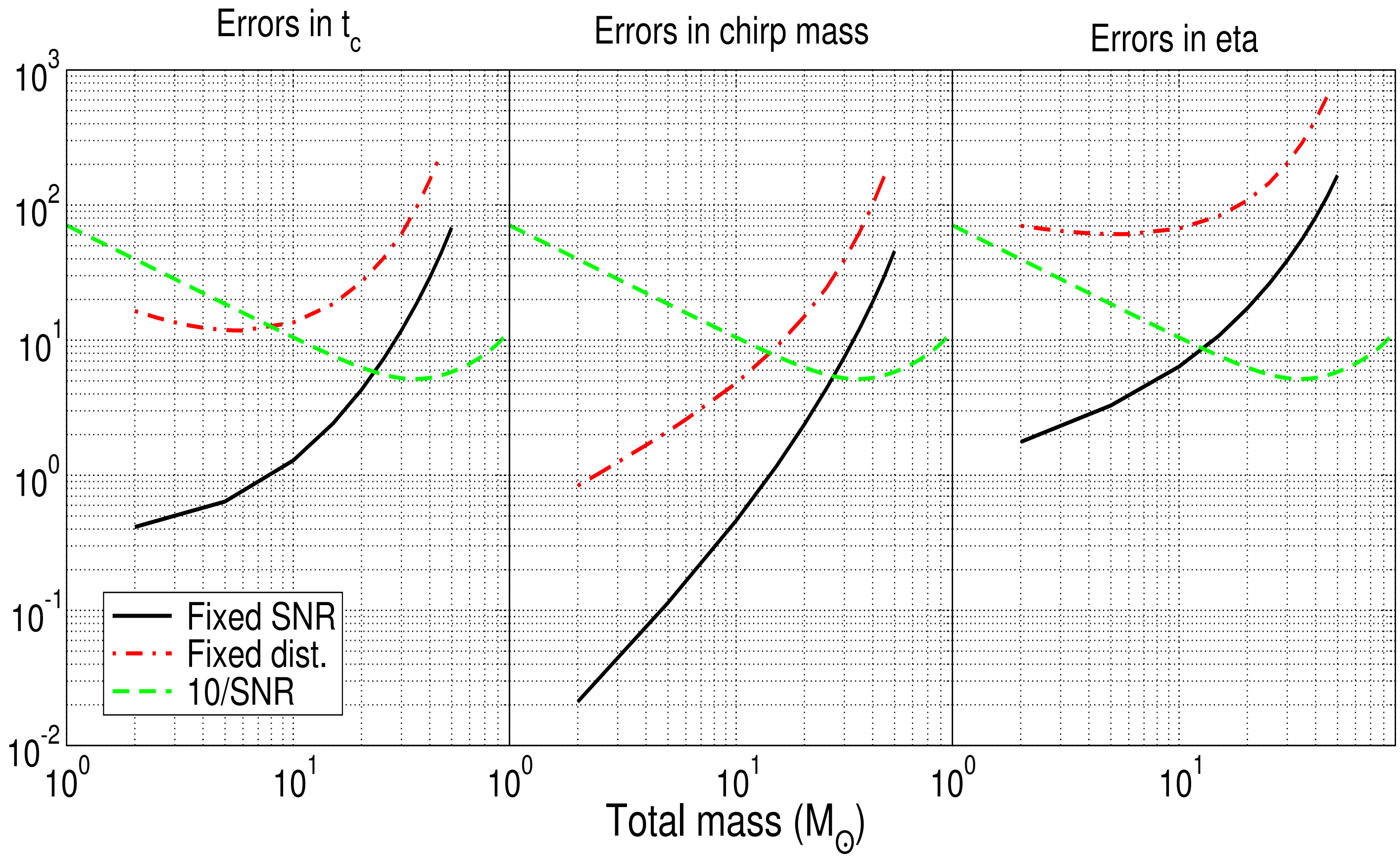
Fixed-SNR vs Fixed-Distance



Effect of sensitivity

For souces at a fixed distance, both BW and sensitivity plays a role in the parameter estimation. Advanced LIGO gains an improvement of 30-60 times compared to initial LIGO. Of this 3-4 times is from BW and 10-15 times is due to its sensitivity over the initial LIGO config.

Errors and SNR



Number of GW cycles

Additional GW cycles at each order

	NS-NS	NS-BH	BH-BH
Newtonian	16031	3576	602
1PN	441	213	59
1.5PN	−211	−181	−51
2PN	9.9	9.8	4.1
2.5PN	−12.2	−20.4	−7.5
3PN	2.6	2.3	2.2
3.5PN	−1.0	−1.9	−0.9

Total and Useful GW cycles

- Total number GW cycles accumulated in the detectors sensitivity band is

$$N_{\rm total} = \int_{F_{\rm begin}}^{F_{\rm end}} dF \left(\frac{1}{2\pi} \frac{d\phi}{dF} \right)$$

- This has no information about the detector sensitivity and depends only on upper and lower cut-offs
- In the context of detection problem, Damour, Iyer and Sathyaprakash, 2000 proposed to use a different quantifier called "Number of useful GW cycles", which is weighted by the detector noise

N_{useful}

$$N_{\rm useful} = \left[\int_{F_{\rm min}}^{F_{\rm max}} \frac{df}{f}\, w(f)\, N(f)\right] \left[\int_{F_{\rm min}}^{F_{\rm max}} \frac{df}{f}\, w(f)\right]^{-1}$$

$$N(F) = \frac{F^2}{dF/dt}\,, \qquad w(f) = \frac{a^2(f)}{h_n^2(f)},$$

with $a(f)$ being the ‘bare amplitude’ appearing in the Fourier domain waveform within the SPA, $|\tilde{h}(f)| \simeq a(f)/\sqrt{\dot{F}}$ and $h_n^2 \equiv f\, S_h(f)$.

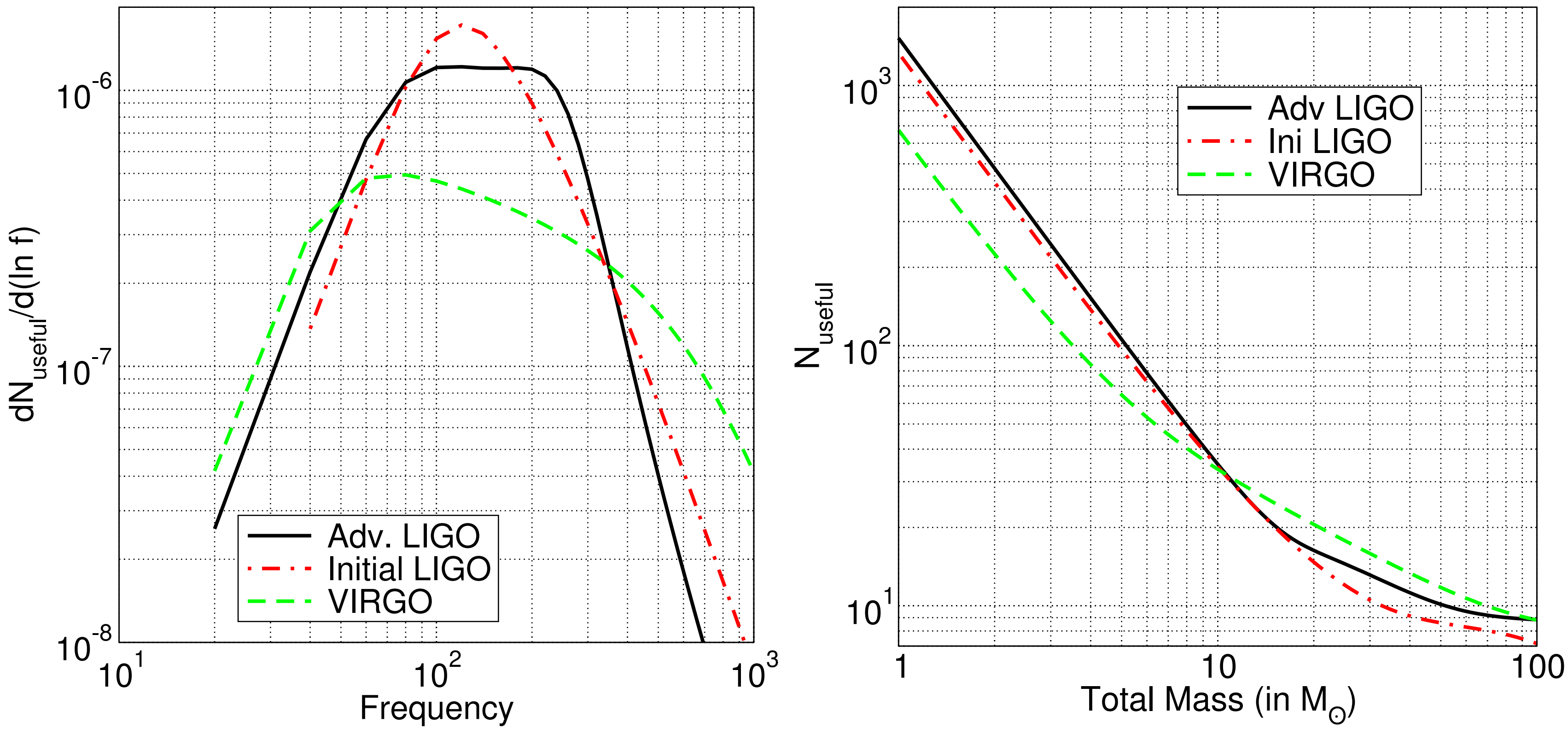
- Unlike the total number of cycles, the number of *useful cycles* contains information about both the detector and the source

Useful cycles for diff detectors

Detector	NS-NS	NS-BH	BH-BH
VIRGO	140 (5137)	64 (1112)	19 (185)
Adv. LIGO	284 (5137)	61 (1112)	14 (185)
Ini. LIGO	251 (1616)	59 (331)	13 (53)

- The greater errors observed for massive systems can be attributed to lesser number of useful cycles the detector accumulates
- Except for the NS-NS case, the errors for fixed SNR and useful cycles shows correlation. Inspite of the lesser useful cycles, VIRGO shows better PE compared to other detectors for the NS-NS case

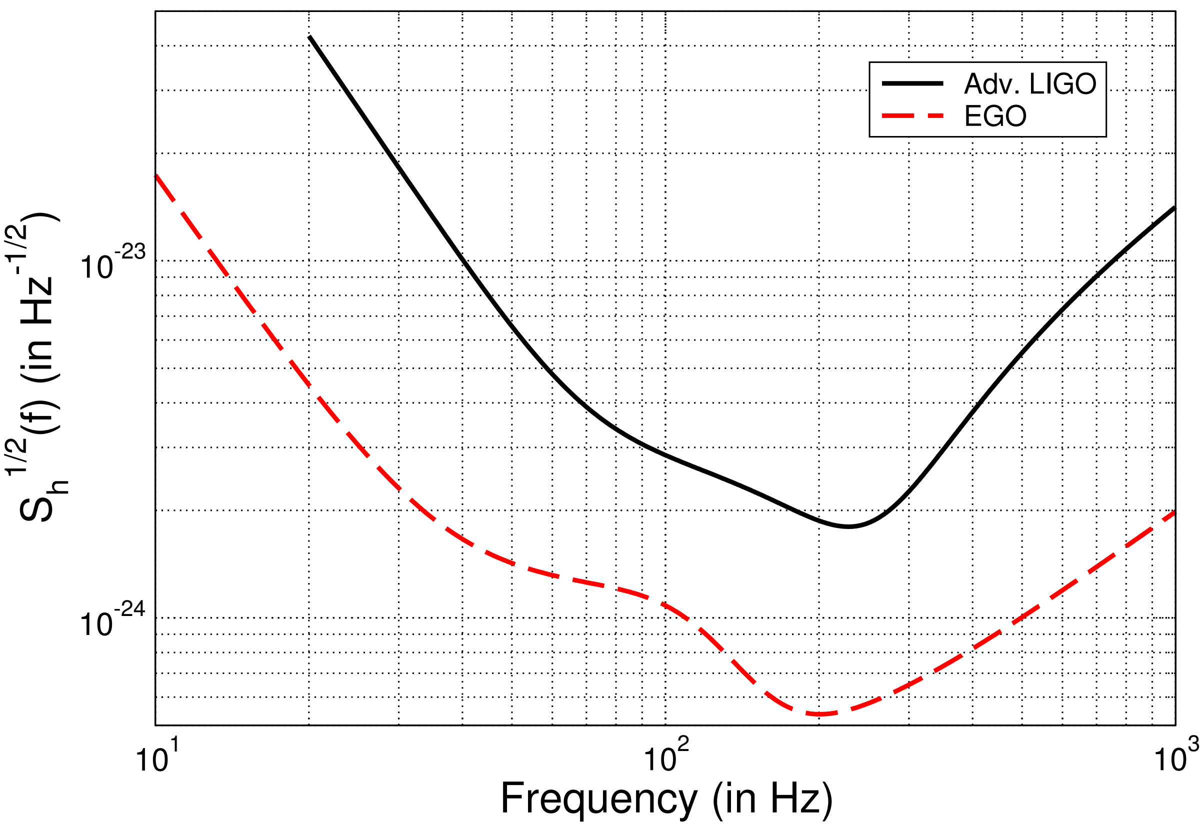
Useful cycles and detector performance



- The plot shows that for binaries with total mass $\leq 10 M_\odot$ due to its broader BW, VIRGO has lesser number of useful cycles!

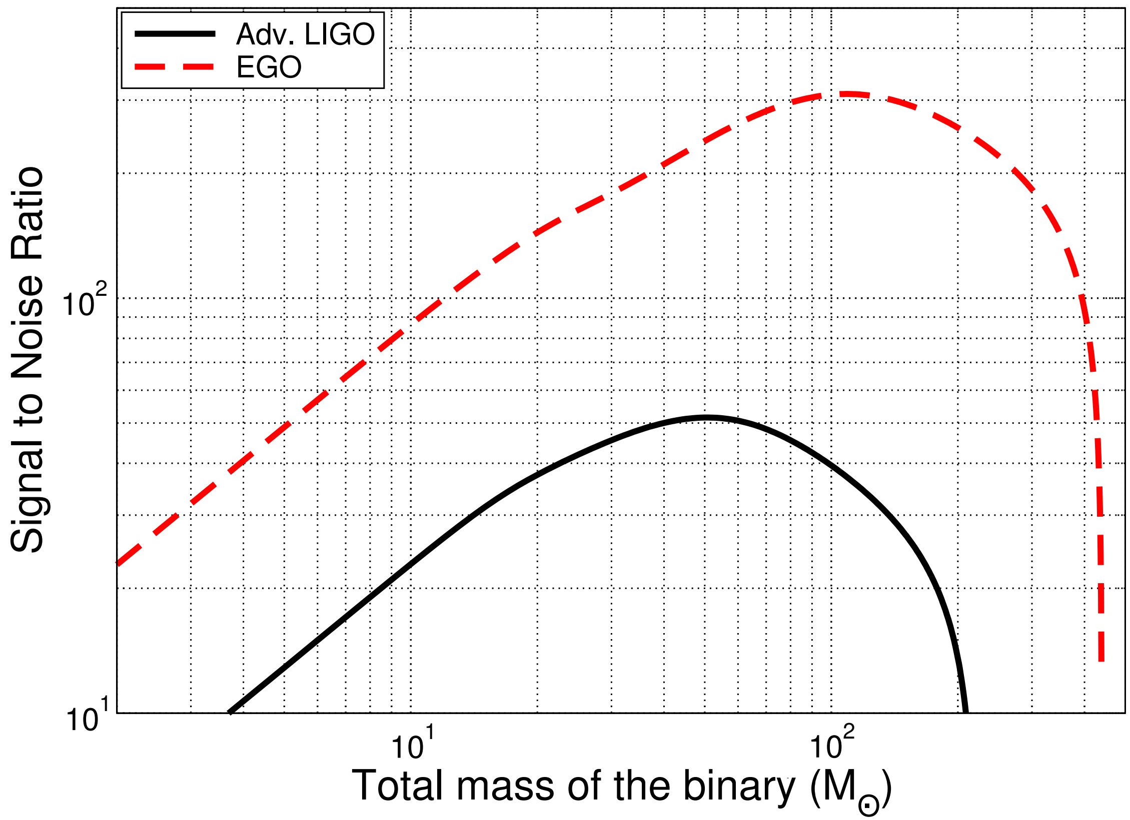
Parameter estimation using EGO

European Gravitational Observatory: A third generation GW interferometer envisaged by the European GW community dedicated to Khz frequencies..



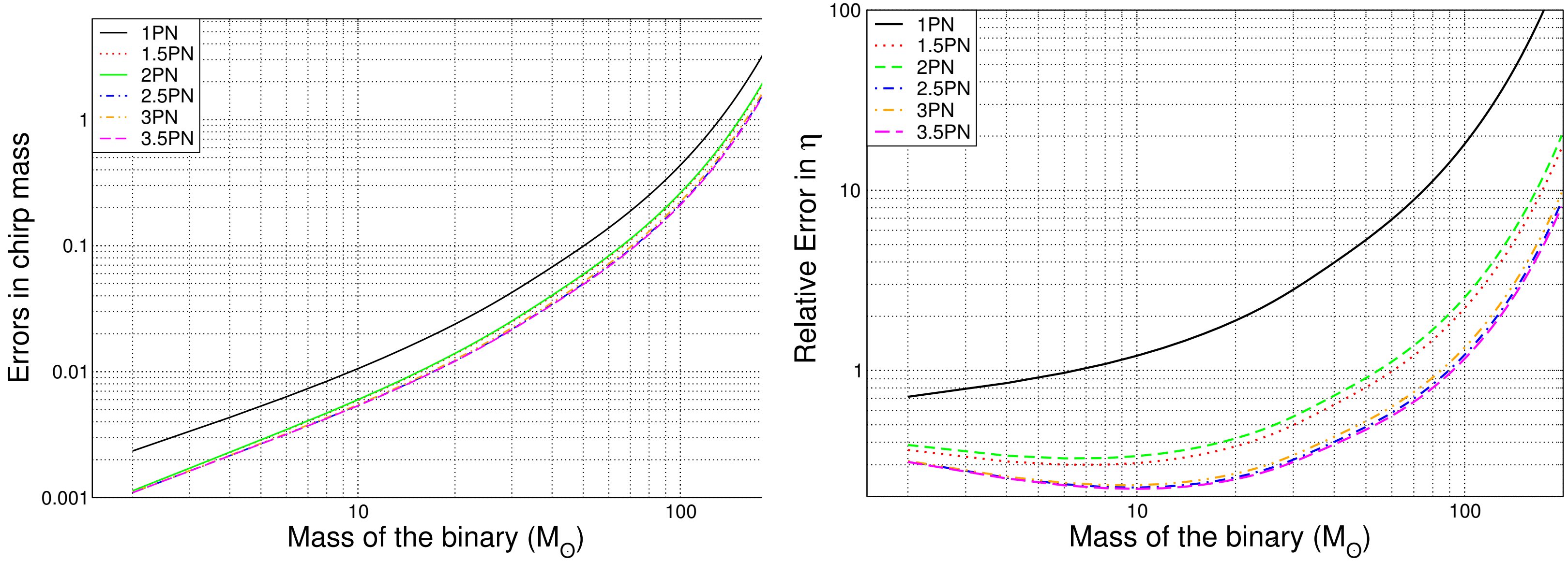
Noise PSD of Adv LIGO and EGO

Parameter estimation using EGO



SNR of Adv LIGO and EGO for equal mass binaries

PN convergence in the EGO context



K G Arun, Ph D Thesis, 2006 (Unpublished)

- Very accurate parameter estimation possible with EGO: Order of magnitude better than Adv LIGO
- Strong field tests possible with EGO:
  Arun, Iyer, Qusailah and Sathyaprakash(2006)
  Qusailah (PhD Thesis 2006)
  Van den Broeck and Sengupta (2006)

Part II

Based on

Parameter estimation of coalescing supermassive black hole binaries with *LISA*

K G Arun

Phys. Rev. **D 74**, 024025 (2006), gr-qc/0605021.

Laser Interferometer Space Antenna

- NASA + ESA = LISA
- Space based detector in the freq range $10^{-5} - 1$Hz.
- Three Space craft constellation, distance b/w any two detectors is 5 million kilometers.
- Proposed launch by 201?+
- Complements the high freq observations made by the Ground based detectors.
- Science goals include observation of supermassive BHs, Strong field tests of Gravity
  Berti,Buonanno & Will (2005);
  KGA, Iyer, Qusailah & Sathyaprakash (2006).
- Future upgrades of LISA: BBO, DECIGO

LISA features

- Three arms at 60 degrees
  $\Rightarrow$ one can construct two detector outputs by combining the data of two
  $\Rightarrow$ one detector and two detector configuration.
- Orbital motion
  $\Rightarrow$ encodes information about angular coordinates of the source
  $\Rightarrow$ even with single detector configuration location and orientation of the source can be measured.
- Location and orientation measurement by orbital modulation

LISA features

If one does not average the pattern functions, the waveform can be written

$$\tilde{h}_\alpha(f) = \frac{\sqrt{3}}{2}\mathcal{A}\, f^{-7/6}\, e^{i\Psi(f)} \left\{\frac{5}{4}\tilde{A}_\alpha(t(f))\right\} e^{-i\left(\varphi_{p,\alpha}(t(f))+\varphi_D(t(f))\right)} ,$$

where $\varphi_{p,\alpha}(t(f))$ and $\varphi_D(t(f)$ are the polarization phase and Doppler phase respectively. $\tilde{A}_\alpha(t(f))$ correspond to the amplitude modulations induced by the LISA's orbital motion. which depends on the pattern functions $F_+^\alpha(t)$ and $F_\times^\alpha(t)$ and hence vary with time.

- Calculation now is more involved than for the ground based detectors.
- Three cases:
  using a pattern averaged waveform,
  without pattern averaging for one detector and
  without pattern averaging for 2 detector network.

Parameter estimation with LISA

- The essential equations related to the response of LISA are given below. 'Barred' quantities are in the fixed-solar system based coordinate systems and those 'unbarred' are in the rotating LISA frame.
- Assume that noise is symmetric in each pair of the LISA arms and hence treat LISA to be consisting of two independent Michelson interferometers in the shape of an equilateral triangle. Compared to the ground based detector case, the resultant waveform will have an overall $\frac{\sqrt{3}}{2}$ factor to account for the equilateral geometry. The pattern functions are given by

$$
\begin{aligned}
F_{\rm I}^{+}(\theta_S,\phi_S,\psi_S) &= \frac{1}{2}(1+\cos^2\theta_S)\cos 2\phi_S\cos 2\psi_S \\
&\quad -\cos\theta_S\sin 2\phi_S\sin 2\psi_S\ , \\
F_{\rm I}^{\times}(\theta_S,\phi_S,\psi_S) &= \frac{1}{2}(1+\cos^2\theta_S)\cos 2\phi_S\sin 2\psi_S \\
&\quad +\cos\theta_S\sin 2\phi_S\cos 2\psi_S\ ,
\end{aligned}
$$

Parameter estimation with LISA

$$F_{\mathrm{II}}^{+}(\theta_S,\phi_S,\psi_S) = F_{\mathrm{I}}^{+}(\theta_S,\phi_S-\frac{\pi}{4},\psi_S)\,,$$
$$F_{\mathrm{II}}^{\times}(\theta_S,\phi_S,\psi_S) = F_{\mathrm{I}}^{+}(\theta_S,\phi_S-\frac{\pi}{4},\psi_S)\,.$$

- In the above (θ_S,ϕ_S) denotes the source location and ψ_S the polarization angle defined as

$$\tan\psi_S(t) = \frac{\hat{\mathbf{L}}\cdot\mathbf{z}-(\hat{\mathbf{L}}\cdot\mathbf{n})(\mathbf{z}\cdot\mathbf{n})}{\mathbf{n}\cdot(\hat{\mathbf{L}}\times\mathbf{z})}\,,$$

  $\hat{\mathbf{L}}$, $\mathbf{z}$ and $-\mathbf{n}$ being the unit vectors along the orbital angular momentum, the unit normal to *LISA*'s plane and the GW direction of propagation, respectively.

Parameter estimation with LISA

- The waveform polarization and Doppler phases in the above equations are given by :

$$\varphi_{p,\alpha}(t) = \tan^{-1}\left[\frac{2(\hat{\mathbf{L}}\cdot\mathbf{n})F^{\times}_{\alpha}(t)}{(1+(\hat{\mathbf{L}}\cdot\mathbf{n})^2)F^{+}_{\alpha}(t)}\right] ,$$

$$\varphi_D(t) = \frac{2\pi f}{c} R\sin\bar{\theta}_S \cos(\bar{\phi}(t)-\bar{\phi}_S) ,$$

where $\alpha = \mathrm{I}, \mathrm{II}$, with $R = 1\,\mathrm{AU}$ and $\bar{\phi}(t) = \bar{\phi}_0 + 2\pi t/T$. Here $T = 1$ year is the orbital period of *LISA*, and $\bar{\phi}_0$ is a constant that specifies the detector's location at time $t = 0$. For nonprecessing binaries $\hat{L}^a$ points in a fixed direction $(\bar{\theta}_L, \bar{\phi}_L)$.

- To express the angles $(\theta_S, \phi_S, \psi_S)$ evaluated with respect to the rotating detector-based coordinate system as function of the angles $(\bar{\theta}_S, \bar{\phi}_S, \bar{\theta}_L, \bar{\phi}_L)$ evaluated with respect to the fixed solar-system based coordinate system, one uses the following relations Cutler 98:

Parameter estimation with LISA

$$
\begin{aligned}
\cos\theta_S(t) &= \frac{1}{2}\cos\bar\theta_S - \frac{\sqrt{3}}{2}\sin\bar\theta_S\cos(\bar\phi(t)-\bar\phi_S)\ ,\\
\phi_S(t) &= \alpha_0 + \frac{2\pi t}{T} + \tan^{-1}\left[\frac{\sqrt{3}\cos\bar\theta_S + \sin\bar\theta_S\cos(\bar\phi(t)-\bar\phi_S)}{2\sin\bar\theta_S\sin(\bar\phi(t)-\bar\phi_S)}\right],
\end{aligned}
$$

where α_0 is a constant specifying the orientation of the arms at $t = 0$. We take $\alpha_0 = 0$ and $\bar\phi_0 = 0$, corresponding to a specific choice of the initial position and orientation of the detector.

$$
\begin{aligned}
\mathbf{z}\cdot\mathbf{n} &= \cos\theta_S\ ,\\
\hat{\mathbf{L}}\cdot\mathbf{z} &= \frac{1}{2}\cos\bar\theta_L - \frac{\sqrt{3}}{2}\sin\bar\theta_L\cos\left(\bar\phi(t)-\bar\phi_L\right)\ ,\\
\hat{\mathbf{L}}\cdot\mathbf{n} &= \cos\bar\theta_L\cos\bar\theta_S + \sin\bar\theta_L\sin\bar\theta_S\cos\left(\bar\phi_L-\bar\phi_S\right)\ ,\\
\mathbf{n}\cdot(\hat{\mathbf{L}}\times\mathbf{z}) &= \frac{1}{2}\sin\bar\theta_L\sin\bar\theta_S\sin\left(\bar\phi_L-\bar\phi_S\right)\\
&\quad -\frac{\sqrt{3}}{2}\cos\bar\phi(t)\left(\cos\bar\theta_L\sin\bar\theta_S\sin\bar\phi_S - \cos\bar\theta_S\sin\bar\theta_L\sin\bar\phi_L\right)\\
&\quad -\frac{\sqrt{3}}{2}\sin\bar\phi(t)\left(\cos\bar\theta_S\sin\bar\theta_L\cos\bar\phi_L - \cos\bar\theta_L\sin\bar\theta_S\cos\bar\phi_S\right).
\end{aligned}
$$

Parameter estimation with LISA

- For 3.5PN accurate expression for $t(f)$ one uses the following relation

$$2\pi\, t(f) \;=\; \frac{d\,\psi(f)}{df}.$$

This can be rewritten as

$$t(f) \;=\; t_c - \sum_{k=0}^{7} t_k^v\, v^k,$$

- For calculations where LISA is assumed to be a two detector network, we calculate the SNR and Fisher matrix using

$$\rho^{\rm Network} = \sqrt{\rho_{\rm I}^2 + \rho_{\rm II}^2},$$
$$\Gamma_{ab}^{\rm Network} = \Gamma_{ab}^{\rm I} + \Gamma_{ab}^{\rm II}.$$

The errors for the two detector case are obtained inverting the total Fisher matrix following the standard procedure.

Earlier Work on Parameter estimation with LISA

- Cutler (1996):
  Parameter estimation using 1.5PN phasing in the LISA context.
- Hughes (2002):
  Parameter estimation with 2PN phasing including spin-spin interaction. Monte-Carlo simulation
- Vecchio (2004):
  1.5PN spin-orbit coupling, no precession Vs simple precession
- Berti, Buonanno and Will (2004,2005):
  Testing alternate theories gravity
- KGA, Iyer, Qusailah and Sathyaprakash, (2006):
  Testing PN gravity using 3.5PN phasing formula.
- Porter and Cornish & Wikham, Stroer and Vecchio (2006):
  MCMC simulations with 2PN phasing
- Lang and Hughes (2006):
  Spin-Spin + simple precession

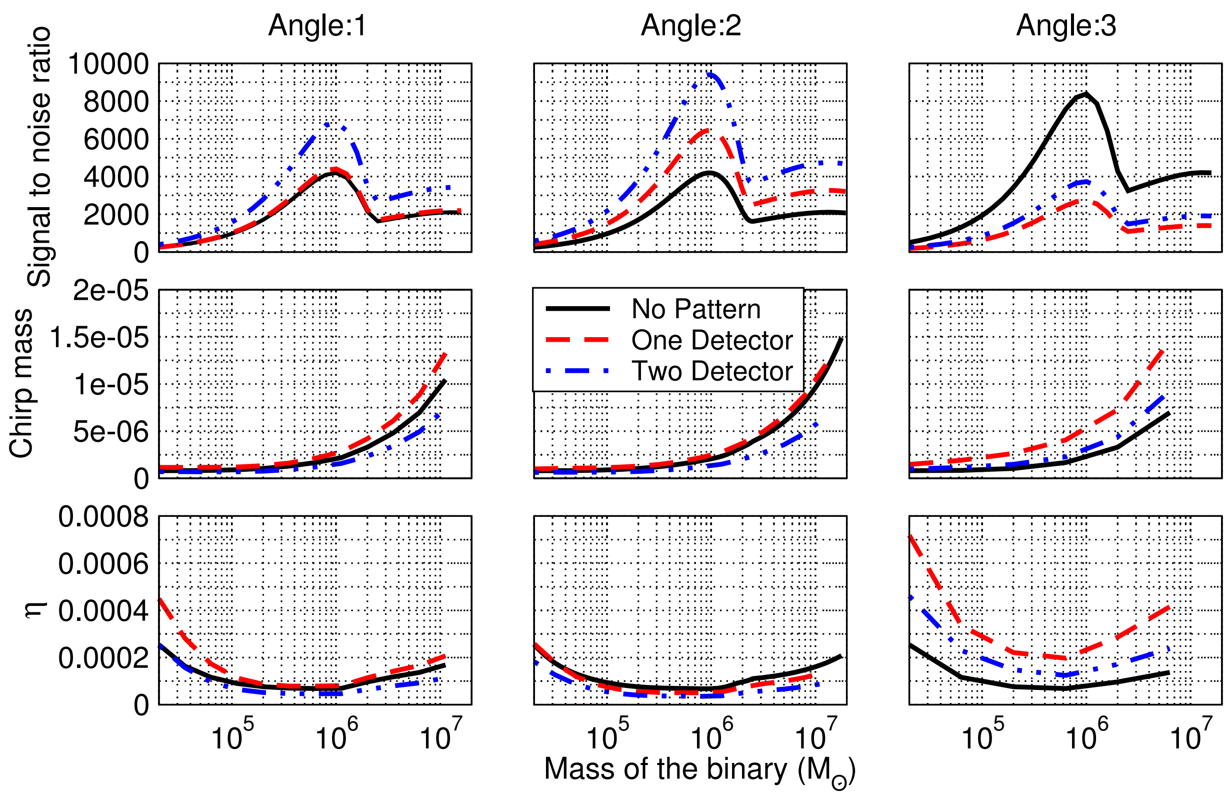
Choice of upper and lower cut-off frequencies

- The upper limit of integration is $f_{\rm fin} = {\rm Min}[f_{\rm lso}, f_{\rm end}]$, where $f_{\rm lso}$ is the frequency of the innermost stable circular orbit for the test particle case, $f_{\rm lso} = (6^{3/2}\,\pi\,m)^{-1}$ and $f_{\rm upper}$ corresponds to the upper cut-off of the LISA noise curve $f_{\rm end} = 1{\rm Hz}$.
- We have chosen the lower limit of frequency $f_{\rm start} = {\rm Max}[f_{\rm in}, f_{\rm lower}]$ where $f_{\rm in}$ is calculated by assuming the signal to last for one year in the LISA sensitivity band

PE using non-pattern averaged WF

- Four more parameters corresponding to the luminosity distance and orientation/location are added to the parameter space:
  $\{t_c\,,\phi_c\,,\mathcal{M}\,,\eta\,,D_L\,,\bar{\mu}_L\,,\bar{\mu}_S\,,\bar{\phi}_S\,,\bar{\phi}_L\}$
- Dimension of the parameter space increased to 9.
- Increased dimensionality
  $\Rightarrow$ increased errors in estimation of the existing 4 parameters.
- But there is an increase/decrease in SNR due to the inclusion of pattern functions also.
  Increase/decrease depends on the orientation of the source.

Results including the pattern functions



(K G Arun, 2006)

Parameter estimation with LISA

- Angle:1 corresponds to $\{\bar{\mu}_L = 0.5, \bar{\mu}_S = -0.8, \bar{\phi}_L = 3, \bar{\phi}_S = 1\}$.
- Angle:2 is $\{\bar{\mu}_L = 0.2, \bar{\mu}_S = -0.6, \bar{\phi}_L = 3, \bar{\phi}_S = 1\}$ and
- Angle:3 $\{\bar{\mu}_L = 0.8, \bar{\mu}_S = 0.3, \bar{\phi}_L = 2, \bar{\phi}_S = 5\}$. The errors thus depend very much on the position and orientation of the source in the sky.

Inferences

- Among the different effects, the change (increase/decrease) in SNR is the most dominant effect..
- The improvement in going from 2PN to 3.5PN waveform depends very much on the source's location and orientation.
- General conclusions cannot be drawn.
- Monte-Carlo methods may have to be used to study PN trends in this case.
- But irrespective of the location and orientation, the higher order PN terms do NOT improve the estimation of D_L and angular resolution.

PN convergence with pattern averaged WF

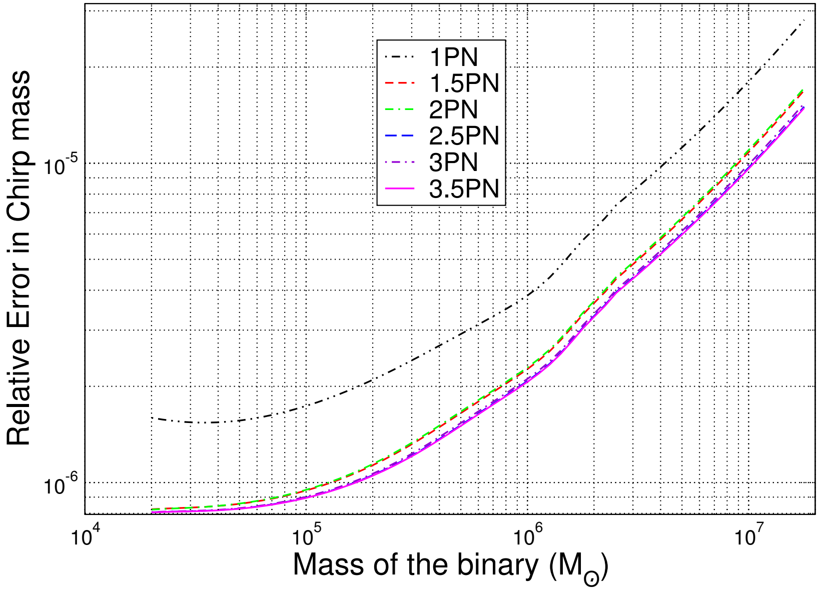
- Justification: Earlier MC simulations using 2PN phasing showed that the results with the two detector case without pattern averaging is very close to that of pattern averaged waveform.
  (Berti, Buonanno & Will, 2005)
- Hence we believe the trends obtained using the pattern averaged waveform will give useful insights about the problem, though it has to be supported with a MC simulation in future.
- Similar to the ground-based detector case for sources in the LISA band also, 3.5PN phasing improves estimation of mass parameters. Improvement could be as high as 13% for $\mathcal{M}$ and 45% for η (with a lower cut-off of 10^{-5} Hz).

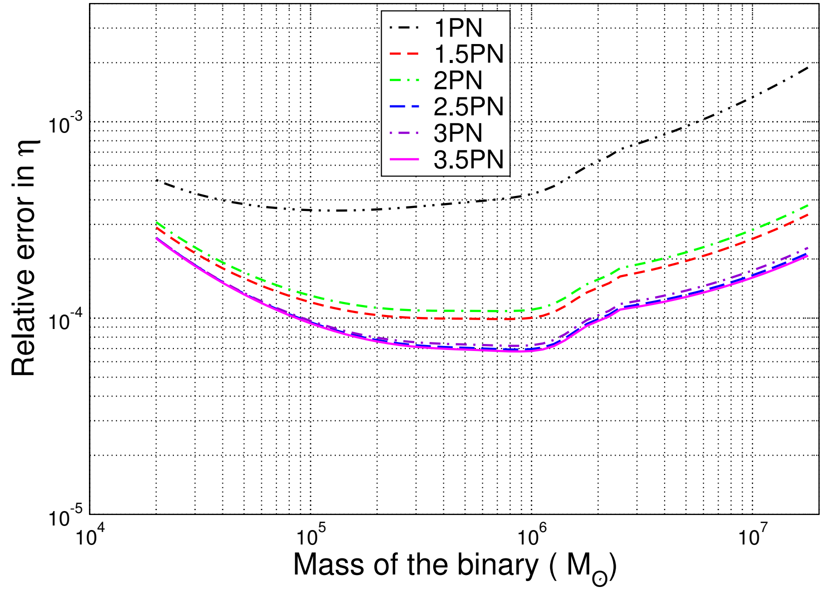
LISA errors across PN orders

PN Order	Δt_c (sec)	$\Delta\mathcal{M}/\mathcal{M}$ (10^{-6})	$\Delta\eta/\eta$ (10^{-4})	$N_{\rm cycles}$
1PN	0.2474	6.217	6.287	2414.03
1.5PN	0.3149	3.648	1.427	2310.26
2PN	0.3074	3.694	1.572	2305.52
2.5PN	0.3947	3.320	0.9882	2314.48
3PN	0.3435	3.377	1.033	2308.73
3.5PN	0.4399	3.300	0.9661	2308.13

(K G Arun, 2006)

Results for LISA: PN convergence



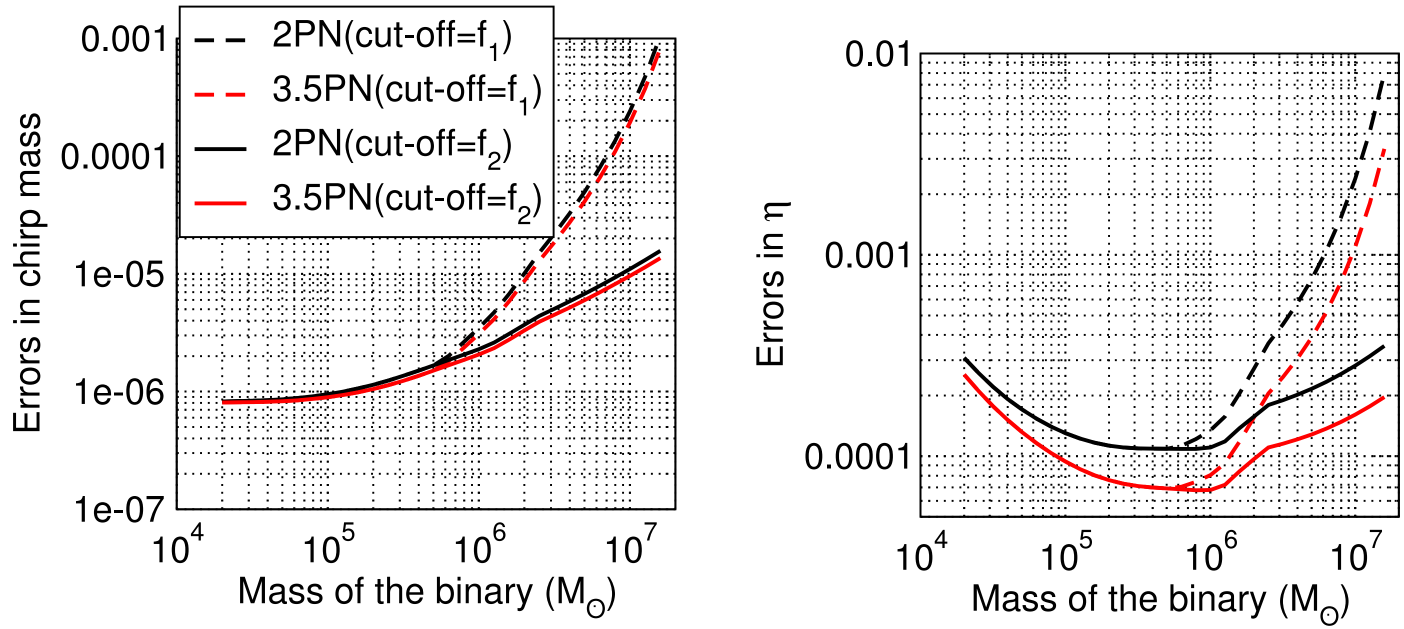


(K G Arun, 2006)

Errors in chirp mass and η for different PN orders

- % improvement in $\mathcal{M}$ for $2 \times 10^6 M_\odot$=11% (∼20%)
- % improvement in η for $2 \times 10^6 M_\odot$=40% (∼60%)

Effect of lower cut-off for LISA



$f_1 = 10^{-4}$ Hz, $f_2 = 10^{-5}$ Hz

(K G Arun, 2006)

Improvement is smaller with smaller lower cut-off $\Rightarrow$ convergence is better.

Errors for unequal mass coalescences

- Fisher matrix inversion for Extreme mass ratio inspirals are very much ill-conditioned.
- We consider IMBH-SMBH coalescences to understand the unequal mass case.
- For a $10^4\ M_\odot$-$10^7\ M_\odot$ system, the SNR is about a hundred and improvement with a 10^{-5} Hz cut-off is about 20% for chirp mass and 62% for symmetric mass ratio.
- Larger improvements for asymmetric systems is not special to LISA.

Conclusions - Ground Based LI

- This study emphasizes the significance of higher PN order modelling of the ICBs for the parameter estimation.
- Relative to 2PN phasing the 3.5PN phasing provides a better estimate of the mass parameters $\mathcal{M}$ and η.
- At fixed SNR, VIRGO has the least errors due to its larger BW, followed by Adv. LIGO and Initial LIGO
- Errors oscillate at each half a PN order in the phasing formula in going from 1PN to 3.5PN
- For sources at fixed distances, Adv. LIGO provides the most accurate estimates due to its better sensitivity. VIRGO performs better than initial LIGO

Conclusions ..Grd based LI

- Number of useful cycles can be used to gauge the detector performance as well as partially understand variation of PE with PN order.
- The correlation of PE with detector sensitivity, detector BW and number of useful cycles has been reasonably understood.
- This analysis can have implications for future theoretical studies of the ICBs as well as designing future generation interferometers.

Conclusions - LISA

- Higher order phasing terms are very much significant for LISA as well.
- The improvement is sensitive to the lower cut-off LISA has (for massive systems).
- Compared to equal mass case, improvement is larger for inspirals with intermediate mass ratios.
- Percentage improvements are very much sensitive to the location and orientation of the sources when orbital motion of LISA is put in.

Parameter Estimation with Full Waveform

- Early work include Sintes and Vechio (LIGO and LISA), 2000; Hellings and Moore (LISA), 2002
- More complete study by Van Den Broeck and Sengupta (2006) (Adv LIGO, EGO)
- Use of high order amplitude corrrected waveforms lead to dramatic improvement in quality of parameter estimation
- With restricted WF errors are steep functions of total mass of the binary. Hence accurate PE is possible for relatively light stellar mass binaries
- Amplitude corrected waveforms allow for high accuracy parameter extraction for total masses up to several hundred solar masses at distances of 100 Mpc
- On the Table in next slide the results are displayed. Change in signal-to-noise ratio and improvement of parameter estimation for Advanced LIGO ($(5, 50)\, M_\odot$) and EGO ($(10, 100)\, M_\odot$) with increasing p in $(p, 2.5)$PN waveforms.

Parameter Estimation with Full Waveform

Van Den Broeck and Sengupta 2006

AdvLIGO,	$(5, 50)M_\odot$					
p	SNR	$\Delta\mathcal{M}/\mathcal{M}$ (%)	$\Delta\delta$ (%)	Δt_c (ms)	$\Delta\beta$	$\Delta\sigma$
0	76.5	13.75	514.4	234.3	51.93	790.
0.5	85.0	0.905	4.498	6.468	2.143	8.52
1	74.1	0.674	3.662	5.446	1.691	6.90
1.5	69.0	0.463	4.376	5.273	1.124	6.06
2	65.5	0.458	3.444	4.205	1.144	5.70
2.5	64.0	0.471	2.318	3.822	1.611	4.45

EGO,	$(10, 100)M_\odot$					
p	SNR	$\Delta\mathcal{M}/\mathcal{M}$ (%)	$\Delta\delta$ (%)	Δt_c (ms)	$\Delta\beta$	$\Delta\sigma$
0	461.9	2.140	81.75	74.91	8.160	125.
0.5	513.0	0.145	0.758	2.085	0.347	1.39
1	446.4	0.107	0.625	1.769	0.272	1.14
1.5	417.7	0.075	0.748	1.753	0.184	1.02
2	396.8	0.076	0.589	1.405	0.194	0.97
2.5	387.7	0.076	0.401	1.262	0.192	0.75

Future directions

- Future work towards a critical understanding of PE could involve addressing aspects like variety in the space of waveforms, change in covariances among different parameters and effects of additional parameters in the phasing formula.
- Implications of 3.5PN phasing formula in estimation of distances and orientation of the binary using a detector network.
- Implications of Full-Waveform for parameter estimation
- Tighter bounds with Monte-Carlo simulations???

The phasing formula: Corrected Coefficients

$$\alpha_5 = \pi\left(\frac{38645}{756} + \frac{38645}{252}\log\left(\frac{v}{v_{\rm lso}}\right) + \frac{5}{3}\eta\left[1 + 3\log\left(\frac{v}{v_{\rm lso}}\right)\right]\right),$$

$$\alpha_5 = \pi\left(\frac{38645}{756} + \frac{38645}{252}\log\left(\frac{v}{v_{\rm lso}}\right) - \frac{65}{9}\eta\left[1 + 3\log\left(\frac{v}{v_{\rm lso}}\right)\right]\right),$$

$$\alpha_7 = \pi\left(\frac{77096675}{254016} + \frac{1014115}{3024}\eta - \frac{36865}{378}\eta^2\right).$$

$$\alpha_7 = \pi\left(\frac{77096675}{254016} + \frac{378515}{1512}\eta - \frac{74045}{756}\eta^2\right).$$

Errors at diff. PN orders: fixed SNR

TABLE I: Convergence of measurement errors from 1PN to 3.5PN at a SNR of 10 for the three prototypical binary systems: NS-NS, NS-BH and BH-BH using the phasing formula, in steps of 0.5PN. For each of the three detector noise curves the table presents Δt_c (in msec), $\Delta\phi_c$ (in radians), $\Delta\mathcal{M}/\mathcal{M}$ and $\Delta\eta/\eta$.

	NS-NS				NS-BH				BH-BH			
PN Order	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$	Δt_c	$\Delta\phi_c$	$\Delta\mathcal{M}/\mathcal{M}$	$\Delta\eta/\eta$
Advanced LIGO												
2PN	0.4623	1.392	0.0143%	1.764%	0.7208	1.848	0.0773%	2.669%	1.404	2.850	0.6240%	10.79%
2.5PN	0.5090	1.359	0.0134%	1.334%	0.9000	1.219	0.0686%	1.515%	1.819	1.574	0.5300%	5.934%
3PN	0.4938	1.331	0.0135%	1.348%	0.8087	1.131	0.0698%	1.571%	1.544	1.580	0.5466%	6.347%
3.5PN	0.5193	1.279	0.0133%	1.319%	0.9966	0.9268	0.0679%	1.457%	2.078	1.161	0.5241%	5.739%
Initial LIGO												
2PN	0.4109	1.816	0.0423%	3.007%	1.148	3.597	0.2903%	6.316%	2.900	7.179	2.851%	32.82%
2.5	0.4605	1.650	0.0384%	2.129%	1.467	1.975	0.2491%	3.305%	3.836	3.119	2.351%	16.48%
3PN	0.4402	1.618	0.0389%	2.170%	1.286	1.798	0.2554%	3.474%	3.159	3.123	2.446%	17.94%
3.5PN	0.4754	1.517	0.0383%	2.099%	1.666	1.324	0.2456%	3.151%	4.512	1.912	2.314%	15.77%
VIRGO												
2PN	0.1562	0.7515	0.0098%	1.085%	0.5918	1.561	0.0611%	2.215%	1.395	2.667	0.5199%	9.625%
2.5PN	0.1743	0.7045	0.0091%	0.7957%	0.7384	1.039	0.0541%	1.263%	1.787	1.545	0.4417%	5.370%
3PN	0.1671	0.6920	0.0092%	0.8083%	0.6632	0.9672	0.0551%	1.309%	1.532	1.547	0.4552%	5.724%
3.5PN	0.1797	0.6562	0.0091%	0.7858%	0.8183	0.7968	0.0536%	1.215%	2.024	1.173	0.4369%	5.201%

Percentage improvements: NS-NS

Interferometer	NS-NS			
	t_c	ϕ_c	$\ln\mathcal{M}$	$\ln\eta$
Adv. LIGO	-12.33	8.118	6.993	25.23
Ini. LIGO	-15.70	16.470	9.456	30.20
VIRGO	-15.05	12.68	7.143	27.58

Improvement: NS-BH

Interferometer	NS-BH			
	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
Adv. LIGO	-38.26	49.85	12.16	45.41
Ini. LIGO	-45.12	63.19	15.40	50.11
VIRGO	-38.27	48.96	12.28	45.15

Improvement:BH-BH

Interferometer	BH-BH			
	t_c	ϕ_c	$\ln \mathcal{M}$	$\ln \eta$
Adv. LIGO	−48.01	59.26	16.01	46.81
Ini. LIGO	−55.59	73.37	18.84	51.95
VIRGO	−45.09	56.02	15.97	45.96

Noise curves used

- Adv.LIGO:

$$
\begin{aligned}
S_h(f) &= 10^{-49}\left[x^{-4.14} - 5x^{-2} + \frac{111(1 - x^2 + x^4/2)}{(1 + x^2/2)}\right],\ f \geq f_s \\
&= \infty,\ f < f_s,
\end{aligned}
$$

- VIRGO:

$$
\begin{aligned}
S_h(f) &= 9 \times 10^{-46}\left[(6.23x)^{-5} + 2x^{-1} + 1 + x^2\right],\ f \geq f_s \\
&= \infty,\ f < f_s,
\end{aligned}
$$

- Initial LIGO:

$$
\begin{aligned}
S_h(f) &= 3.24 \times 10^{-46}\left[(4.49x)^{-56} + 0.16x^{-4.52} + 0.52 + 0.32x^2\right],\ f \geq \\
&= \infty,\ f < f_s,
\end{aligned}
$$