Black Hole Perturbations

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Plan

1. Introduction

2. Perturbation of Schwarzschild Black Hole

3. Perturbation of Kerr Black Hole

4. MiSaTaQuWa force for radiation reaction

5. Adiabatic approximation to radiation reaction

1. Introduction

- Astrophysical black holes

  many candidates in the universe

  $\sim 10 M_\odot$ stellar BHs (X-ray binaries, etc.)

  $10^6 \sim 10^8 M_\odot$ supermassive BHs (galactic center, AGN, Quasars, etc.)

- Uniqueness (assuming 4-dim GR)

  Black hole is characterized only by mass (M) and angular momentum (J) --- Kerr solution

Can we actually 'see' black holes?

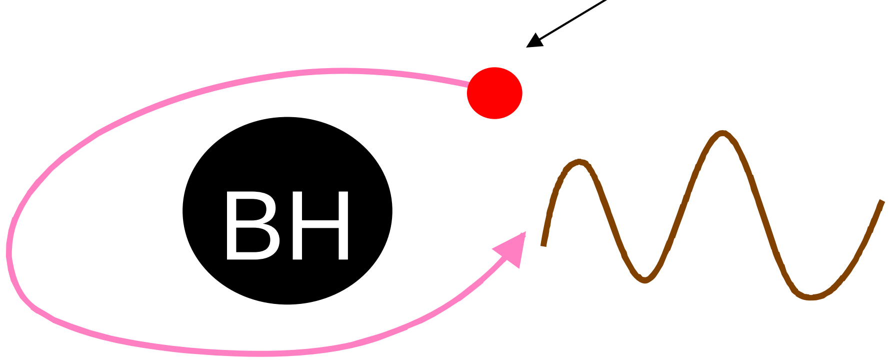
Yes, by observing gravitational waves from BHs.

Gravitational waves from EMRI

(Extreme Mass Ratio Inspiral)

BH: $M \gtrsim 10^6 M_\odot$

NS, BH, etc.: $M \sim M_\odot$



gravitational waves

expected to be detected by LISA

- probe spacetime around BH
- probe properties (mass & spin) of BH

BH perturbation approach is most suited for EMRI

2. Perturbation of Schwarzschild BH

G=c=1 units

- metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= g_{\mu\nu}dx^\mu dx^\nu$$ --- static, spherically symmetric

- perturbation

$$g_{\mu\nu} \to g_{\mu\nu} + \varepsilon h_{\mu\nu}$$

$$G_{\mu\nu}[g] = 0 \to G_{\mu\nu}[g+\varepsilon h] = 8\pi\varepsilon T_{\mu\nu} \to \delta G_{\mu\nu}[h] = 8\pi T_{\mu\nu}$$

$$-\Box \bar{h}_{\mu\nu} + \left(\nabla_\mu \bar{f}_\nu + \nabla_\nu \bar{f}_\mu\right) - R_\mu{}^\alpha{}_\nu{}^\beta \bar{h}_{\alpha\beta} = 16\pi T_{\mu\nu}$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^\alpha{}_\alpha \ , \ \ \bar{f}_\mu \equiv \nabla^\nu \bar{h}_{\mu\nu}$$

Regge-Wheeler-Zerilli formalism

Fourier-harmonic expansion: $h_{\mu\nu} \sim h_{\omega lm}(r) e^{-i\omega t} \hat{O}_{\mu\nu}\left[Y_{lm}(\theta,\phi)\right]$

2-sphere metric: $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \equiv \gamma_{AB} dx^A dx^B$

- scalar spherical harmonics

$$Y_{lm} \qquad \left[\overset{(2)}{\Delta} + l(l+1)\right] Y_{lm}(\theta,\phi) = 0$$

- vector harmonics

$D_A Y_{lm}$, $\varepsilon_{AB} D^B Y_{lm}$; ε_{AB} ··· unit antisymmetric tensor

$$\varepsilon_{\theta\phi} = -\varepsilon_{\phi\theta} = \sqrt{\gamma},$$

$$\varepsilon_{\theta\theta} = \varepsilon_{\phi\phi} = 0$$

- tensor harmonics

$$\gamma_{AB} Y_{lm} \ , \ \left[D_A D_B - \frac{1}{2} g_{AB} \overset{(2)}{\Delta}\right] Y_{lm} \ ,$$

$$\left[\varepsilon_{AC} D^C D_B + \varepsilon_{BC} D^C D_A\right] Y_{lm}$$

decomposition of metric perturbation

Even parity (7 d.o.f.)

$$h_{00}^{(+)},\quad h_{0r}^{(+)},\quad h_{rr}^{(+)} \propto Y_{lm}$$

$$h_{0A}^{(+)},\quad h_{rA}^{(+)} \propto D_A Y_{lm}$$

$$h_{AB}^{(+,L)} = \gamma_{AB} Y_{lm},\quad h_{AB}^{(+,TL)} \propto \left[D_A D_B - \frac{1}{2}\gamma_{AB} \overset{(2)}{\Delta} \right] Y_{lm}$$

Odd parity (3 d.o.f.)

$$h_{00}^{(-)} = h_{0r}^{(-)} = h_{rr}^{(-)} = 0$$

$$h_{0A}^{(-)},\quad h_{rA}^{(-)} \propto \varepsilon_{AB} D^B Y_{lm}$$

$$h_{AB}^{(-)} \propto [\varepsilon_{AC} D^C D_B + \varepsilon_{BC} D^C D_A] Y_{lm}$$

Regge-Wheeler gauge

4 gauge d.o.f.

$$x^{\mu} \to x^{\mu} + \xi^{\mu} \quad \Rightarrow \quad h_{\mu\nu} \to h_{\mu\nu} - (\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu})$$

Even parity (7-3=4)

$$\xi^{0}_{(+)}, \quad \xi^{r}_{(+)} \propto Y_{lm}, \quad \xi^{A}_{(+)} \propto D^{A} Y_{lm}$$

$$\Longrightarrow \qquad h^{(+)}_{0A} = h^{(+)}_{rA} = h^{(+,TL)}_{AB} = 0$$

4 non-zero components: $h^{(+)}_{00},\ h^{(+)}_{or},\ h^{(+)}_{rr},\ h^{(+,L)}_{AB} (\propto \gamma_{AB})$

Odd parity (3-1=2)

$$\xi^{A}_{(-)} \propto \varepsilon^{AB} D_{B} Y_{lm}$$

$$\Longrightarrow \qquad h^{(-)}_{AB} = 0$$

2 non-zero components: $h^{(-)}_{0A},\ h^{(-)}_{rA}$

Perturbation equations in RW gauge

Odd parity:

$$h_{mn}^{(-)} dx^m dx^n = -[h_0 dt dx^A + h_1 dr dx^A]\varepsilon_{AB} D^B Y_{lm} e^{-i\omega t}$$

$\delta G_{AB} = 0:$

$$\frac{i\omega h_0}{1-2M/r} + \frac{d}{dr}\left[(1-2M/r)h_1\right] = 0$$

$\delta G_{rA} = 0:$

$$\frac{i\omega}{1-2M/r}\left[r^2 \frac{d}{dr}\left(\frac{h_0}{r^2}\right) + i\omega h_1\right] + (l-1)(l+2)\frac{h_1}{r^2} = 0$$

$\delta G_{0A} = 0:$ redundant because of Bianchi id.

remaining $\delta G_{\mu\nu} = 0$ are trivial.

Introducing the variable, $X^{(-)} \equiv (1 - 2M/r)\dfrac{h_1}{r}$

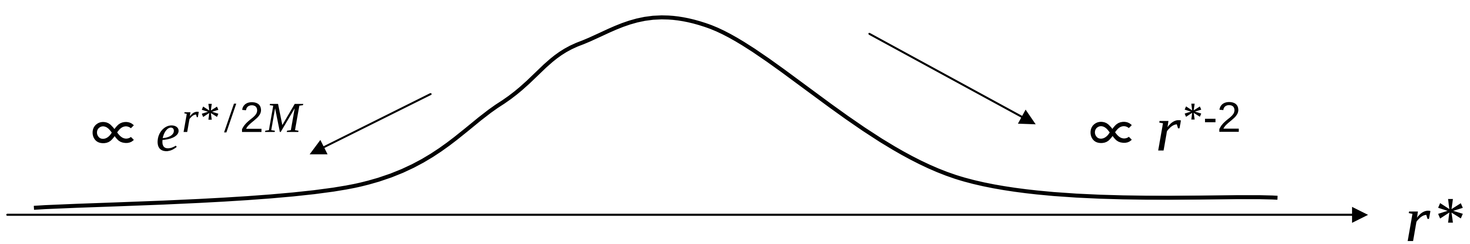
$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V^{(-)}(r)\right] X^{(-)} = 0$$

Regge & Wheeler ('57)

$$r^* \equiv r + 2M \ln(r/2M - 1)$$ --- tortoise coordinate

$$r^* \to \begin{cases} -\infty \\ \infty \end{cases} \Leftrightarrow r \to \begin{cases} 2M \\ \infty \end{cases}$$

$$V^{(-)}(r) = (1 - 2M/r)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)$$ --- RW potential



peak at $r \sim 3M$

Even parity:

$$h^{(+)}_{mn}dx^m dx^n = [(1-2M/r)H_0 dt^2 + 2H_1 dtdr + H_2(1-2M/r)^{-1} + r^2 K g_{AB}dx^A dx^B]Y_{lm}e^{-i\omega t}$$

Non-trivial eqs. are

$$\delta G_{0r}=0,\ \delta G_{0A}=0,\ \delta G_{rA}=0,\ \delta G_{AB}=0,\ \delta G_{rr}=0$$

Much more complicated than odd parity eqs.

But Zerilli found a master variable $X^{(+)}$ Zerilli ('70)

$$\left[\frac{d^2}{dr*^2}+\omega^2-V^{(+)}(r)\right]X^{(+)}=0$$

$$V^{(+)}(r)=(1-2M/r)\frac{\lambda^2(\lambda+2)r^3+6\lambda^2 Mr^2+36\lambda M^2 r+72M^3}{r^3(\lambda r+6M)^2}$$

--- Zerilli potential

$$\lambda \equiv l(l+1)-2=(l-1)(l+2)$$

Chandrasekhar transformation

RW eq. and Zerilli eq. are found to be equivalent to each other.

Chandrasekhar ('83)

$$V^{(\pm)} = \lambda(\lambda+2) f + (6M)^2 f^2 \pm 6M \frac{df}{dr*}$$

$$f \equiv \frac{1-2M/r}{r(\lambda r + 6M)}$$

↓

$$\left(\lambda(\lambda+2) \mp 12iMw\right) X^{(\pm)}$$

$$= \left[\lambda(\lambda+2) + \frac{72M^2(1-2M/r)}{r(\lambda r+6M)} \pm 12M \frac{d}{dr*} \right] X^{(\mp)}$$

This fact is related to Starobinsky-Teukolsky id.
found among Newman-Penrose variables.

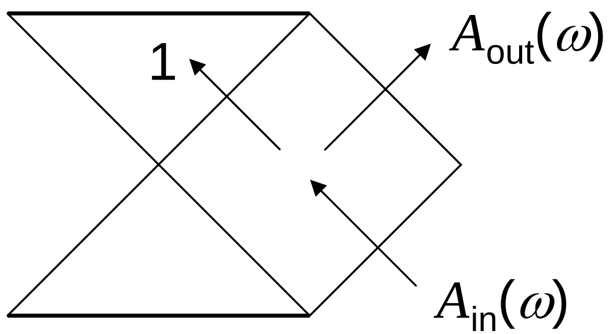
Ingoing & upgoing solutions

Since RW and Zerilli eqs. are equivalent, we focus on RW eq.

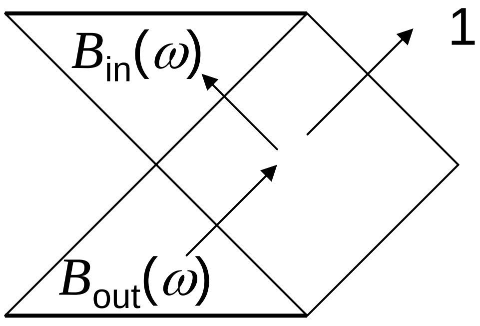
- 2 independent solutions

$$X_{\rm in}^{(-)} \to \begin{cases} e^{-i\omega r^*} & ; \quad r^* \to -\infty \\ A_{\rm in}(\omega) e^{-i\omega r^*} + A_{\rm out}(\omega) e^{i\omega r^*} & ; \quad r^* \to \infty \end{cases}$$

$$X_{\rm up}^{(-)} \to \begin{cases} B_{\rm in}(\omega) e^{-i\omega r^*} + B_{\rm out}(\omega) e^{i\omega r^*} & ; \quad r^* \to -\infty \\ e^{i\omega r^*} & ; \quad r^* \to \infty \end{cases}$$



$X_{\rm in}(r) e^{-i\omega t}$



$X_{\rm up}(r) e^{-i\omega t}$

Asymptotic amplitudes of $X^{(+)}$ and $X^{(-)}$

At $r^*\to\pm\infty$, $X^{(+)}$ and $X^{(-)}$ are related as

$$\left(\lambda(\lambda+2)\mp 12iMw\right)X^{(\pm)}=\left[\lambda(\lambda+2)\pm 12M\frac{d}{dr^*}\right]X^{(\mp)}$$

This gives

$$X_{\rm in}^{(+)}\to\begin{cases} e^{-i\omega r^*} & ;\ r^*\to-\infty \\ \dfrac{\bar{C}_0}{C_0}A_{\rm in}(\omega)e^{-i\omega r^*}+A_{\rm out}(\omega)e^{i\omega r^*} & ;\ r^*\to\infty \end{cases}$$

$$X_{\rm up}^{(+)}\to\begin{cases} B_{\rm in}(\omega)e^{-i\omega r^*}+\dfrac{\bar{C}_0}{C_0}B_{\rm out}(\omega)e^{i\omega r^*} & ;\ r^*\to-\infty \\ \dfrac{\bar{C}_0}{C_0}e^{i\omega r^*} & ;\ r^*\to\infty \end{cases}$$

$$C_0\equiv\lambda(\lambda+2)-12i\,M\omega$$

Wronskian (flux conservation)

(hereafter, X=$X^{(-)}$)

$$W[X_{\rm in},\bar{X}_{\rm in}] \equiv X_{\rm in}\frac{d}{dr*}\bar{X}_{\rm in} - \bar{X}_{\rm in}\frac{d}{dr*}X_{\rm in}$$
$$=2i\omega = 2i\omega\left(|A_{\rm in}(\omega)|^2 - |A_{\rm out}(\omega)|^2\right)$$

$$W[\bar{X}_{\rm up},X_{\rm up}] \equiv \bar{X}_{\rm up}\frac{d}{dr*}X_{\rm up} - X_{\rm up}\frac{d}{dr*}\bar{X}_{\rm up}$$
$$=2i\omega = 2i\omega\left(|B_{\rm out}(\omega)|^2 - |B_{\rm in}(\omega)|^2\right)$$

valid for real ω

$$\Rightarrow \quad |A_{\rm in}(\omega)|^2 - |A_{\rm out}(\omega)|^2 = |B_{\rm out}(\omega)|^2 - |B_{\rm in}(\omega)|^2 = 1$$

$$W[X_{\rm in},X_{\rm up}] \equiv X_{\rm in}\frac{d}{dr*}X_{\rm up} - X_{\rm up}\frac{d}{dr*}X_{\rm in}$$
$$=2i\omega A_{\rm in}(\omega) = 2i\omega B_{\rm out}(\omega)$$

valid for complex ω

$$\Rightarrow \quad A_{\rm in}(\omega) = B_{\rm out}(\omega)$$

Causal (retarded) Green function

When there is a source term (eg, orbiting particle),

$$\left[\frac{d^2}{dr^{*2}}+\omega^2-V(r)\right]X=S(r)$$

$$V(r)=V^{(-)}(r)$$
$$X(r)=X^{(-)}(r)$$

natural causal boundary condition is

$$X\to\begin{cases}Z_{\rm H}(\omega)e^{-i\omega r^*} & ;\ r^*\to-\infty \\ Z_{\infty}(\omega)e^{i\omega r^*} & ;\ r^*\to\infty\end{cases}$$

(ingoing at horizon)

(outgoing at infinity)

- radial Green function satisfying the boundary condition

$$\left[\frac{d^2}{dr^{*2}}+\omega^2-V(r)\right]G(\omega;r^*,r^{*\prime})=\delta(r^*-r^{*\prime})$$

$$G(\omega;r^*,r^{*\prime})=\frac{\theta(r^*-r^{*\prime})X_{\rm up}(r^*)X_{\rm in}(r^{*\prime})+\theta(r^{*\prime}-r^*)X_{\rm in}(r^*)X_{\rm up}(r^{*\prime})}{\mathrm{W}[X_{\rm in},X_{\rm up}]}$$

$$=2i\omega A_{\rm in}(\omega)=2i\omega B_{\rm out}(\omega)$$

Amplitude at infinity and at horizon

$$X(\omega; r^*) = \int_{-\infty}^{\infty} dr^{*\prime}\, G(\omega; r^*, r^{*\prime}) S(r^{*\prime})$$

$$= \frac{1}{2i\omega A_{\rm in}} \left\{ X_{\rm up}(r^*) \int_{-\infty}^{r^*} dr^{*\prime}\, X_{\rm in}(r^{*\prime}) S(r^{*\prime}) + X_{\rm in}(r^*) \int_{r^*}^{\infty} dr^{*\prime}\, X_{\rm up}(r^{*\prime}) S(r^{*\prime}) \right\}$$

⇓

$$X(r^*; \omega) \to \begin{cases} Z_{\rm H}(\omega) e^{-i\omega r^*} & ; \quad r^* \to -\infty \\ Z_{\infty}(\omega) e^{i\omega r^*} & ; \quad r^* \to \infty \end{cases}$$

$$Z_{\infty}(\omega) = \frac{1}{2i\omega A_{\rm in}(\omega)} \int_{-\infty}^{\infty} dr^*\, X_{\rm in}(r^*) S(r^*)$$

$$A_{\rm in}(\omega) = B_{\rm out}(\omega)$$

$$Z_{\rm H}(\omega) = \frac{1}{2i\omega A_{\rm in}(\omega)} \int_{-\infty}^{\infty} dr^*\, X_{\rm up}(r^*) S(r^*)$$

For waves at infinity, we only need to know $X_{\rm in}$

For waves at horizon, we only need to know $X_{\rm up}$

Quasi-normal modes

homogeneous solutions of X such that

$$X_{QNM} \sim \begin{cases} e^{-i\omega r^*} & ; r^* \to -\infty \\ e^{i\omega r^*} & ; r^* \to +\infty \end{cases}$$

ingoing at horizon

outgoing at infinity

consider X_{in},

$$X_{\mathrm{in}} \to \begin{cases} e^{-i\omega r^*} & ; \ r^* \to -\infty \\ A_{\mathrm{in}}(\omega)e^{-i\omega r^*} + A_{\mathrm{out}}(\omega)e^{i\omega r^*} & ; \ r^* \to \infty \end{cases}$$

QNMs: X_{in} satisfying $A_{\mathrm{in}}(\omega)=0$ (X_{up} satisfying $B_{\mathrm{out}}(\omega)=0$)

⇨ ω: complex with $\mathrm{Im}(\omega)<0$

Proof:

- ω is complex.

  ∵ If ω =real, it contradicts with flux conservation

$$|A_{\text{in}}(\omega)|^2 - |A_{\text{out}}(\omega)|^2 = -|A_{\text{out}}(\omega)|^2 = 1$$

- Im(ω)<0

  ∵ If Im(ω)>0, X_{in} would damp exponentially at both $r* \to \mp\infty$

  ($e^{\mp i\omega r*} \sim e^{\pm(\text{Im}\ \omega)r*}$ at $r* \to \mp\infty$)

  Then from $\int_{-\infty}^{\infty} dr*\, \bar{X}\left[-\frac{d^2}{dr*^2} - \omega^2 + V\right] X = 0$

$$\int_{-\infty}^{\infty} dr*\left[\frac{d}{dr*}\bar{X}\frac{d}{dr*}X + V\,\bar{X}X\right] = \omega^2 \int_{-\infty}^{\infty} dr*\left|X\right|^2 > 0$$

  This is inconsistent with the fact that ω is complex.

From reality of $V(r)$, there is a symmetry $\overline{X(-\bar{\omega}, r^*)} = X(\omega, r^*)$

$$\hat{L}\left[X(\omega, r^*)\right] \equiv \left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r)\right] X(\omega, r^*) = 0$$

$\Rightarrow$ $\overline{A_{\mathrm{in}}(-\bar{\omega})} = A_{\mathrm{in}}(\omega)$

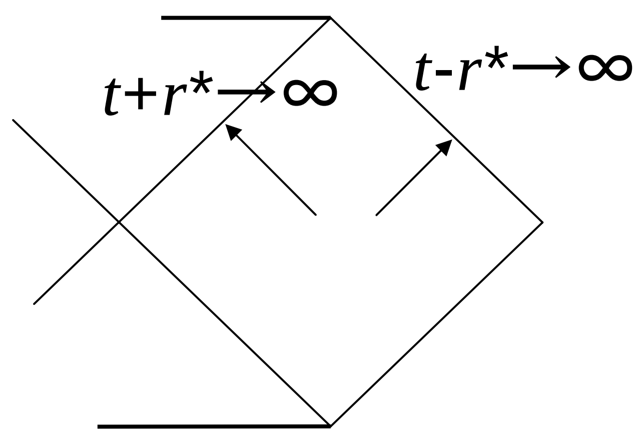
If ω is a QNM, then so is $-\bar{\omega}$

$\Rightarrow$ $$\omega = \pm\sigma - i\Gamma \quad (\Gamma > 0)$$

- QNMs damp exponentially in time
  (~ stability of Schwarzschild BH)

$$X_{QNM} e^{-i\omega t} \sim \begin{cases} e^{-i\omega(t+r^*)} & ; \quad r^* \to -\infty \\ e^{+i\omega(t-r^*)} & ; \quad r^* \to +\infty \end{cases}$$

$\Rightarrow$ $X_{QNM} e^{-i\omega t} \to e^{-\Gamma(t \pm r^*)}$ as $t \pm r^* \to \infty$

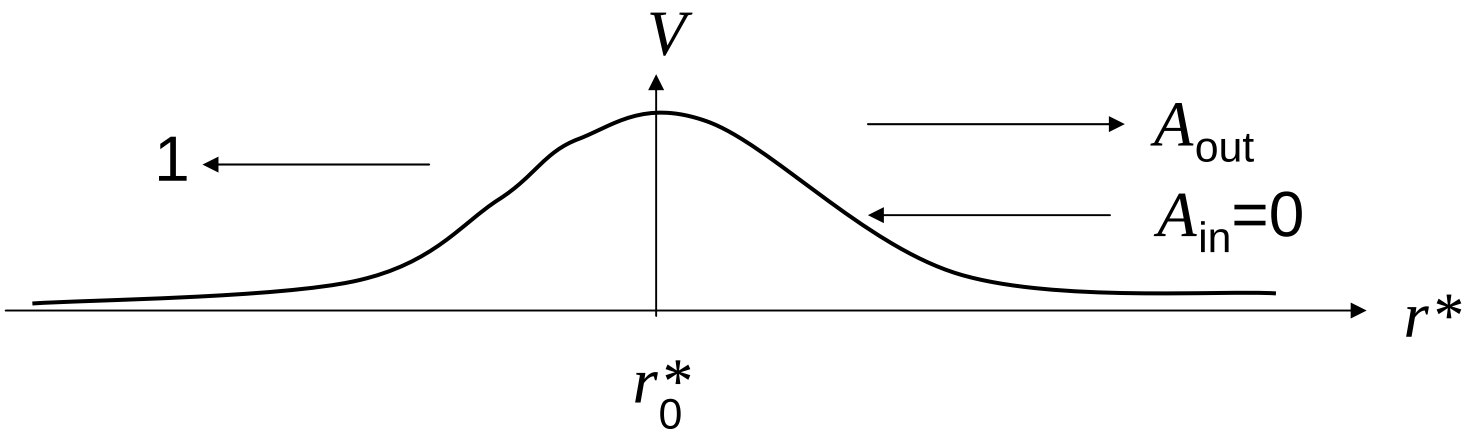


WKB solution of QNMs

Schutz & Will ('85)

QNMs ~ resonance with V

⇨ we expect $\sigma^2 \sim V_{\rm max}$



expand ω^2-V around V=$V_{\rm max}$≡V_0

$$\omega^2 - V = \omega^2 - V_0 + \frac{1}{2}\left|V_0''\right|(r^* - r_0^*)^2 + \cdots$$

set $$r^* - r_0^* = \frac{x}{(2|V_0''|)^{1/4}}$$

$$\Longrightarrow \quad \left[\frac{d^2}{dx^2}+Q(x)\right]X=0\,; \qquad Q(x)=Q_0+\frac{x^2}{4}\,, \quad Q_0=\frac{w^2-V_0}{\sqrt{2|V'|}}$$

This is Weber's equation with $\nu+1/2=iQ_0$

$$\left[\frac{d^2}{dz^2}+\nu+\frac{1}{2}-\frac{z^2}{4}\right]X=0 \qquad \left(z=e^{-\frac{\pi}{4}i}x\right)$$

- solution is a parabolic cylinder fcn.

$$X=a\,D_\nu(z)+b\,D_{-\nu-1}(iz)$$

$$\sim\begin{cases}\tilde{c}(-x)^\nu e^{i\frac{x^2}{4}}+\tilde{d}(-x)^{-\nu-1}e^{-i\frac{x^2}{4}} & ;\ x\to-\infty\\ a\,e^{-\frac{\pi}{4}i\nu}x^\nu e^{i\frac{x^2}{4}}+b\,e^{-\frac{\pi}{4}i(\nu+1)}x^{-\nu-1}e^{-i\frac{x^2}{4}} & ;\ x\to\infty\end{cases}$$

$$\tilde{c}=a\,e^{\frac{3\pi}{4}i\nu}+b\,e^{\frac{\pi}{4}i\nu}\frac{\sqrt{2\pi}}{\Gamma(\nu+1)}\,, \qquad \tilde{d}=b\,e^{\frac{3\pi}{4}i(\nu+1)}-a\,e^{\frac{\pi}{4}i(\nu-3)}\frac{\sqrt{2\pi}}{\Gamma(-\nu)}$$

• WKB solution with 'outgoing' b.c. at $x \to \mp\infty$

$$X \sim \begin{cases} e^{-i\int_0^x Q^{1/2}(x')dx'} \sim e^{-i\frac{x^2}{4}} \;;\; x \to -\infty \\ e^{+i\int_0^x Q^{1/2}(x')dx'} \sim e^{i\frac{x^2}{4}} \;;\; x \to \infty \end{cases}$$

To match to this WKB solution, we must require

$$b = 0 \quad \text{and} \quad \tilde{d} = b\, e^{\frac{3\pi}{4}i(\nu+1)} - a\, e^{\frac{\pi}{4}i(\nu-3)} \frac{\sqrt{2\pi}}{\Gamma(-\nu)} = 0$$

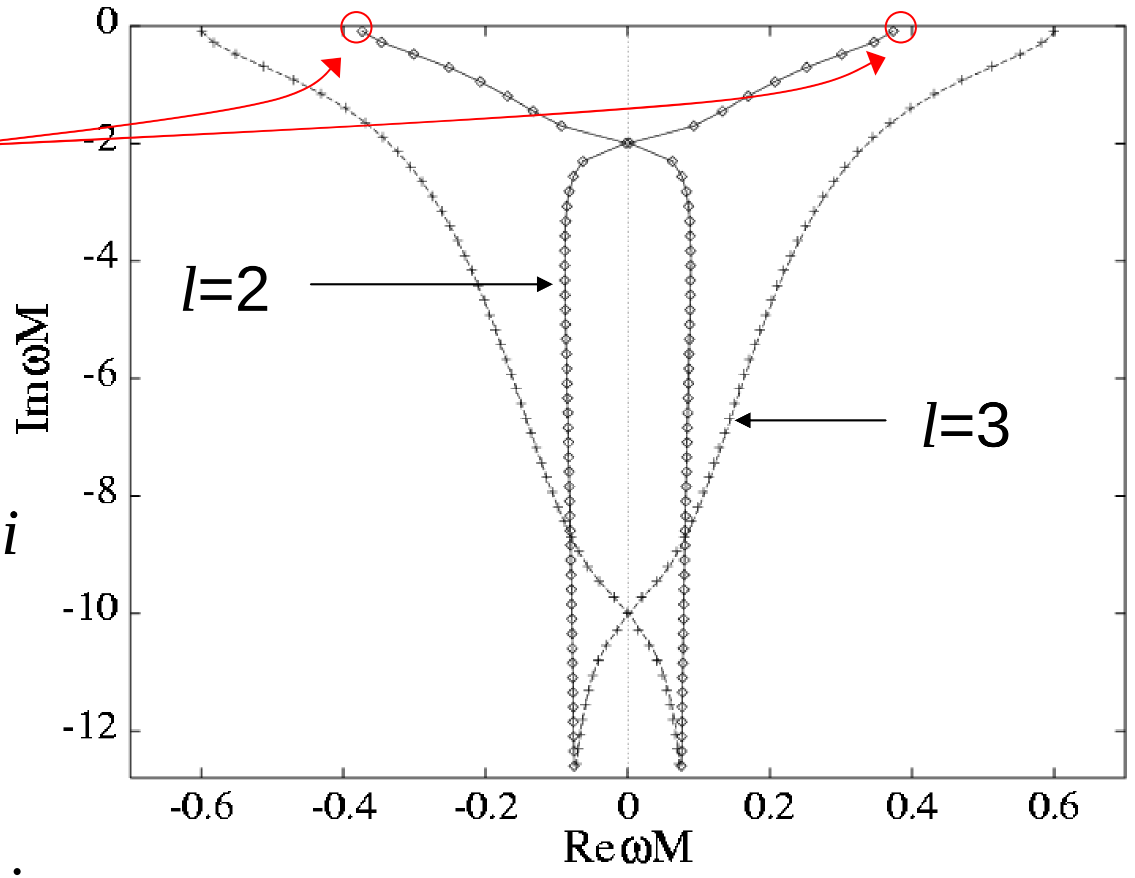
⇨ $b=0,\ \nu = n\ (n=0,1,2,\cdots)$

$$n + \frac{1}{2} = \frac{\omega^2 - V_0}{\sqrt{2|V_0''|}}\, i \quad \longrightarrow \quad \omega^2 = V_0 - \sqrt{2|V_0''|}\left(n + \frac{1}{2}\right) i$$

⇨ $$\omega \approx \frac{1}{3\sqrt{3}M}\left(\pm\ell - \left(n + \frac{1}{2}\right) i\right)$$

numerical (~exact) result

l=2 fundamental mode (*n*=0)



Andersson & Linnaeus ('92)

Numerical:

$M\omega_{l=2,n=0}$
$=0.37367 - 0.08896\, i$

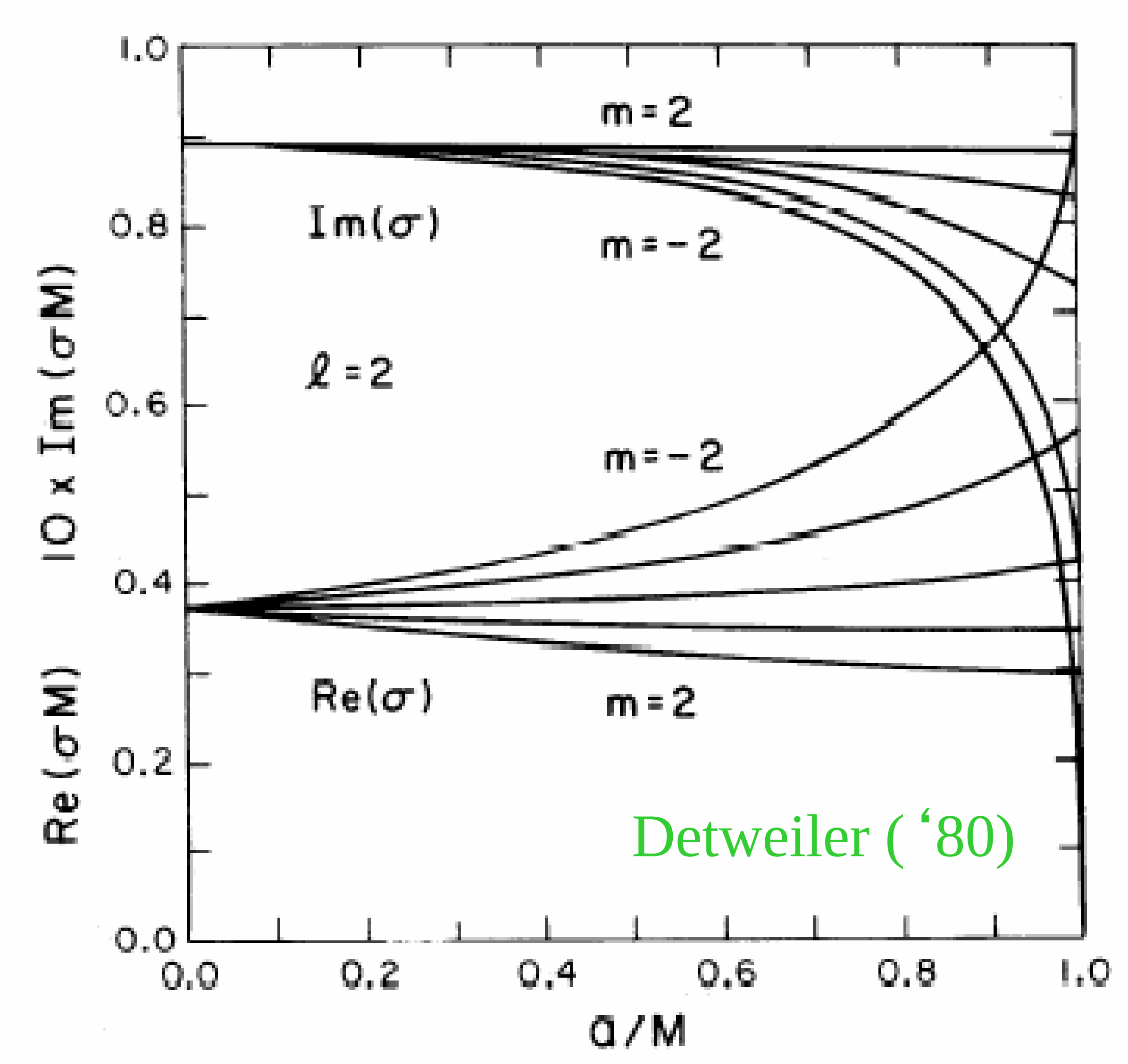
⇕

WKB:

$M\omega_{l=2,n=0}$
$=0.38490 - 0.09623\, i$

$$\nu_{2,0} = \frac{\omega_{2,0}}{2\pi} \sim 10\mathrm{kHz}\,\frac{M_\odot}{M} = 10^{-2}\,\mathrm{Hz}\,\frac{10^6 M_\odot}{M}$$ ··· detectable by LISA

Quasi-normal modes of Kerr BH



Q-value (=σ/Γ) is larger for BHs with higher spin

Excitation of QNMs

QNM fit to numerical data for 2BH head-on collision

l =2 waveform fitted by the first two (*n*=0,1) least damped modes

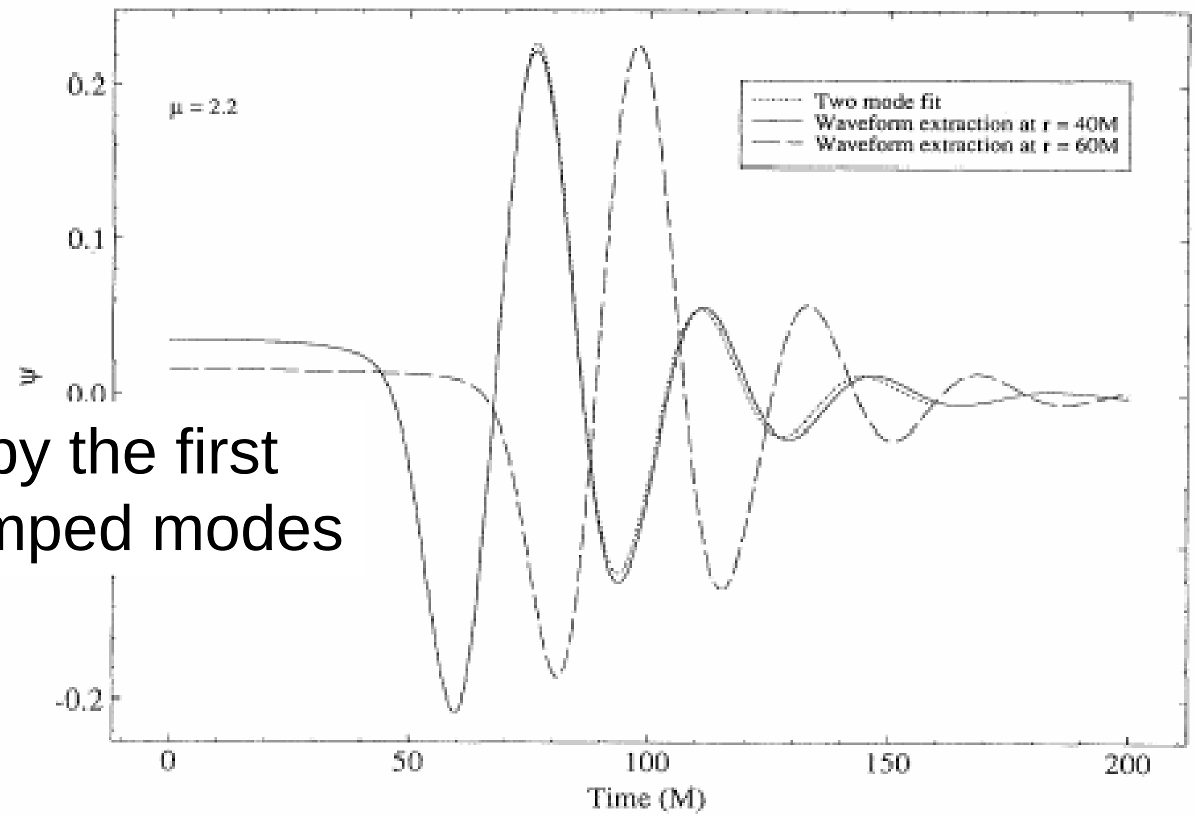


FIG. 1. $l=2$ wave forms for the case $\mu=2.2$. The solid line shows the wave form extracted at $R=40M$ and the long-dashed line shows the wave form at $R=60M$. The short-dashed line shows the quasinormal mode fit.

Anninos, Hobil, Seidel, Smarr & Suen ('93)

Why waveform can be fitted by QNMs

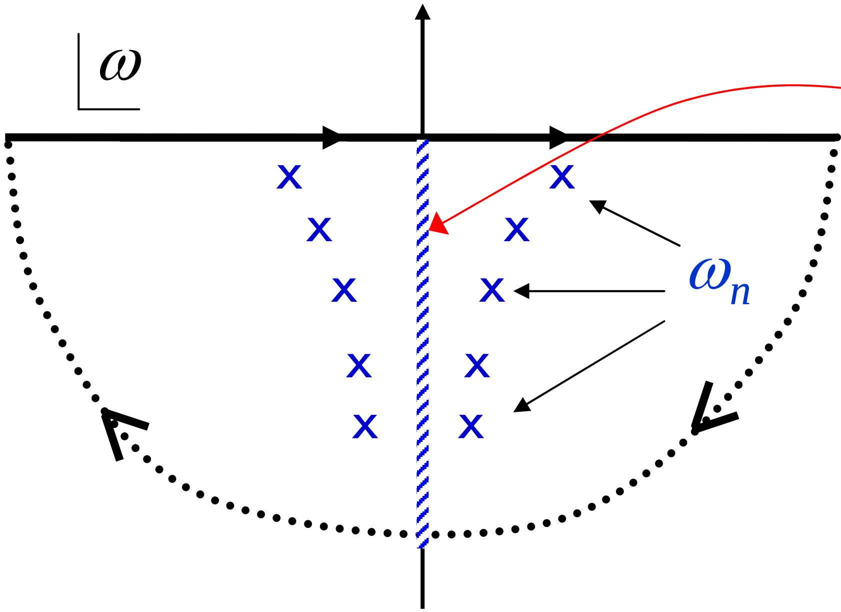
waveform at infinity:

$$X_\infty(t-r^*) \sim \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega(t-r^*)}}{2i\omega A_{\rm in}(\omega)} \int_{-\infty}^{\infty} dr^{*\prime} X_{\rm in}(\omega, r^{*\prime}) S(\omega, r^{*\prime})$$

QNMs ~ poles in the ω-plane

t-r^* >0

$$\sim \sum_n \frac{e^{-i\omega_n(t-r^*)}}{\omega_n \partial_\omega A_{\rm in}(\omega)\big|_{\omega=\omega_n}} \int_{-\infty}^{\infty} dr^{*\prime} X_{\rm in}(\omega_n, r^{*\prime}) S(\omega_n, r^{*\prime})$$



In reality, there is a cut along Re(ω)=0

leads to power-law tail at late times