3. Perturbation of Kerr BH

Kerr geometry:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar \sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma} \sin^2\theta\right) \sin^2\theta d\phi^2,$$

$$\Sigma = r^2 + a^2 \cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2.$$

horizon at $\Delta = 0$ $(r = r_{\pm})$

- Unfortunately, it is technically formidable to deal with the metric perturbation of Kerr BH because of coupling between r and θ
- Nevertheless, there exists a formalism (Newman-Penrose formalism) to deal with perturbations of Kerr geometry.

Complex null-tetrad and Weyl tensor

Newman & Penrose ('62)

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu$$

where $l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = 0 \qquad -l_\mu n^\mu = m_\mu \bar{m}^\mu = 1$

flat space example:

$$ds^2 = -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$l^\mu = (1,1,0,0), \quad n^\mu = \frac{1}{2}(1,-1,0,0), \quad m^\mu = \frac{1}{\sqrt{2}\, r}\left(0,\ 0\ ,\ 1\ ,\ \frac{i}{\sin\theta}\right)$$

Weyl tensor components (10 d.o.f. ⇒ 5 complex d.o.f.)

$$\Psi_0 = -C_{lmlm} \qquad \Psi_1 = -C_{lnlm} \qquad \Psi_2 = -\frac{1}{2}\left(C_{lnln} - C_{lnm\bar{m}}\right)$$

$$\Psi_3 = -C_{ln\bar{m}n} \qquad \Psi_4 = -C_{n\bar{m}n\bar{m}}$$

Lorentz transformation of tetrad frame

(6 d.o.f. = 3 complex d.o.f.)

(1) $\boldsymbol{l} \to \boldsymbol{l},\ \ \boldsymbol{m} \to \boldsymbol{m} + a\,\boldsymbol{l},\ \ \boldsymbol{n} \to \boldsymbol{n} + a\,\bar{\boldsymbol{m}} + \bar{a}\,\boldsymbol{m} + a\bar{a}\,\boldsymbol{l}$

(2) $\boldsymbol{n} \to \boldsymbol{n},\ \ \boldsymbol{m} \to \boldsymbol{m} + b\,\boldsymbol{n},\ \ \boldsymbol{l} \to \boldsymbol{l} + b\,\bar{\boldsymbol{m}} + \bar{b}\,\boldsymbol{m} + b\bar{b}\,\boldsymbol{n}$

(3) $\boldsymbol{l} \to \Lambda\boldsymbol{l},\ \ \boldsymbol{n} \to \boldsymbol{n}/\Lambda,\ \ \boldsymbol{m} \to e^{i\alpha}\boldsymbol{m}$

transformation of Weyl components

(1)

$$\begin{pmatrix}\Psi'_0\\ \Psi'_1\\ \Psi'_2\\ \Psi'_3\\ \Psi'_4\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ \bar{a} & 1 & 0 & 0 & 0\\ \bar{a}^2 & 2\bar{a} & 1 & 0 & 0\\ \bar{a}^3 & 3\bar{a}^2 & 3\bar{a} & 1 & 0\\ \bar{a}^4 & 4\bar{a}^3 & 6\bar{a}^2 & 4\bar{a} & 1\end{pmatrix}\begin{pmatrix}\Psi_0\\ \Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$$

(2)

$$\begin{pmatrix}\Psi'_0\\ \Psi'_1\\ \Psi'_2\\ \Psi'_3\\ \Psi'_4\end{pmatrix} = \begin{pmatrix} b^4 & 4b^3 & 6b^2 & 4b & 1\\ b^3 & 3b^2 & 3b & 1 & 0\\ b^2 & 2b & 1 & 0 & 0\\ b & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\Psi_0\\ \Psi_1\\ \Psi_2\\ \Psi_3\\ \Psi_4\end{pmatrix}$$

In particular, under (2)

$$\boldsymbol{n} \to \boldsymbol{n}, \quad \boldsymbol{m} \to \boldsymbol{m} + b\,\boldsymbol{n}, \quad \boldsymbol{l} \to \boldsymbol{l} + b\,\bar{\boldsymbol{m}} + \bar{b}\,\boldsymbol{m} + b\bar{b}\,\boldsymbol{n}$$

we have

$$\Psi_0 \to \Psi_0{}' = P(b) \equiv \Psi_4 b^4 + 4\Psi_3 b^3 + 6\Psi_2 b^2 + 4\Psi_1 b + \Psi_0 1$$

$$4\Psi_1 \to 4\Psi_1{}' = P'(b) = 4\Psi_4 b^3 + 12\Psi_3 b^2 + 12\Psi_2 b + 4\Psi_1 1$$

Petrov type D:

$$P \equiv \Psi_4 (b - b_1)^2 (b - b_2)^2$$

Kerr is type D

One can make Ψ_0=0 by (2) with $b=b_1$

In this case,

$$4\Psi_1' = P'(b) = 2\Psi_4 \left\{ (b-b_1)(b-b_2)^2 + (b-b_1)^2 (b-b_2) \right\}$$

Hence we have Ψ_1=0 simultaneously.

The result is

$$\begin{pmatrix}\Psi_0\\ \Psi_1\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \qquad \begin{pmatrix}\Psi_2\\ \Psi_3\end{pmatrix}=\begin{pmatrix}\frac{1}{12}Q''\\ \frac{1}{24}Q\end{pmatrix}=\begin{pmatrix}\frac{1}{6}\Psi_4(b_1-b_2)^2\\ \frac{1}{2}\Psi_4(b_1-b_2)^2\end{pmatrix}$$

If we further apply transformation (1)

$$\Psi'_2=\Psi_2$$

$$\Psi'_3=\frac{\bar{a}}{4}P''(b_1)+\frac{1}{24}P'''(b_1)=\frac{1}{2}\Psi_4(b_1-b_2)\left(1+\bar{a}(b_1-b_2)\right)$$

$$\Psi'_4=\frac{\bar{a}^2}{2}P''(b_1)+\frac{\bar{a}}{6}P'''(b_1)+\frac{1}{24}P^{(4)}(b_1)=\Psi_4\left(1+\bar{a}(b_1-b_2)\right)^2$$

$\Psi_3=\Psi_4=0$ for $\bar{a}=\dfrac{1}{b_2-b_1}$ ⇨ Only $\Psi_2 \neq 0$

(***l*** and ***n*** are called repeated principal null directions)

When we consider the perturbation,

Ψ_0 and Ψ_4 are both gauge-invariant and Lorentz-invariant.

(Ψ_1 and Ψ_3 are not Lorentz-invariant.)

For Kerr geometry,

- Kinnersley's null-tetrad

$$l^\mu = \Delta^{-1}\left(r^2 + a^2\ ,\ \Delta\ ,\ 0\ ,\ a\right)$$

$$n^\mu = \left(2\Sigma\right)^{-1}\left(r^2 + a^2\ ,\ -\Delta\ ,\ 0\ ,\ a\right)$$

$$m^\mu = \frac{1}{\sqrt{2}\left(r + ia\cos\theta\right)}\left(ia\sin\theta\ ,\ 0\ ,\ 1\ ,\ \frac{i}{\sin\theta}\right)$$

$$l^\mu = \frac{dx^\mu}{dv}$$ affinely parametrized outgoing null geodesics

- Weyl

$$\Psi_2 = \frac{M}{(r - i\,a\cos\theta)^3} \qquad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$$

Teukolsky equation

$${}_s\psi = \begin{pmatrix} \Psi_0 \\ \bar{z}^4 \Psi_4 \end{pmatrix} \quad \begin{matrix} s=2 \\ s=-2 \end{matrix}$$

$$z = r + i a \cos\theta$$

Teukolsky equation

$${}_s L\left[{}_s\psi\right] = 4\pi \Sigma \, {}_s\hat{T}$$

$${}_s\hat{T} = {}_s\tau_{\mu\nu}\left[T^{\mu\nu}\right]$$

⇧

2nd order diff operator

This may be solved by separation of variables

$${}_s\psi = \sum_{\Lambda=(\omega,l,m)} {}_sR_\Lambda(r)\, {}_sZ_\Lambda(\theta,\varphi) e^{-i\omega t}$$

⮤ spin-weighted spheroidal harmonics

$$ {}_s\psi = \sum_{l,m} \int_{-\infty}^{+\infty} d\omega \; {}_sR_{lm\omega}(r) \; \underbrace{{}_sS_{lm}^{a\omega}(\theta) e^{im\varphi}}_{{}_sZ_{lm\omega}(\theta,\varphi)} e^{-i\omega t} $$

$$ \left[\frac{1}{\sin\theta} \partial_\theta \left(\sin\theta \, \partial_\theta \right) + \cdots \right] {}_sS_{lm}^{a\omega}(\theta) = -\lambda \; {}_sS_{lm}^{a\omega}(\theta) $$

… eigenvalue eq.

$\lambda = (l - s)(l + s + 1)$ in the limit $a\omega \to 0$

$$ \left[\Delta^{-s} \partial_r \left(\Delta^{s+1} \partial_r \right) + \cdots - \lambda \right] {}_sR_\Lambda(r) = {}_sT_\Lambda $$

… radial Teukolsky eq.

$$ {}_sT_\Lambda = 4\pi \int \sqrt{-g} \, dt \, d\Omega \; {}_s\bar{Z}_\Lambda \; {}_s\hat{T} $$

source term

$$ {}_s\hat{T} = {}_s\tau_{\mu\nu}\left[T^{\mu\nu}\right] \qquad \partial_t = i\omega,\ \ \partial_\varphi = i\,m $$

$$ {}_2\tau_{\mu\nu} = \frac{1}{\bar{z}^4 z}\left[\frac{1}{\sqrt{2}}\left(\mathcal{L}^\dagger_{-1}\frac{\bar{z}^4}{z^2}\mathcal{D}_0 + \mathcal{D}_0\frac{\bar{z}^4}{z^2}\mathcal{L}^\dagger_{-1}\right) z^2\left(l_\mu m_\nu + m_\mu l_\nu\right)\right. $$
$$ \left. - \mathcal{L}^\dagger_{-1}\bar{z}^4\mathcal{L}^\dagger_0\frac{1}{z}\,l_\mu l_\nu - 2\mathcal{D}_0\bar{z}^4\mathcal{D}_0\, z\, m_\mu m_\nu\right] $$

$$ {}_{-2}\tau_{\mu\nu} = -\frac{1}{\bar{z}^4 z}\left[\frac{\Delta}{2\sqrt{2}}\left(\mathcal{L}_{-1}\frac{\bar{z}^4}{z^2}\mathcal{D}^\dagger_{-1} + \mathcal{D}^\dagger_{-1}\frac{\bar{z}^4}{z^2}\mathcal{L}_{-1}\right)\Sigma^2\left(n_\mu\bar{m}_\nu + \bar{m}_\mu n_\nu\right)\right. $$
$$ \left. + \mathcal{L}_{-1}\bar{z}^4\mathcal{L}_0\bar{z}\,\Sigma\, n_\mu n_\nu + \frac{\Delta^2}{2}\mathcal{D}^\dagger_0\bar{z}^4\mathcal{D}^\dagger_0\frac{\bar{z}^2}{z}\bar{m}_\mu\bar{m}_\nu\right] $$

$$ \mathcal{L}_{sm\omega} \equiv \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + s\cot\theta $$

$$ \mathcal{D}_{sm\omega} \equiv \partial_r + \frac{i\omega\left(r^2+a^2\right) - i\,m\,a}{\sin\theta} + \frac{2s\left(r-M\right)}{\Delta} $$

† operation:

$m \rightarrow -m$

$\omega \rightarrow -\omega$

Teukolsky-Starobinsky identity

Relation between ${}_{2}R$ & ${}_{-2}R$ (Ψ_0 & Ψ_4)

$$\begin{cases} \Delta^2(\mathcal{D}^\dagger)^4[\Delta^2\,{}_2R] = \bar{C}\;{}_{-2}R \\ (\mathcal{D})^4[{}_{-2}R] = C\;{}_2R \end{cases}$$

$$\mathcal{D} = \partial_r - i\frac{K}{\Delta}\,, \quad \mathcal{D}^\dagger = \partial_r + i\frac{K}{\Delta}$$

$$K = (r^2+a^2)\omega - am$$

$$\Delta = r^2 - 2Mr + a^2$$

$$\Rightarrow \begin{cases} (\mathcal{D})^4\Delta^2(\mathcal{D}^\dagger)^4[\Delta^2\,{}_2R] = |C|^2\;{}_2R \\ \Delta^2(\mathcal{D}^\dagger)^4\Delta^2(\mathcal{D})^4[{}_{-2}R] = |C|^2\;{}_{-2}R \end{cases}$$

$$|C|^2 = (Q_2{}^2 + 4a\omega m - 4a^2\omega^2)\left[(Q_2-2)^2 + 36a\omega m - 36a^2\omega^2\right]$$
$$+(2Q_2-1)(96a^2\omega^2 - 48a\omega m) + 144\omega^2(M^2-a^2)$$

$$\mathrm{Im}[C] = 12M\omega \qquad Q_s = \lambda_s + s(s+1) = Q_{-s}$$

$a \to 0$ limit: $C(a=0) = \lambda(\lambda+2) + 12iM\omega = \bar{C}_0$

Chandrasekhar transformation (for a=0)

Chandrasekhar ('83)

- For the Schwarzschild case, we have two independent formalisms: RWZ formalism & Teukolsky formalism

How are they related?

Radial s=-2 Teukolsky eq.

$$\left[\Lambda_+\Lambda_- + A(r)\Lambda_- - B(r)\right]Y = 0; \quad Y = r^3{}_{-2}R(r)$$

$$\Lambda_\pm = \frac{d}{dr*} \pm i\omega$$

$$A(r) = 2\frac{d}{dr*}\ln\frac{r^4}{\Delta} = \frac{4(r-3M)}{r^2}$$

$$\Delta = r(r-2M)$$

$$B(r) = \frac{\Delta}{r^5}(\lambda r + 6M)$$

$$\lambda = (l-1)(l+2)$$

Setting $Y = \frac{\Delta}{r^3}\Lambda_+\left[\frac{r^2}{\Delta}\Lambda_+(rX)\right]$, one finds X satisfies RW eq.:

$$\left[\Lambda_+\Lambda_- - V(r)\right]X = 0$$

Thus

$${}_{-2}R = r^3 Y = \Delta\Lambda_+\left[\frac{r^2}{\Delta}\Lambda_+(rX)\right] = \frac{\Delta^2}{r^2}J_+J_+[rX]$$

$$\Rightarrow \quad {}_{-2}R = \frac{\Delta^2}{r^2}J_+J_+[rX]$$

$$J_\pm = \frac{d}{dr} \pm i\omega\frac{r^2}{\Delta}$$

$$J_\pm = \begin{cases} \mathcal{D}^\dagger(a=0) \\ \mathcal{D}(a=0) \end{cases}$$

Inverse transformation:

$$C_0 X = \frac{r^5}{\Delta}\Lambda_-\left[\frac{r^2}{\Delta}\Lambda_-\left(\frac{{}_{-2}R}{r^2}\right)\right] = r^3 J_- J_-\left[\frac{{}_{-2}R}{r^2}\right]$$

$$C_0 = \lambda(\lambda+2) - 12iM\omega$$

No such transformation is known for Kerr case

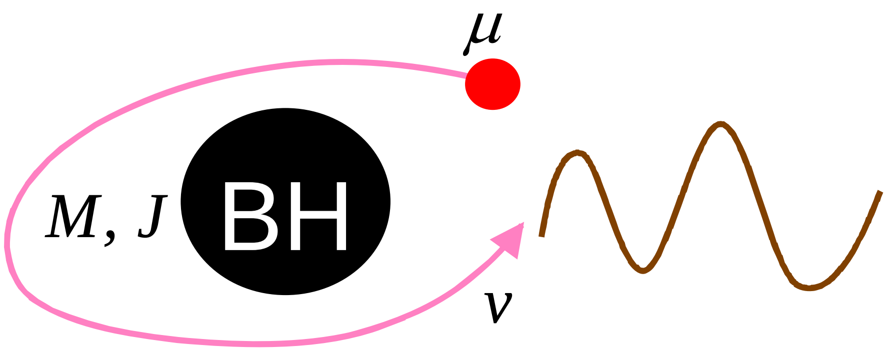
Gravitational radiation from EMRI

- $G^{\mu\nu}[\boldsymbol{g}] = 8\pi G\, T^{\mu\nu}$

$$g_{\mu\nu} = g^{BH}_{\mu\nu} + h_{\mu\nu}$$

✧ $M >> \mu$

✧ v/c can be large

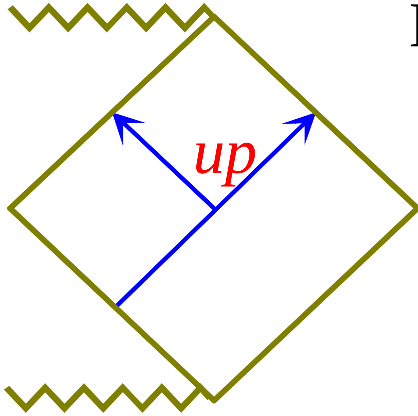


$$\Rightarrow \quad \delta G^{\mu\nu}[h] = 8\pi G\, T^{\mu\nu}$$

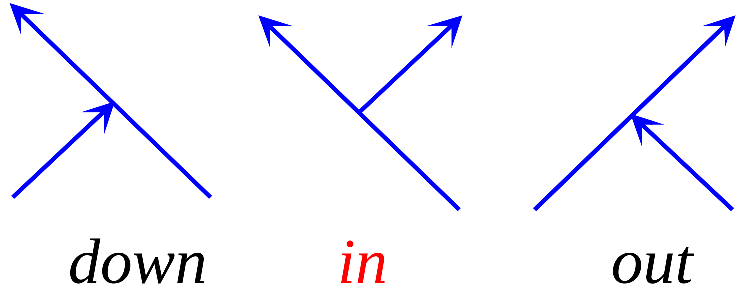
compact star ($\mu \sim M_{\odot}$) can be approximated by a point particle:

$$T^{\mu\nu} = \mu \int d\tau \frac{dz^{\mu}}{d\tau}\frac{dz^{\nu}}{d\tau}\frac{\delta^{(4)}\left(x - z(\tau)\right)}{\sqrt{-g}}$$

Green function method



Boundary conditions for homogeneous modes



$$ {}_sR_\Lambda(r) \approx \frac{1}{{}_sW_\Lambda} R^{up}_\Lambda(r) \int dr' \, R^{in}_\Lambda(r') \, {}_sT_\Lambda(x') + \cdots $$

Wronskian

$$ {}_sW_\Lambda \approx {}_sR^{up}_\Lambda \; \overleftrightarrow{\partial}_r \; {}_sR^{in}_\Lambda $$

Teukolsky eq. is not real ⇨ no sym. under complex conjugation:

$$ {}_sR^{in}_\Lambda \neq {}_s\bar{R}^{out}_\Lambda $$

But it is symmetric under c.c.+($s \to -s$): $\;{}_sR^{in}_\Lambda = {}_{-s}\bar{R}^{out}_\Lambda$

At $r \to \infty$ $\quad \dfrac{{}_{-2}\psi}{r^4} = \Psi_4 \sim \dfrac{1}{2}\left(\ddot{h}_+ - i\,\ddot{h}_\times\right)$ ➡ $\left(\dfrac{dE}{dt}\right)^\infty_\Lambda = \dfrac{1}{4\pi\omega^2}\left|\dfrac{{}_{-2}R_\Lambda}{r^3}\right|^2$

Mano-Takasugi-Suzuki formalism

(Mano et al.(1996)&(1997))

$$ {}_s\psi \approx \sum_{\Lambda} \frac{1}{{}_sW_\Lambda} {}_s\Omega^{up}_\Lambda(x) \int \sqrt{-g}\, d^4x' \, {}_s\bar{\tilde{\Omega}}^{in}_\Lambda(x') \, {}_sT(x') + \cdots $$

$$ \Omega^{up}_\Lambda = R^{up}_\Lambda(r) \, {}_sZ_\Lambda(\theta,\varphi) e^{-i\omega t} \qquad \bar{\tilde{\Omega}}^{in}_\Lambda = R^{in}_\Lambda(r) \, {}_s\bar{Z}_\Lambda(\theta,\varphi) e^{i\omega t} $$

- ${}_sR^{in}_\Lambda(r)$ at small *r* and ${}_sR^{up}_\Lambda(r)$ at large *r* are not difficult to obtain by series expansion.

- To evaluate the Wronskian ${}_sW_\Lambda \approx {}_sR^{up}_\Lambda \overleftrightarrow{\partial}_r \, {}_sR^{in}_\Lambda$ we need to evaluate ${}_sR^{in}_\Lambda(r)$ at large *r*.
  (or ${}_sR^{up}_\Lambda(r)$ at small *r*)

Need to construct a solution valid in the whole range of *r*

MTS method

- Expansion near horizon (for simplicity, set a = 0)

  series in terms of hypergeometric functions

$$R^{in} \sim \zeta_{in}^{\nu}(x) \approx \sum_{n=-\infty}^{\infty} a_n^{\nu} p_{n+\nu}(x), \qquad x = -\left(\frac{r}{2M} - 1\right)$$

$$p_{n+\nu}(x) \approx F(n+\nu-1-i\varepsilon, -n-\nu-2-i\varepsilon, 1-2i\varepsilon; x)$$

ν: eigenvalue (to be determined) $\quad \varepsilon = 2M\omega$

(Newton limit: $\varepsilon \to 0$, $\nu \to l$ or $-(l+1)$)

Teukolsky equation leads to 3-term recursion relation

$$\alpha_n^{\nu} a_{n+1}^{\nu} + \beta_n^{\nu} a_n^{\nu} + \gamma_n^{\nu} a_{n-1}^{\nu} = 0$$

$$\frac{a_n^{\nu}}{a_{n-1}^{\nu}} \xrightarrow[n\to+\infty]{} 0 \qquad \frac{a_n^{\nu}}{a_{n+1}^{\nu}} \xrightarrow[n\to-\infty]{} 0$$

convergence condition determines ν

Analytic continuation of $F(\alpha, \beta, \gamma; x)$ to large x

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;x) = \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)}(1-x)^{-\alpha} F\left(\alpha,\gamma-\beta,\alpha-\beta+1;\frac{1}{1-x}\right) + \frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)}(1-x)^{-\beta} F\left(\beta,\gamma-\alpha,\beta-\alpha+1;\frac{1}{1-x}\right)$$

One finds $\zeta_{in}^{\nu} = \zeta_0^{\nu} + \zeta_0^{-\nu-1}$ with

$$\zeta_0^{\nu} \approx (1-x)^{\nu+1} \sum_{n=-\infty}^{\infty} a_n^{\nu} (1-x)^n F\left(*,*;*;\frac{1}{1-x}\right)$$

- Expansion around $r = \infty$

$$\zeta_C^{\nu} \approx z^{\nu} \sum_{n=-\infty}^{\infty} b_n^{\nu} e^{-iz} z^n \, {}_2F_0(*,*;2iz) \qquad z = \varepsilon(1-x) = \omega r$$

↑ confluent hypergeometric fcn. (Coulomb wave fcn.)

MTS find exactly the same 3-term recursion relation as the one near horizon

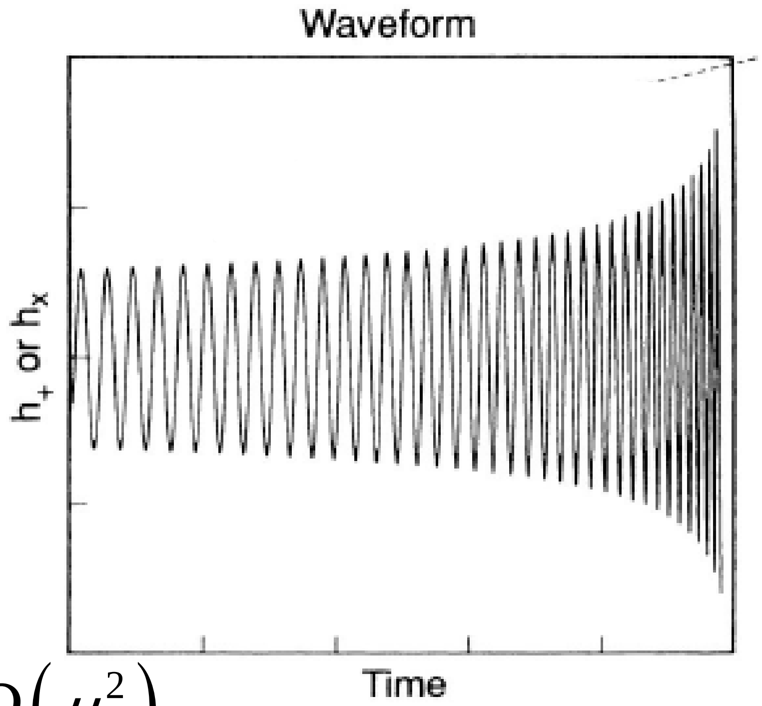
⇨ matching at $z<\infty$ & $(-x)>>1$ determines ζ_C^{ν} ⇨ $\zeta_C^{\nu} \propto \zeta_0^{\nu}$

(matching region: $2M << r < \infty$)

Leading order waveform

Energy balance argument is sufficient: $$\frac{dE_{GW}}{dt} = \frac{dE_{orbit}}{dt}$$

For example, wave form
for quasi-circular orbits is
determined by rate of change
of GW frequency *f*:



$$\frac{df}{dt} = \frac{dE_{orbit}}{dt} \bigg/ \frac{dE_{orbit}}{df}$$

$$\frac{dE_{orbit}}{dt} = 0 + O(\mu) + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

Energy loss rate for circular orbit on the equatorial plane in Kerr background to 4PN order

$$\left\langle \frac{dE}{dt} \right\rangle = \frac{32}{5}\left(\frac{\mu}{M}\right) v'^{10}\left[1 - \frac{1247 v'^2}{336} + \left(4\pi - \frac{11q}{4}\right) v'^3 + \left(-\frac{44711}{9072} + \frac{33q^2}{16}\right) v'^4 + \left(\frac{-8191\pi}{672} - \frac{59q}{16}\right) v'^5 \right.$$

$$+ \left(\frac{6643739519}{69854400} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} - \frac{65\pi q}{6} + \frac{611q^2}{504} - \frac{3424\ \ln 2}{105} - \frac{1712\ \ln v'}{105}\right) v'^6$$

$$+ \left(\frac{-16285\pi}{504} + \frac{162035q}{3888} + \frac{65\pi q^2}{8} - \frac{71q^3}{24}\right) v'^7 + \left(-\frac{323105549467}{3178375200} + \frac{232597\gamma}{4410} - \frac{1369\pi^2}{126} - \frac{359\pi q}{14}\right.$$

$$\left.\left. + \frac{22667q^2}{4536} + \frac{17q^4}{16} + \frac{39931\ \ln 2}{294} - \frac{47385\ \ln 3}{1568} + \frac{232597\ \ln v'}{4410}\right) v'^8 \right].$$

$$q = a / M \qquad\qquad v' = \left(M\underline{\Omega}\right)^{1/3}$$

Orbital freq. as measured by observer at infinity