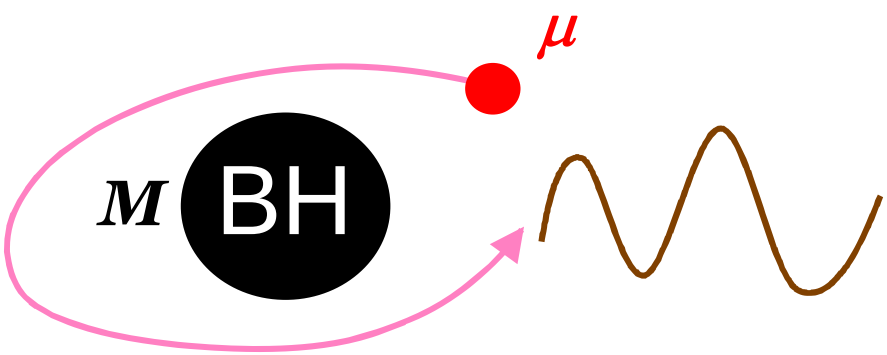
4. MiSaTaQuWa force for radiation reaction

$$G^{\mu\nu}[g] = 8\pi G T^{\mu\nu}$$

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

✧ $M >> \mu$

✧ v/c can be large



Energy-momentum of a point particle

$$T^{\mu\nu}(x) = \mu \int d\tau \, \dot{z}^{\mu} \dot{z}^{\nu} \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} \qquad \left(\dot{z}^{\mu} = \frac{dz^{\mu}}{d\tau} \right)$$

Linear perturbation in μ

$$\delta G^{\mu\nu}\left[\boldsymbol{h}^{(1)}\right] = 8\pi G \boldsymbol{T}^{(1)\mu\nu}$$

$$\boldsymbol{T}^{(1)\mu\nu}(x) = \mu \int d\tau \, \dot{z}^{\mu} \dot{z}^{\nu} \frac{\delta^4(x - z(\tau))}{\sqrt{-g^{(0)}}} \qquad \left(\dot{z}^{\mu} = \frac{dz^{\mu}}{d\tau}\right)$$

geodesic on $g^{(0)}$

background metric

Master variable ζ:

$$\zeta = h^{(1)}_{\mu\nu} \quad \text{or} \quad {}_s\psi^{(1)} \qquad ({}_s\psi \sim \text{ a component of Weyl tensor})$$

$$\zeta = \sum_{lm} \phi_{lm}(t, r) Y_{lm}(\Omega)$$

: expanded in spherical (spheroidal) harmonics

$$L[\zeta] = S\left[\boldsymbol{T}^{(1)}\right]$$

Regge-Wheeler-Zerilli/Teukolsky eq.

From ζ, we can calculate:

- Waveform at infinity.
- $dE/dt|_{GW}$, $dL_z/dt|_{GW}$, etc. $\sim O\left((G\mu)^2\right)$

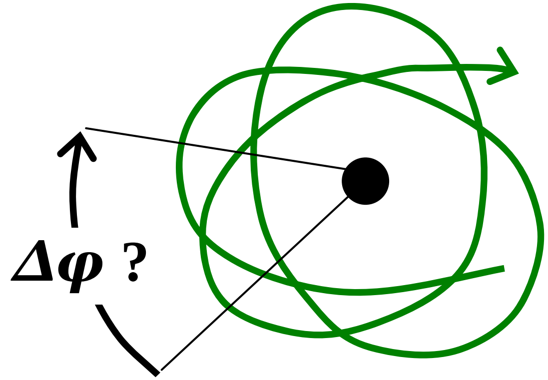
⇒ the orbit deviates from a geodesic on $g^{(0)}$

How can we incorporate this deviation?

- Use dE/dt & dL_z/dt to determine the evolution of the orbital parameters (adiabatic approximation).

But, this cannot predict the phase shift in orbit

It cannot deal with non-adiabatic case.



- Evaluate self-force from $h_{\mu\nu}$ acting on the particle.

Self-force problem

For point particle,

$$\delta G^{\mu\nu}[\boldsymbol{h}] = 8\pi G \boldsymbol{T}^{\mu\nu} \Longrightarrow \boldsymbol{h}_{\mu\nu} \propto \frac{1}{|\boldsymbol{x} - \boldsymbol{z}(\boldsymbol{\tau})|}$$

$$\boldsymbol{h}_{\mu\nu}(x) \ \textbf{diverges at} \ \boldsymbol{x}^{\alpha} = \boldsymbol{z}^{\alpha}(\boldsymbol{\tau})$$

- self-force (back-reaction) in a curved background:

$$\underbrace{\mu \frac{D^2 z^\alpha}{d\tau^2} = F^\alpha[h] \approx \mu\,\delta\Gamma^\alpha_{\mu\nu}[h]\dot{z}^\mu \dot{z}^\nu}_{\sim \text{ geodesic eq. on } \boldsymbol{g}^{(0)} + \boldsymbol{h}} = \mu \frac{1}{2}\left(h^\alpha{}_{\mu;\nu}(x) + \ldots\ldots\right)\dot{z}^\mu \dot{z}^\nu$$

~ geodesic eq. on $\boldsymbol{g}^{(0)} + \boldsymbol{h}$

singular !

● Breakdown of perturbation theory ?

Yes! & No!

- Yes, because a point particle is ill-defined in GR.

↔ Mass is non-renormalizable in GR

$$\lim_{r_0 \to 0}\left(m_{\text{bare}} - \frac{G\, m_{\text{bare}}^2}{r_0} \right)$$ has no well-defined limit.

- No, because $\exists$ regular exact solution (BH) in GR.

↔ Mass renormalization is unnecessary

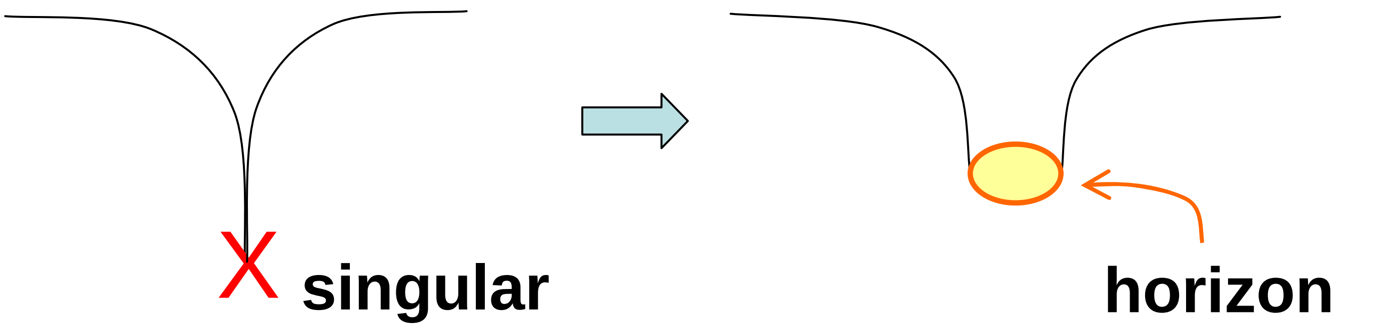
cf. EM theory:

point particle exists $\Longleftrightarrow$ mass is renormalizable

$$m_{\text{phys}} = \lim_{r_0 \to 0}\left(m_{\text{bare}} + \frac{e^2}{r_0} \right)$$: two parameters to tune the limit

Namely, in GR:

- Identify the point particle with a BH solution of mass μ



- Embed the BH geometry in the linearly perturbed

  metric $\boldsymbol{g}_{\mu\nu} = \boldsymbol{g}^{(0)}_{\mu\nu} + \boldsymbol{h}_{\mu\nu}$ **:** matching at $|x\text{-}z(\tau)| >> G\mu$

Matched Asymptotic Expansion

Thorne & Hartle ('84)

Simplest example

background geodesic eq.

Consider a point particle in the flat background

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu}$$

$$h_{\mu\nu}(x) = \eta_{\mu\alpha}\eta_{\nu\beta}\frac{2G\mu\,(2\dot{z}^{\alpha}\dot{z}^{\beta} + \eta^{\alpha\beta})}{\dot{z}^{0}\,|\vec{x} - \vec{z}(\tau_{\mathrm{ret}})|}\,; \quad \ddot{z}^{\alpha}(\tau) = 0$$

In the rest frame $\{X^a\}$ of the particle:

$$h_{ab}(X) = \eta_{ac}\eta_{bd}\frac{2G\mu\,(2\dot{Z}^{c}\dot{Z}^{d} + \eta^{cd})}{|\vec{X}|}\,; \quad \dot{Z}^{a} = (1,0,0,0)$$

This is just the Newtonian part of the Schwarzschild metric.

Thus a Schwarzschild BH of mass μ can be naturally matched to $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$ at $|X| >> G\mu$

⇨ EOM unchanged. No self-force correction to all orders in $G\mu$

In General Curved Background:

- Hadamard decomposition of $G_{(ret)}$ in harmonic (Lorenz) gauge

$$G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}(x,z) = \theta(x^0 - z^0)\left[u^{\mu\nu}{}_{\alpha\beta}\ \delta\big(\sigma(x,z)\big) - v^{\mu\nu}{}_{\alpha\beta}\ \theta\big(-\sigma(x,z)\big)\right]$$

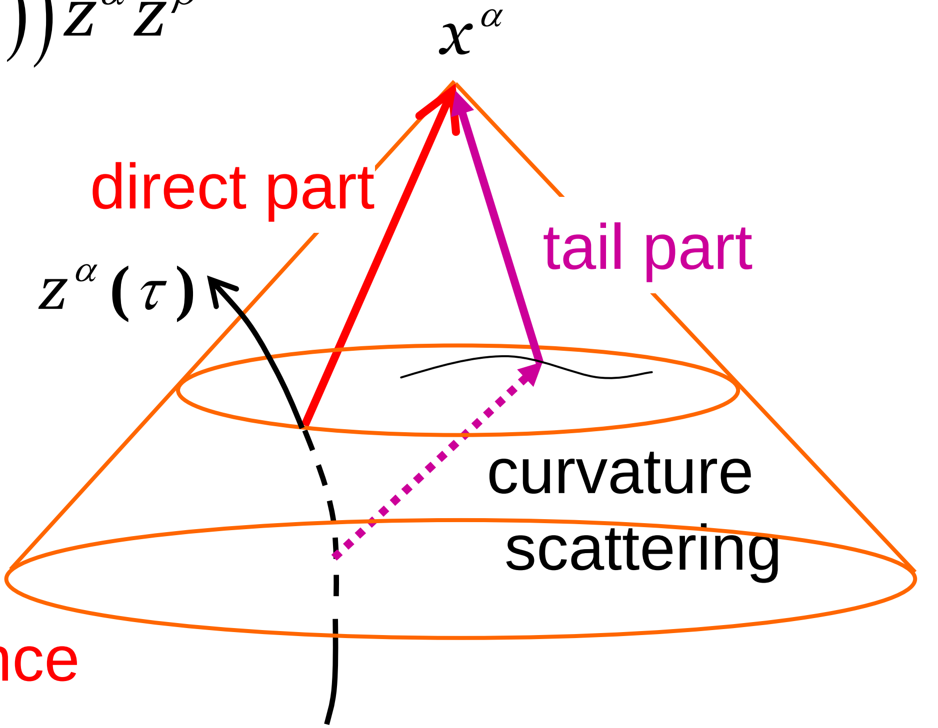
$\sigma(x,z)$: world interval between x and z $\left(\sim \frac{1}{2}(x\text{-}z)^2\right)$

$$h^{\mu\nu}_{(\mathrm{ret})}(x) = \mu \int d\tau\ G^{\mu\nu}_{(\mathrm{ret})\alpha\beta}\big(x, z(\tau)\big)\dot{z}^\alpha \dot{z}^\beta$$

$u^{\mu\nu}{}_{\alpha\beta}$: direct part

$v^{\mu\nu}{}_{\alpha\beta}$: tail part

$$h^{\mu\nu}_{(\mathrm{ret})}(x) = h^{\mu\nu}_{(\mathrm{direct})} + h^{\mu\nu}_{(\mathrm{tail})}$$

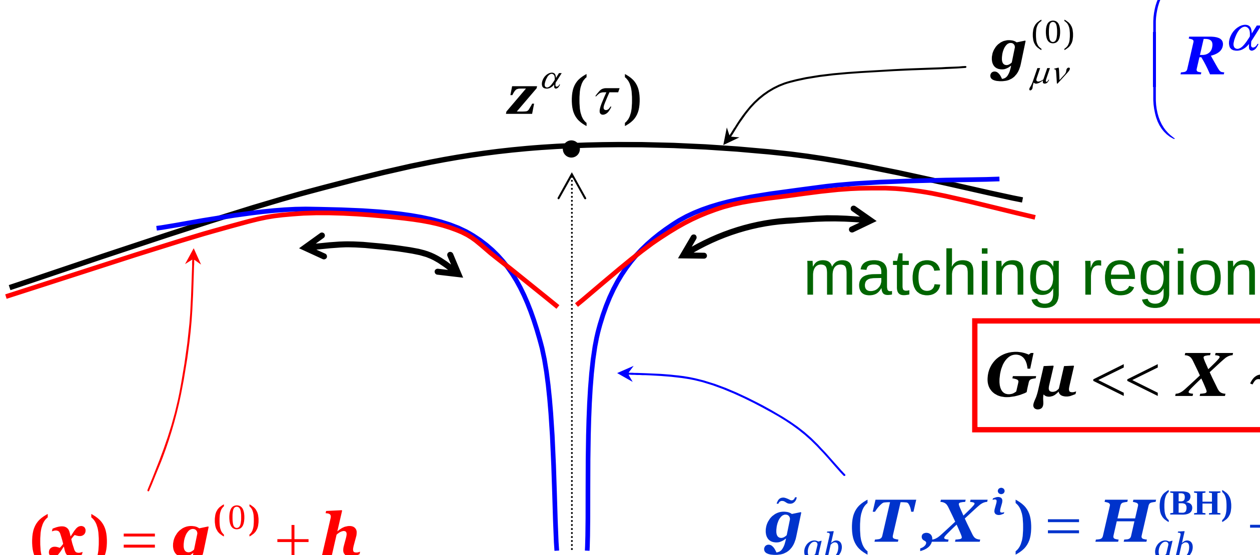


$h^{\mu\nu}_{(\mathrm{direct})}$ contains divergence

$v^{\mu\nu}{}_{\alpha\beta}$ is a solution of source-free eq. but not $h^{\mu\nu}_{(\mathrm{tail})}$

Matched asymptotic expansion

$$\left(R^{\alpha}{}_{\beta\gamma\delta}[g^{(0)}] \sim \frac{1}{L^2} \right)$$



$$G\mu << X \sim \sqrt{G\mu\, L} << L$$

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

(external: valid at $|X| >> G\mu$)

$$\tilde{g}_{ab}(T, X^i) = H^{(BH)}_{ab} + \delta H_{ab}$$

(internal: valid at $|X| << L$)

- coordinate transformation: $g_{ab}(X) = \dfrac{\partial x^{\mu}}{\partial X^{a}} \dfrac{\partial x^{\nu}}{\partial X^{b}} g_{\mu\nu}(x)$

$$\sigma^{;\mu}(x, z(\tau)) \left(\approx -(x^{\mu} - z^{\mu}) \right) = -\left(f^{\mu}_{\ i}(T) X^{i} + f^{\mu}_{\ ij}(T) X^{i} X^{j} + \ldots \right)$$

$\sigma^{;\mu}(x, z(\tau))\, \bar{g}_{\mu\alpha}(x, z) \dot{z}^{\alpha} = 0$; $\bar{g}_{\mu\alpha}$: parallel transport bi-tensor

- identify g_{ab} with $\tilde{g}_{ab}$ in the matching region.

external scheme

$$g_{ab} = g^{(0)}_{ab} + h_{ab}$$

- background Riemann ~ $1/L^2$
- perturbation in $G\mu$

$$g^{(0)}_{ab} = \eta_{ab} + \frac{1}{L}\,{}^{(1)}_{(0)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(0)}h_{ab} + \dots$$

$$h_{ab} = G\mu\left({}^{(0)}_{(1)}h_{ab} + \frac{1}{L}\,{}^{(1)}_{(1)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(2)}h_{ab} + \dots\right)$$

$$+(G\mu)^2\left({}^{(0)}_{(2)}h_{ab} + \frac{1}{L}\,{}^{(1)}_{(2)}h_{ab} + \frac{1}{L^2}\,{}^{(2)}_{(2)}h_{ab} + \dots\right)$$

internal scheme

$$\tilde{g}_{ab} = H^{(BH)}_{ab} + \delta H_{ab}$$

- background Riemann ~ $G\mu / |X|^3$
- perturbation in $1/L$

$$H^{(BH)}_{ab} = \eta_{ab} + G\mu\,{}^{(0)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(0)}_{(2)}H_{ab} + \dots$$

$$\delta H_{ab} = \frac{1}{L}\left({}^{(1)}_{(0)}H_{ab} + G\mu\,{}^{(1)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(1)}_{(2)}H_{ab} + \dots\right)$$

$$+\frac{1}{L^2}\left({}^{(2)}_{(0)}H_{ab} + G\mu\,{}^{(2)}_{(1)}H_{ab} + (G\mu)^2\,{}^{(2)}_{(2)}H_{ab} + \dots\right)$$

matching condition:

$${}^{(n)}_{(m)}H_{ab} = {}^{(n)}_{(m)}h_{ab} + O\left((G\mu)^{(m+1)} / L^{(n+1)}\right)$$

$${}^{(n)}_{(m)}H_{ab} \sim \frac{(G\mu)^{m}}{L^{n}} X^{(n\text{-}m)}$$

Asymptotic matching to $O(G\mu)$

flat part
→background EOM

regular (tail) part
→ EOM with self-force

m \ n	1	$1/L$	$1/L^2$
1	η_{ab}	0	$\dfrac{\|X\|^2}{L^2}$
$G\mu$	$\dfrac{G\mu}{\|X\|}$	0	$\dfrac{G\mu\,\|X\|}{L^2}$

external background

linear perturbation

BH (=point particle)

singular (direct) part
→no effect on EOM

'Center of mass' gauge condition

quadrupolar deformation

Regularized Gravitational Self-force

'MiSaTaQuWa' force: (named by Eric Poisson)

$$F^{\alpha}[h_{(\mathrm{tail})}(x)] \approx \frac{1}{2}(h^{\alpha}{}_{(\mathrm{tail})\,\mu;\nu}(x) + \ldots\ )\,\dot{z}^{\mu}\dot{z}^{\nu}$$

Mino, Sasaki and Tanaka ('97), Quinn and Wald ('99)

Tail part of the metric perturbation

$$h^{\mu\nu}_{(\mathbf{tail})}(x) \approx \int_{-\infty}^{\tau(x)} d\tau'\ v^{\mu\nu}{}_{\alpha\beta}\left(x, z(\tau')\right) T^{\alpha\beta}\left(z(\tau')\right)$$

Regularized self-force is determined by the tail part

E.O.M. with self-force = geodesic on $g^{\mu\nu} + h^{\mu\nu}_{(\mathrm{tail})}$

But $h^{\mu\nu}_{(\mathrm{tail})}(x)$ is NOT a solution of Einstein equations.

⇨ meaning of the metric $g^{\mu\nu} + h^{\mu\nu}_{(\mathrm{tail})}$ was unclear

Detweiler - Whiting's S-R decomposition

(improved over "direct-tail" decomposition) **PRD 67, 024025 (2003)**

$$G^{ret}(x,z) = 2\theta\left(x^0 - z^0\right)\, G^{sym}(x,z)$$

$$G^{sym}(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) - v(x,z)\theta(-\sigma)\right]$$

$$G^{\mathrm{S}}(x,z) = G^{sym}(x,z) + \frac{1}{8\pi}v(x,z) = \frac{1}{8\pi}\left[u(x,z)\delta(\sigma) + v(x,z)\theta(\sigma)\right]$$

$$h^{\mathrm{S}}(x) = \int d^4x'\sqrt{-g}\; G^{\mathrm{S}}(x,x')T(x')$$: satisfies pert eqs.

$$G^{\mathrm{R}}(x,z) = G^{ret}(x,z) - G^{\mathrm{S}}(x,z) = \left(G^{ret}(x,z) - G^{adv}(x,z)\right) - \frac{1}{8\pi}v(x,z)$$

$$h^{\mathrm{R}}(x) = h^{ret}(x) - h^{\mathrm{S}}(x)$$: satisfies source-free pert eqs.

$$h^{\mathrm{R}} - h^{\mathrm{tail}} = O\left((x-z)^2\right)$$ ⇒ Both give the same force

EOM = geodesic on $g^{(0)}_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$ → solution of (linearized) vacuum Einstein eqs.