IHP Trimester on Gravity

Equations of Motion in the ADM Formalism
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ADM Hamiltonian and Routh Functional
- field equations
- coordinate conditions
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- Brill-Lindquist vs Misner-Lindquist BHs
- Hamiltonian, exact to order ρ^2 (Thiesfelder/Faye/S;

3 PN Binary BH Conservative Dynamics
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PN Approach to the Radiation Reaction

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References: see Trimester Homepage

ADM Hamiltonian and Routh Functional

$$g^{1/2}\mathrm{R} = \frac{1}{g^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + g^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$$\delta_a \equiv \delta(\vec{x}-\vec{x}_a)$$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i g_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

3 Cōrdinate Conditions: $g_{ij} = \left(1+\frac{1}{8}\phi\right)^4\delta_{ij} + h^{\mathrm{TT}}_{ij}$

1 CC: $\pi^{ii} = 0,$ $\pi^{ij} = -g^{1/2}(K^{ij} - g^{ij}K),$ $\pi^i_i = \pi^{ij}h^{\mathrm{TT}}_{ij}$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi^{ij}_{\mathrm{TT}} c^3/16\pi G$: canonical conjugate to h^{TT}_{ij}

ADM Hamiltonian

$$H\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right] = -\frac{c^4}{16\pi G}\int d^3x\ \Delta\phi\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \pi^{ij}_{\mathrm{TT}}\right]$$

Routh functional

$$R\left[x^i_a, p_{ai}, h^{\mathrm{TT}}_{ij}, \partial_t h^{\mathrm{TT}}_{ij}\right] = H - \frac{c^3}{16\pi G}\int d^3x\ \pi^{ij}_{\mathrm{TT}}\partial_t h^{\mathrm{TT}}_{ij}$$

$$\frac{\delta \int R(t')dt'}{\delta h^{\mathrm{TT}}_{ij}(x^k, t)} = 0, \quad \dot{p}_{ai} = -\frac{\partial R}{\partial x^i_a}, \quad \dot{x}^i_a = \frac{\partial R}{\partial p_{ai}}$$

conservative Routh functional:

$$R_c(t) = \frac{1}{2}[R_{on}(t) + R_{on}(-t)]$$

on-field-shell Routh functional:

$$R_{on}(t) = R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}], \partial_t h_{ij}^{\mathrm{TT}}[x_a^k, p_{ak}]\right]$$

$$\dot{p}_{ai}(t) = -\frac{\delta \int R_c(t')dt'}{\delta x_a^i(t)}, \qquad \dot{x}_a^i(t) = \frac{\delta \int R_c(t')dt'}{\delta p_{ai}(t)}$$

$$\frac{\delta \int R_c(t')dt'}{\delta z(t)} = \frac{\partial R_c}{\partial z(t)} - \frac{d}{dt}\frac{\partial R_c}{\partial \dot{z}(t)} + \ldots, \quad z = (x_a^i, p_{ai})$$

conservative ordinary Hamiltonian $\bar{H}_{co}(\bar{x}^i_a, \bar{p}_{ai})$:

$$\dot{\bar{p}}_{ai} = -\frac{\partial \bar{H}_{co}}{\partial \bar{x}^i_a}, \qquad \dot{\bar{x}}^i_a = \frac{\partial \bar{H}_{co}}{\partial \bar{p}_{ai}}$$

$$\bar{H}_{co}(x,p) \equiv R_c(x,p,\dot{x}(x,p),\dot{p}(x,p)), \quad L_c = p_{ai}\dot{x}^i_a - R_c$$

$$\begin{aligned} L_c(x,p,\dot{x},\dot{p}) &= \bar{L}_{co}(x,p) + \frac{\delta L_c}{\delta x^i_a}\frac{\partial R_c}{\partial \dot{p}_{ai}} - \frac{\delta L_c}{\delta p_{ai}}\frac{\partial R_c}{\partial \dot{x}^i_a} + \text{Double Zero} \\ &= \bar{L}_{co}(\bar{x},\bar{p}) + DZ, \quad \bar{x}^i_a = x^i_a + \delta x^i_a, \ \bar{p}_{ai} = p_{ai} + \delta p_{ai} \end{aligned}$$

$$\delta x^i_a = \frac{\partial R_c}{\partial \dot{p}_{ai}}, \quad \delta p_{ai} = -\frac{\partial R_c}{\partial \dot{x}^i_a}$$

Radiation field:

$$h_{ij}^{\mathrm{TT}}(\mathbf{x},t) \;=\; \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\left\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{[l]}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}\right.$$

$$\left. +\;\left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\;\epsilon_{pq(k}\;\mathrm{S}^{[l]}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\right\}$$

Multipole moment with tail:

$$\mathrm{M}_{ij}(t-\frac{r_*}{c}) \;=\; \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right)$$

$$+\;\frac{2Gm}{c^3}\int_0^{\infty}dv\;\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{[2]}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4)$$

$$r_* = r + \frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right) + O(1/c^3)$$

Luminosity:

$$\mathcal{L} = \frac{G}{5c^5} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n \hat{\mathcal{L}}_n$$

2PN energy loss:

$$- < \frac{d\mathcal{E}(t - r_*/c)}{dt} > \ = \ < \mathcal{L}(t) >$$

$$\begin{aligned} \mathcal{L} \ &= \ \frac{G}{5c^5} \left\{ \mathrm{M}_{ij}^{[3]} \mathrm{M}_{ij}^{[3]} + \frac{1}{c^2} \left[\frac{5}{189} \mathrm{M}_{ijk}^{[4]} \mathrm{M}_{ijk}^{[4]} + \frac{16}{9} \mathrm{S}_{ij}^{[3]} \mathrm{S}_{ij}^{[3]} \right] \right. \\ &+ \ \left. \frac{1}{c^4} \left[\frac{5}{9072} \mathrm{M}_{ijkm}^{[5]} \mathrm{M}_{ijkm}^{[5]} + \frac{5}{84} \mathrm{S}_{ijk}^{[4]} \mathrm{S}_{ijk}^{[4]} \right] \right\} \end{aligned}$$

PN-Expansions

$$R\left[x_a^i, p_{ai}, h_{ij}^{\mathrm{TT}}, \partial_t h_{ij}^{\mathrm{TT}}\right] - Mc^2 = \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n R_n \left[x_a^i, p_{ai}, \hat{h}_{ij}^{\mathrm{TT}}, \partial_t \hat{h}_{ij}^{\mathrm{TT}}\right]$$

$$h_{ij}^{\mathrm{TT}} = \tfrac{G}{c^4} \hat{h}_{ij}^{\mathrm{TT}}$$

$$\left(\Delta - \frac{\partial_t^2}{c^2}\right) h^{\mathrm{TT}} = \frac{G}{c^4} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n D_n^{\mathrm{TT}}[x, x_a(t), p_a(t), \hat{h}^{\mathrm{TT}}(t), \partial_t \hat{h}^{\mathrm{TT}}(t)]$$

PN expansion

$$\phi = \phi_{(2)} + \phi_{(4)} + \phi_{(6)} + \phi_{(8)} + \cdots$$

$$\pi^{ij} = \pi^{ij}_{(3)} + \pi^{ij}_{(5)} + \cdots$$

$$\pi^{ij}_{(5)} = \tilde{\pi}^{ij}_{(5)} + \pi^{ij\,TT}_{(5)}$$

$$\pi^{ij}_{(3)} = \tilde{\pi}^{ij}_{(3)}$$

$$\phi_{(2)} = \frac{1}{4\pi}\sum_a \frac{\boxed{m_a}}{r_a}$$

$$\phi_{(4)} = -\frac{2}{(16\pi)^2}\sum_a\sum_{b\neq a}\frac{m_a m_b}{r_{ab}\, r_a} + \frac{1}{8\pi}\sum_a \frac{p_a^2}{m_a r_a}$$

$$\phi_{(2)} + \phi_{(4)} = \frac{1}{4\pi}\sum_a \frac{\boxed{E_a}}{r_a}$$

$$E_a = m_a + \frac{1}{2}\frac{p_a^2}{m_a} - \frac{1}{64\pi}\sum_{b\neq a}\frac{m_a m_b}{r_{ab}}$$

$$\tilde{\pi}^{ij}_{(3)} = \frac{1}{16\pi}\sum_a p_{ak}\left\{-\delta_{ij}\left(\frac{1}{r_a}\right)_{,k} + 2\left[\delta_{ik}\left(\frac{1}{r_a}\right)_{,j} + \delta_{jk}\left(\frac{1}{r_a}\right)_{,i}\right] - \frac{1}{2} r_{a,ijk}\right\}$$

FE for h^{TT}_{ij} ; $d=3$:

$$\Box h^{TT}_{ij} = \delta^{TT\,kl}_{ij} \Big[A_{(4)kl} + 2 B_{(6)kl} - \frac{1}{2} h^{TT}_{kl} \sum_a m_a \delta_a$$

$$+ \left(h^{TT}_{kl,m}\,\phi_{(2)}\right)_{,m} + \frac{d}{dt}\left(\phi_{(2)}\, \tilde{\pi}^{kl}_{(3)}\right) \Big]$$

$$A_{(4)ij} = -\sum_a \frac{p_{ai}\,p_{aj}}{m_a}\,\delta_a - \frac{1}{4}\,\phi_{(2),i}\,\phi_{(2),j}$$

$$B_{(6)ij} = \frac{1}{4} \sum_a \frac{\vec{p}_a^{\,2}\, p_{ai}\, p_{aj}}{m_a^3}\,\delta_a + \frac{5}{8}\,\phi_{(2)} \sum_a \frac{p_{ai}\,p_{aj}}{m_a}\,\delta_a$$

$$+ \left(\frac{1}{16\pi}\right)\phi_{(2),ij} \sum_a \frac{\vec{p}_a^{\,2}}{m_a r_a} + \left(\frac{1}{16\pi}\right)\tilde{\pi}^{ij}_{(3)} \sum_a p_{ak}\left(\frac{1}{r_a}\right)_{,k}$$

$$+ \frac{1}{2}\left(\frac{1}{16\pi}\right)\tilde{\pi}^{ij}_{(3),k}\left(8\sum_a p_{ak}\,\frac{1}{r_a} - \sum_a p_{al}\, r_{a,kl}\right)$$

$$+ 4\left(\frac{1}{16\pi}\right)\tilde{\pi}^{jk}_{(3)} \sum_a \left[p_{ak}\left(\frac{1}{r_a}\right)_{,i} - p_{ai}\left(\frac{1}{r_a}\right)_{,k}\right]$$

$$- \frac{3}{4}\left(\frac{1}{16\pi}\right)^2 \phi_{(2),ij} \sum_a \sum_{b\neq a} \frac{m_a m_b}{r_{ab}\, r_a} + \frac{5}{64}\,\phi_{(2)}\,\phi_{(2),i}\,\phi_{(2),j}$$

$$\delta^{TT\,kl}_{ij} = \frac{1}{2}\Big[(\delta_{il} - \Delta^{-1}\partial_i\partial_l)(\delta_{jk} - \Delta^{-1}\partial_j\partial_k) + (\delta_{ik} - \Delta^{-1}\partial_i\partial_k)$$

$$(\delta_{jl} - \Delta^{-1}\partial_j\partial_l) - (\delta_{kl} - \Delta^{-1}\partial_k\partial_l)(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)\Big]$$

$$h^{TT}_{(4)ij} = \frac{1}{4}\frac{1}{16\pi}\sum_a \frac{1}{m_a r_a}\Big\{\left[\vec{p}_a^{\,2} - 5(\vec{n}_a\cdot\vec{p}_a)^2\right]\delta_{ij} + 2p_{ai}p_{aj}$$

$$+\left[3(\vec{n}_a\cdot\vec{p}_a)^2 - 5\vec{p}_a^{\,2}\right]n_a^i n_a^j + 6(\vec{n}_a\cdot\vec{p}_a)(n_a^i p_{aj} + n_a^j p_{ai})\Big\}$$

$$+\frac{1}{8}\left(\frac{1}{16\pi}\right)^2 \sum_a\sum_{b\neq a} m_a m_b \Big\{-\frac{32}{s_{ab}}\left(\frac{1}{r_{ab}} + \frac{1}{s_{ab}}\right)n_{ab}^i n_{ab}^j$$

$$+2\left(\frac{r_a + r_b}{r_{ab}^3} + \frac{12}{s_{ab}^2}\right)n_a^i n_b^j + 16\left(\frac{2}{s_{ab}^2} - \frac{1}{r_{ab}^2}\right)(n_a^i n_{ab}^j + n_a^j n_{ab}^i)$$

$$+\left[\frac{5}{r_{ab} r_a} - \frac{1}{r_{ab}^3}\left(\frac{r_b^2}{r_a} + 3r_a\right) - \frac{8}{s_{ab}}\left(\frac{1}{r_a} + \frac{1}{s_{ab}}\right)\right]n_a^i n_a^j$$

$$+\left[\frac{5r_a}{r_{ab}^3}\left(\frac{r_a}{r_b} - 1\right) - \frac{17}{r_{ab} r_a} + \frac{4}{r_a r_b} + \frac{8}{s_{ab}}\left(\frac{1}{r_a} + \frac{4}{r_{ab}}\right)\right]\delta_{ij}\Big\}$$

gr-qc/9712075 (JS)

$$s_{ab} = r_a + r_b + r_{ab}$$

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up to 2.5 PN : metric coefficients are known

$$g_{ij} = \delta_{ij} + g_{ij}^{(2)} + g_{ij}^{(4)} + g_{ij}^{(5)}$$

shift : $N_i = N_i^{(3)} + N_i^{(5)} + N_i^{(6)}$

lapse : $N = 1 + N^{(2)} + N^{(4)} + N^{(6)} + N^{(7)}$

N 1PN 2PN 2.5PN

Calculation of H

$(16\pi G = c = 1)$

ADM approach in d space dimensions :

$$\sqrt{g}\,R = \frac{1}{\sqrt{g}}\left(g_{ik}g_{jl}\pi^{ij}\pi^{kl} - \frac{1}{d-1}\left(g_{ij}\pi^{ij}\right)^2\right) + \sum_a \left(m_a^2 + g_a^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$$-2D_j\pi^{ij} = \sum_a g_a^{ij}p_{aj}\,\delta_a \qquad \left(\delta_a \equiv \delta(\vec{x}-\vec{x}_a)\right)$$

ADMTT gauge :

$$g_{ij} = A(\phi)\,\delta_{ij} + h_{ij}^{TT}$$

$$\pi^{ij} = \tilde{\pi}^{ij}(V^k) + \pi^{ij}_{TT}$$

$$A(\phi) = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{\frac{4}{d-2}}$$

$$\tilde{\pi}^{ij}(V^k) = \partial_i V^j + \partial_j V^i - \frac{2}{d}\delta^{ij}\partial_k V^k$$

Conformally flat d-dim. time-symmetric spacelike hypersurface:

$$g_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

Hamiltonian constraint for 2 black holes instantaneously at rest:

$$-\left(1 + \frac{d-2}{4(d-1)}\phi\right)\Delta\phi = \frac{16\pi G}{c^2}\left(\bar{m}_1\delta_1 + \bar{m}_2\delta_2\right)$$

Ansatz: $\phi = \frac{2G}{c^2}\frac{d-1}{d-2}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$

Renormalization: $\bar{m}_a = m_a + \varepsilon_a$

$$\varepsilon_a = \frac{d-1}{4(d-2)}\frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2}\right)}\frac{G}{c^2}\frac{\alpha_a^2}{r_{aa}^{d-2}}$$

$d = 3$: $\varepsilon_a = \frac{1}{2}\frac{G\alpha_a^2}{r_{aa}}$

Regularization: $\frac{1}{r_a^{d-2}}\delta_a = 0$, $1 < d < 2$

Brill-Lindquist vs Misner-Lindquist BHs

$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

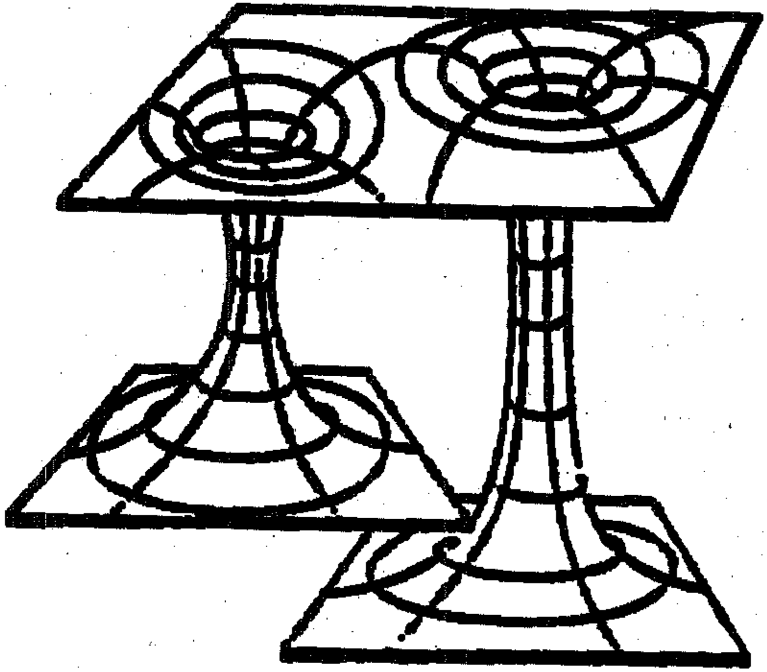
$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right) \quad \text{unique solution}$$

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2 r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2 r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2 r_{ab}/G}\right)^2}-1\right)$$

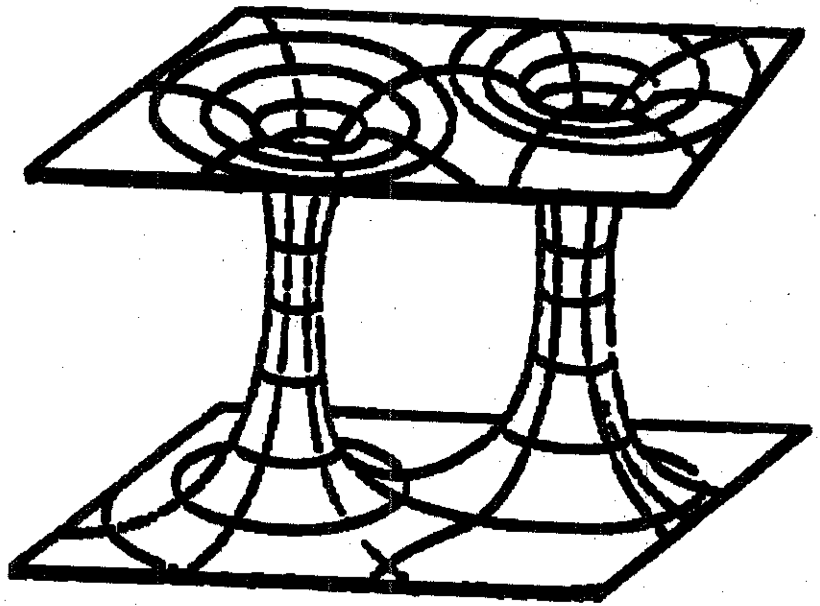
$$H_{\mathrm{BL}}=(\alpha_1+\alpha_2)\,c^2=(m_1+m_2)\,c^2-G\,\frac{\alpha_1\alpha_2}{r_{12}}$$

Konfigurationen binärer Schwarzer Löcher

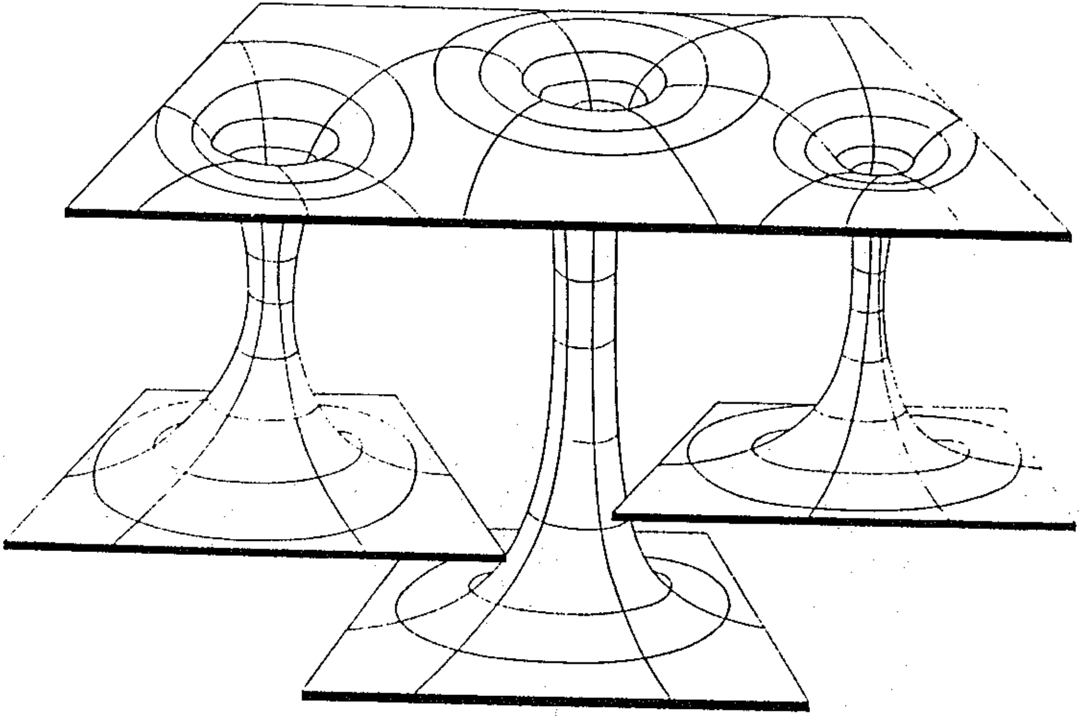
Brill-Lindquist



Misner-Lindquist

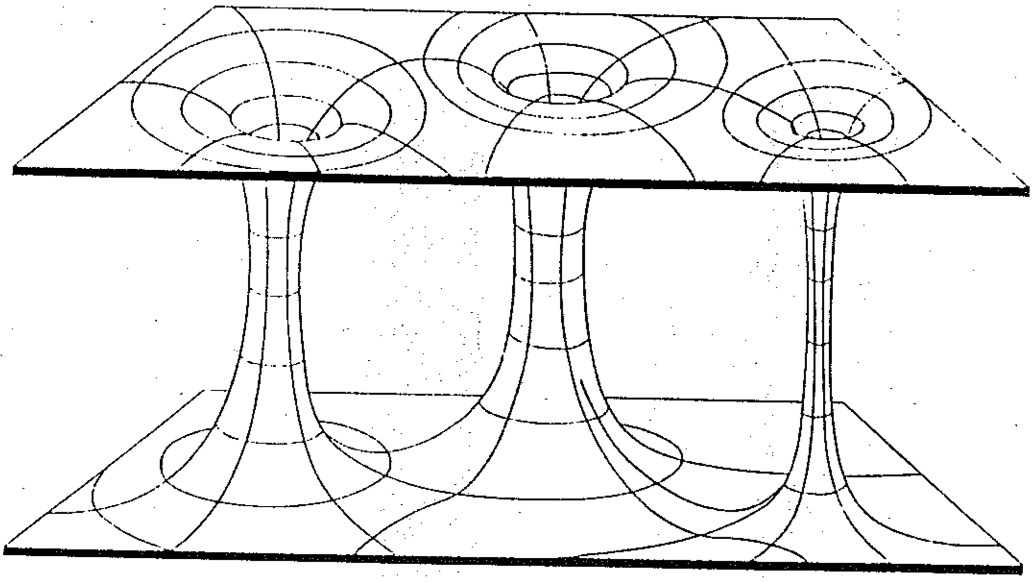


Brill / Lindquist 1963



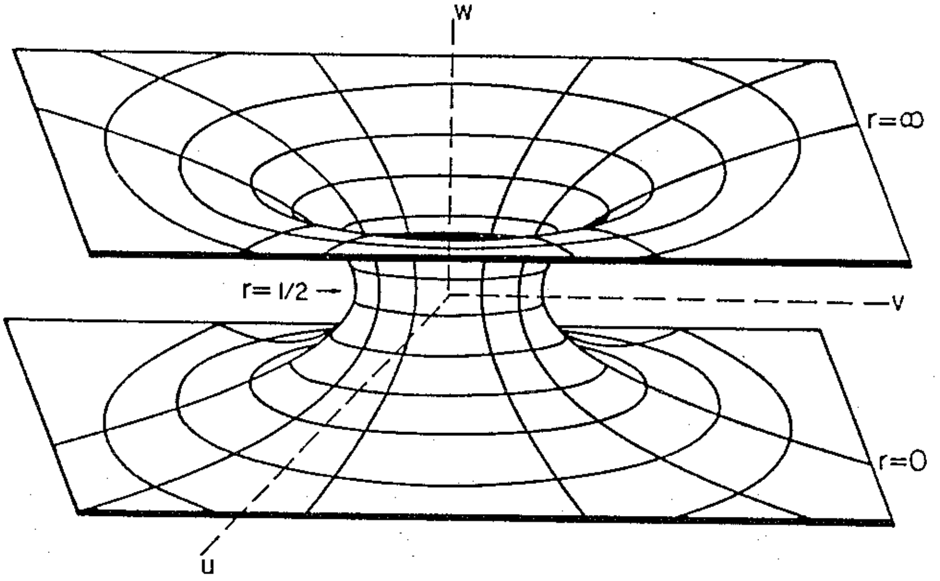
Misner / Lindquist 1963

MISNER

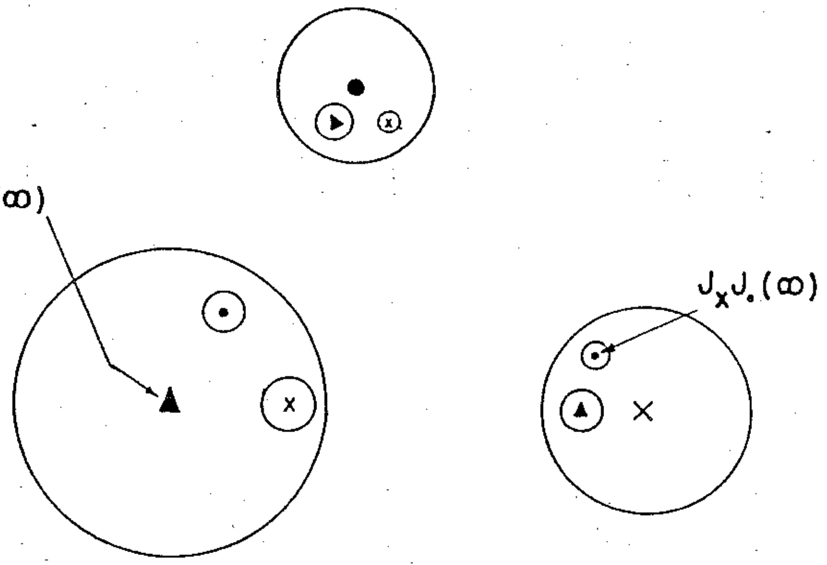


$$dl^2 = \left(1 + \frac{m}{2r}\right)^4 \left(dr^2 + r^2 d\Omega\right) = \left(1 + \frac{m}{2r'}\right)^4 \left(dr'^2 + r'^2 d\Omega'\right)$$

$$\vec{X}' = \left(\frac{m}{2}\right)^2 \frac{\vec{X}}{r^2}$$



method of inversion to achieve isometry



Static Hamiltonians

$$\nu \equiv \frac{m_1 m_2}{(m_1+m_2)^2} \leq \frac{1}{4}$$

$$\hat{H}^{BL} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \frac{1}{8}(1+3\nu)\frac{1}{r^4} + O(1/c^8$$

$$\hat{H}^{ML} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \frac{1}{8}(1+2\nu)\frac{1}{r^4} + O(1/c^8$$

$\uparrow$

ω_{static} - ambiguit

$$\hat{H} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \left(\frac{1}{8} + \left(\frac{3}{8} + \omega_{static}\right)\nu\right)\frac{1}{r^4}$$

dimensional regularization: $\omega_{static} = 0$

Brill-Lindquist: $-(1+\frac{1}{8}\varphi)\Delta\varphi = m_1\delta(\vec{x}-\vec{x}_1) + m_2\delta(\vec{x}-\vec{x}$

$$\varphi^{BL} = \frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}$$

Misner-Lindquist: $-(1+\frac{1}{8}\varphi)\Delta\varphi = \sum_{n=1}^{\infty}\left(\mu_n\delta(\vec{x}-\vec{y}_n) + \nu_n\delta(\vec{x}-\vec{z}_n\right.$

$$\varphi^{ML} = \frac{a_1}{r_1} + \frac{b_1}{r_2} + \sum_{n=2}^{\infty}\left(\frac{a_n}{|\vec{x}-\vec{y}_n|} + \frac{b_n}{|\vec{x}-\vec{z}_n|},\right.$$

$$H^{3PN}_{\text{ambiguous}} = \omega_{\text{static}}(m_1 + m_2)\frac{G^4 m_1^2 m_2^2}{c^6 r_{12}^4} = \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

$$\omega_{\text{static}} = 0 \quad \text{BL}, \quad \omega_{\text{static}} = -\frac{1}{8} \quad \text{ML} \quad \text{(shift of positions}$$

$$H^{3PN}_{\text{induced}} = -\omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3} \quad \text{relative to BL)}$$

Thus, coordinate (gauge) transformation (given in CMS):

$$\delta H = \omega_{\text{static}}\left(\mathbf{p}^2 - 3(\mathbf{n}\cdot\mathbf{p})^2\right)\frac{G^3 M^2}{c^6 r_{12}^3} - \omega_{\text{static}}\frac{G^4 M^3 \mu^2}{c^6 r_{12}^4}$$

Conformal-flat assumption only ($h_{ij}^{\mathrm{TT}} = 0$). Then:

No solution of the constraint equations by dim. reg.

$$\Delta\psi = -\frac{1}{8}\pi^i_j\pi^j_i\frac{1}{\psi^7} - \frac{2\pi G}{c^2}\sum_a m_a\delta_a\frac{1}{\psi}(1 + \frac{p_a^2}{m_a^2\psi^4})^{1/2}$$

$$\partial_j\pi^j_i = -\frac{8\pi G}{c^3}\sum_a p_{ai}\delta_a, \qquad \psi \equiv 1 + \frac{\phi}{8}$$

$$\psi = 1 + \frac{G}{2c^2}(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}) + u$$

$$\psi_a\alpha_a = m_a(1 + \frac{p_a^2}{m_a^2\psi_a^4})^{1/2} \qquad \textbf{(I)}$$

conf. flat binary Hamiltonian to order p^2

p^2-Approximation:

$$\Psi_a = 1 + \frac{G}{2c^2} \sum_{b \neq a} \frac{\alpha_b}{r_{ab}}$$

$$U_a = + \frac{1}{32\pi} \int d^3x \pi^i_j \pi^j_i \frac{1}{r_a \Psi^7}$$

$$\Psi = 1 + \frac{G}{2c^2} \sum_a \frac{\alpha_a}{r_a}$$

$$H_{\mathrm{ADM}} = - \frac{c^4}{2\pi G} \int d^3x \Delta\psi$$

$$= c^2 \sum_a \frac{m_a}{\Psi_a} (1 - \frac{U_a}{\Psi_a} + \frac{p_a^2}{2c^2 m_a^2 \Psi_a^4}) + \frac{c^4}{16\pi G} \int d^3x \pi^i_j \pi^j_i \frac{1}{\Psi^7}$$

Head-on motion:

$$\boxed{\mathcal{I} = \frac{c^4}{16\pi G}\int d^3x \Psi^{-7}\pi^i_j\pi^j_i = \frac{9Gp^2}{16c^2r_{12}}\tilde{\mathcal{I}}}$$

$$a = \frac{G\alpha}{2c^2 r_{12}}, \quad \alpha = \alpha_1 = \alpha_2$$

$$\begin{aligned}
\tilde{\mathcal{I}} = 35\Bigg(& -Li_2\left(\frac{2a-\sqrt{4a+1}+1}{2a}\right)+ \\
& Li_2\left(\frac{(a+1)\left(2a-\sqrt{4a+1}+1\right)}{2a^2}\right) + Li_2\left(-\frac{\sqrt{4a+1}+1}{2a}\right) - \\
& Li_2\left(-\frac{3a+(a+1)\sqrt{4a+1}+1}{2a^2}\right) + Li_2\left(\frac{1-\sqrt{4a^2+1}}{2a+2}\right) + \\
& Li_2\left(-\frac{\sqrt{4a^2+1}-1}{2a}\right) - Li_2\left(-\frac{(a+1)\left(\sqrt{4a^2+1}-1\right)}{2a^2}\right) - \\
& Li_2\left(\frac{\sqrt{4a^2+1}+1}{-2a-2}\right) + Li_2\left(\frac{\sqrt{4a^2+1}-1}{-2a+\sqrt{4a^2+1}-3}\right) - \\
& Li_2\left(\frac{\sqrt{4a^2+1}+1}{-2a+\sqrt{4a^2+1}-1}\right) - Li_2\left(-\frac{-2a+\sqrt{4a^2+1}+1}{2a}\right) + \\
& Li_2\left(-\frac{-2a^2+a+(a+1)\sqrt{4a^2+1}+1}{2a^2}\right)\Bigg)a^2 - \\
& (2(1651507200a^{17}+1789132800a^{16}+2656202752a^{15}+2217930752a^{14}+ \\
& 1724572672a^{13}+1088699392a^{12}+567862016a^{11}+258822400 \\
& a^{10}+94768832a^9+26728000a^8+3185744a^7-1599984a^6- \\
& 758772a^5-101534a^4+1023a^3+29a^2-172a-25))/ \\
& (45(4a+1)^5(4a^2+1)^5a) - \frac{1}{30(4a+1)^{11/2}a^2} \\
& \Big((4300800a^9+5734400a^8+2944928a^7+636816a^6- \\
& 9416a^5-37268a^4-9614a^3-1373a^2-132a-6) \\
& \left(\log\left(\sqrt{4a+1}-1\right)-\log\left(\sqrt{4a+1}+1\right)\right)\Big) + \frac{1}{30(4a^2+1)^{11/2}a^2} \\
& \Big((2150400a^{14}+2867200a^{12}+1550080a^{10}+ \\
& 428160a^8+60816a^6+6172a^4-317a^2-6) \\
& \left(\log\left(2a-\sqrt{4a^2+1}+1\right)-\log\left(2a+\sqrt{4a^2+1}+1\right)\right)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{35}{2} a^2 \Bigg(-2 \log^2\left(\frac{\sqrt{4 a^2+1}-1}{2 a^2}\right)+\log^2\left(\frac{(a+1)\left(\sqrt{4 a^2+1}-1\right)}{2 a^2}\right)+\\
&\quad 2 \log^2\left(\frac{\sqrt{4 a^2+1}+1}{2 a^2}\right)-\log^2\left(\frac{(a+1)\left(\sqrt{4 a^2+1}+1\right)}{2 a^2}\right)+\\
&\quad \log^2\left(-\frac{2 a^2+a-(a+1) \sqrt{4 a^2+1}+1}{2 a^2}\right)-\\
&\quad \log^2\left(\frac{-2 a^2+a+(a+1) \sqrt{4 a^2+1}+1}{2 a^2}\right)+2 \log(2) \log\left(\sqrt{4 a+1}-1\right)-\\
&\quad 2 \log(2) \log\left(\sqrt{4 a+1}+1\right)-2 \log\left(\sqrt{4 a+1}-1\right) \log\left(2 a+\sqrt{4 a+1}+1\right)+\\
&\quad 2 \log\left(\sqrt{4 a+1}+1\right) \log\left(2 a+\sqrt{4 a+1}+1\right)-\\
&\quad 4 \log(2) \log\left(2 a-\sqrt{4 a^2+1}+1\right)+2 \log(2) \log\left(2 a-\sqrt{4 a^2+1}+3\right)-\\
&\quad 2 \log\left(2 a-\sqrt{4 a^2+1}+3\right) \log\left(\sqrt{4 a^2+1}-1\right)+\\
&\quad 2 \log(2) \log\left(\sqrt{4 a^2+1}-1\right)-\log(a+1) \Big(-2 \log\left(\sqrt{4 a+1}+1\right)-\\
&\qquad 2 \log\left(2 a+\sqrt{4 a+1}+1\right)+2 \log\left(2 a-\sqrt{4 a^2+1}+1\right)-\\
&\qquad 2 \log\left(2 a-\sqrt{4 a^2+1}+3\right)+2 \log\left(\sqrt{4 a^2+1}-1\right)+2 \log(2)\Big)+\\
&\quad 4 \log\left(2 a-\sqrt{4 a^2+1}+1\right) \log\left(\sqrt{4 a^2+1}+1\right)-\\
&\quad 4 \log(2) \log\left(\sqrt{4 a^2+1}+1\right)+\log(a)\\
&\quad\ \Big(2 \log\left(\sqrt{4 a+1}-1\right)-4 \log\left(\sqrt{4 a+1}+1\right)-2 \log\left(2 a+\sqrt{4 a+1}+1\right)-\\
&\qquad 2 \log\left(2 a-\sqrt{4 a^2+1}+1\right)+2 \log\left(\sqrt{4 a^2+1}+1\right)+4 \log(2)\Big)-\\
&\quad 2 \log\left(2 a-\sqrt{4 a^2+1}+3\right) \log\left(\frac{(a+1)\left(\sqrt{4 a^2+1}+1\right)}{2 a^2}\right)+\\
&\quad 2 \log(2) \log\left(\frac{(a+1)\left(\sqrt{4 a^2+1}+1\right)}{2 a^2}\right)+2 \log^2(2)\Bigg)
\end{aligned}
$$

$$\boxed{\begin{aligned}\mathcal{I}_r &= \frac{c^4}{16\pi G}\int d^3x\Psi^{-7}r_1^{-1}\pi^i_j\pi^j_i \\ &= \frac{c^4}{16\pi G}\int d^3x\Psi^{-7}r_2^{-1}\pi^i_j\pi^j_i = \frac{9Gp^2}{16c^2r_{12}^2}\tilde{\mathcal{I}}_r\end{aligned}}$$

$$\begin{aligned}
\tilde{\mathcal{I}}_r = \bigl(&6606028800\,a^{20} + 7156531200\,a^{19} + 10949951488\,a^{18} + 9286320128\,a^{17} + \\
&7549222912\,a^{16} + 4963815424\,a^{15} + 2798237696\,a^{14} + \\
&1423418368\,a^{13} + 687249920\,a^{12} + 277778944\,a^{11} + \\
&61045760\,a^{10} - 2298240\,a^9 - 3610560\,a^8 - 476240\,a^7 + \\
&91800\,a^6 + 38468\,a^5 + 10694\,a^4 + 2324\,a^3 + 432\,a^2 + 54\,a + 3\bigr)/ \\
&\left(720\,a^5\,(4\,a+1)^5\,\left(4\,a^2+1\right)^5\right) + \frac{1}{960\,a^7\,(4\,a+1)^{11/2}} \\
&\Bigl(\bigl(17203200\,a^{13} + 22937600\,a^{12} + 12626432\,a^{11} + 3768064\,a^{10} + 759296\,a^9 + \\
&166720\,a^8 + 44616\,a^7 + 8580\,a^6 - 1716\,a^5 - 2262\,a^4 - 924\,a^3 - \\
&198\,a^2 - 22\,a - 1\bigr)\,\left(\log\left(\sqrt{4\,a+1}-1\right) - \log\left(\sqrt{4\,a+1}+1\right)\right)\Bigr) + \\
&\frac{35}{8}\Bigl(a\,\log\left(\sqrt{4\,a+1}-1\right)\,\log(a) - 2\,a\,\log\left(\sqrt{4\,a+1}+1\right)\,\log(a) - \\
&a\,\log\left(2\,a+\sqrt{4\,a+1}+1\right)\,\log(a) + a\,\log\left(\sqrt{4\,a^2+1}+1\right)\,\log(a) + \\
&a\,\log(2)\,\log(a) - a\,\log(2)\,\log(a+1) + a\,\log(2)\,\log\left(\sqrt{4\,a+1}-1\right) + \\
&a\,\log(a+1)\,\log\left(\sqrt{4\,a+1}+1\right) - a\,\log(2)\,\log\left(\sqrt{4\,a+1}+1\right) + \\
&a\,\log(a+1)\,\log\left(2\,a+\sqrt{4\,a+1}+1\right) - \\
&a\,\log\left(\sqrt{4\,a+1}-1\right)\,\log\left(2\,a+\sqrt{4\,a+1}+1\right) + a\,\log\left(\sqrt{4\,a+1}+1\right) \\
&\log\left(2\,a+\sqrt{4\,a+1}+1\right) - a\,\log(a+1)\,\log\left(\sqrt{4\,a^2+1}-1\right)\Bigr) - \\
&\Bigl(\bigl(8601600\,a^{18} + 11468800\,a^{16} + 6200320\,a^{14} + 1712640\,a^{12} + \\
&161472\,a^{10} + 56064\,a^8 + 132\,a^6 - 150\,a^4 - 22\,a^2 - 1\bigr) \\
&\left(\log\left(2\,a-\sqrt{4\,a^2+1}+1\right) - \log\left(2\,a+\sqrt{4\,a^2+1}+1\right)\right)\Bigr)/ \\
&\left(960\,a^7\,\left(4\,a^2+1\right)^{11/2}\right)
\end{aligned}$$

$$-\frac{35}{16}\,a\left(2\,\log^2\left(\frac{\sqrt{4\,a^2+1}-1}{2\,a^2}\right)-\log^2\left(\frac{(a+1)\left(\sqrt{4\,a^2+1}-1\right)}{2\,a^2}\right)-\right.$$

$$2\log^2\left(\frac{\sqrt{4\,a^2+1}+1}{2\,a^2}\right)+\log^2\left(\frac{(a+1)\,\left(\sqrt{4\,a^2+1}+1\right)}{2\,a^2}\right)-$$

$$\log^2\left(\frac{a\left(-2\,a+\sqrt{4\,a^2+1}-1\right)+\sqrt{4\,a^2+1}-1}{2\,a^2}\right)+$$

$$\log^2\left(\frac{a\left(-2\,a+\sqrt{4\,a^2+1}+1\right)+\sqrt{4\,a^2+1}+1}{2\,a^2}\right)+$$

$$2\,\log\left(a-\frac{1}{2}\,\sqrt{4\,a^2+1}+\frac{1}{2}\right)\,\log\left(\frac{(a+1)\left(\sqrt{4\,a^2+1}-1\right)}{2\,a^2}\right)-$$

$$2\,\log\left(a-\frac{1}{2}\,\sqrt{4\,a^2+1}+\frac{1}{2}\right)\,\log\left(\frac{\sqrt{4\,a^2+1}+1}{2\,a}\right)-$$

$$2\,Li_2\left(\frac{2\,a-\sqrt{4\,a+1}+1}{2\,a}\right)+2\,Li_2\left(-\frac{\sqrt{4\,a+1}+1}{2\,a}\right)+$$

$$2\,Li_2\left(-\frac{(a+1)\,\left(-2\,a+\sqrt{4\,a+1}-1\right)}{2\,a^2}\right)+2\,Li_2\left(\frac{1-\sqrt{4\,a^2+1}}{2\,a+2}\right)+$$

$$2\,Li_2\left(-\frac{\sqrt{4\,a^2+1}-1}{2\,a}\right)-2\,Li_2\left(-\frac{(a+1)\left(\sqrt{4\,a^2+1}-1\right)}{2\,a^2}\right)-$$

$$2\,Li_2\left(\frac{\sqrt{4\,a^2+1}+1}{-2\,a-2}\right)-2\,Li_2\left(-\frac{-2\,a+\sqrt{4\,a^2+1}+1}{2\,a}\right)-$$

$$2\,Li_2\left(\frac{2\,a^2}{2\,a^2-\sqrt{4\,a^2+1}\,a+a-\sqrt{4\,a^2+1}+1}\right)-$$

$$2\,Li_2\left(-\frac{a\,\left(\sqrt{4\,a+1}+3\right)+\sqrt{4\,a+1}+1}{2\,a^2}\right)+$$

$$2\,Li_2\left(-\frac{2\,a^2}{a\left(-2\,a+\sqrt{4\,a^2+1}+1\right)+\sqrt{4\,a^2+1}+1}\right)+$$

$$\left.2\,Li_2\left(-\frac{a\,\left(-2\,a+\sqrt{4\,a^2+1}+1\right)+\sqrt{4\,a^2+1}+1}{2\,a^2}\right)\right)$$

3PN Binary BH Conservative Dynamics

$$
\begin{aligned}
H(t) &= m_1 c^2 + m_2 c^2 + H_N + \frac{1}{c^2} H_{[1PN]} \\
&+ \frac{1}{c^4} H_{[2PN]} + \frac{1}{c^6} H_{[3PN]} + \ldots \\
&+ \frac{1}{c^5} H_{[2.5PN]}(t) + \frac{1}{c^7} H_{[3.5PN]}(t) + \ldots
\end{aligned}
$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2 / M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \leq \nu \leq 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMS: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} \equiv \mathbf{p}_1/\mu,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

! Fixation of ω_k via center-of-mass motio

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \quad : \quad \vec{P} = \frac{H}{c^2}\frac{d\vec{X}}{dt} = \frac{d}{dt}\left(\frac{H\vec{X}}{c^2}\right)$$

Poincaré algebra :

$$\{P_i, H\} = \{J_i, H\} = 0$$

$$\{J_i, P_j\} = \varepsilon_{ijk} P_k \quad , \quad \{J_i, J_j\} = \varepsilon_{ijk} J_k$$

$$\{J_i, G_j\} = \varepsilon_{ijk} G_k \quad , \quad \{G_i, G_j\} = -\frac{1}{c^2}\varepsilon_{ijk} J_k$$

$$\{G_i, P_j\} = \frac{1}{c^2} H \delta_{ij} \quad , \quad \{G_i, H\} = P_i$$

138 eqs. for
78 unknown coeff.

$$\underline{\omega_k = \frac{41}{24}}$$

$$K_i \equiv G_i - t P_i$$

$$G_i = \frac{HX^i}{c^2}$$

$$\omega_s = \begin{cases} 0 & \text{Brill-Lindquist} \\ -\frac{1}{8} & \text{Misner-Lindquist} \end{cases}$$

Dynamical Invariants

radial action $i_r(E, j)$:

$$i_r(E, j) = \frac{1}{2\pi} \oint dr\ p_r$$

phase of revolution Φ:

$$\frac{\Phi}{2\pi} = 1 + k = -\frac{\partial}{\partial j} i_r(E, j)$$

orbital period P:

$$\frac{P}{2\pi GM} = \frac{\partial}{\partial E} i_r(E, j)$$

Solution for Global Quantities

action: $\frac{S}{\mu GM} = -E\hat{t} + j\varphi + \int dr \sqrt{R(r,E,j)}$

$$p_r^2 = R(r,E,j) \quad , \quad E = \hat{H}$$

$$p^2 = p_r^2 + \frac{j^2}{r^2}$$

$$R(r,E,j) = A + \frac{2B}{r} + \frac{C}{r^2} + \sum_{n=1}^{5} \frac{D_n}{r^{2+n}}$$

$$i_r(E,j) = \frac{2}{2\pi} \int_{r_{min}}^{r_{max}} dr \sqrt{R(r,E,j)}$$

orbital period: $P = 2\pi GM \frac{\partial}{\partial E} i_r(E,j)$

periastron advance: $\frac{\Phi}{2\pi} - 1 = -\frac{\partial}{\partial j} i_r(E,j)$

Delaunay action variables : $n \equiv j + i_r \; , \; j$

$$\hat{H}^{NR}(n,j) = -\frac{1}{2n^2}\Big\{ 1 + \frac{1}{c^2}\Big[\frac{6}{jn} - \frac{1}{4}(15-\nu)\frac{1}{n^2}\Big] +$$

$$\frac{1}{c^4}\Big[\frac{5}{2}(7-2\nu)\frac{1}{j^3 n} + \frac{27}{j^2 n^2} - \frac{3}{2}(35-4\nu)\frac{1}{jn^3}$$

$$+ \frac{1}{8}(145 - 15\nu + \nu^2)\frac{1}{n^4}\Big] +$$

$$\frac{1}{c^6}\Big[\cdots\Big]\Big\}$$

radial angular frequency : $\omega_{radial} = \frac{2\pi}{P} = \frac{1}{GM}\,\frac{\partial \hat{H}^{NR}(n,j)}{\partial n}$

periastron ang. frequ. : $\omega_{periastron} = \frac{\Delta\phi}{P} = \frac{2\pi k}{P} = \frac{1}{GM}\,\frac{\partial \hat{H}^{NR}(n,j)}{\partial j}$

Circular motion

$i_r = 0 \; : \; n = j$

$$E_{circ} = \hat{H}^{NR}(j)$$

$$\omega_{circ} = \omega_{radial} + \omega_{periastron} = 2\pi\,\frac{1+k}{P}$$

$$\omega_{circ} = \frac{1}{GM}\,\frac{dE_{circ}}{dj}$$

periastron advance k and orbital period P:

$$\begin{aligned} k &= \frac{1}{c^2}\frac{3}{j^2}\left\{1+\frac{1}{c^2}\left[\frac{5}{4}(7-2\nu)\frac{1}{j^2}+\frac{1}{2}(5-2\nu)\,E\right]\right. \\ &+ \left.\frac{1}{c^4}\left[a_1(\nu)\frac{1}{j^4}+a_2(\nu)\frac{E}{j^2}+a_3(\nu)\,E^2\right]\right\} \end{aligned}$$

$$\begin{aligned} \frac{P}{2\pi GM} &= \frac{1}{(-2E)^{3/2}}\left\{1-\frac{1}{c^2}\frac{1}{4}(15-\nu)E\right. \\ &+ \frac{1}{c^4}\left[\frac{3}{2}(5-2\nu)\frac{(-2E)^{3/2}}{j}-\frac{3}{32}(35+30\nu+3\nu^2)\,E^2\right] \\ &+ \left.\frac{1}{c^6}\left[a_2(\nu)\frac{(-2E)^{3/2}}{j^3}-3a_3(\nu)\frac{(-2E)^{5/2}}{j}+a_4(\nu)\,E^3\right]\right\} \end{aligned}$$

$$
\begin{aligned}
a_1(\nu) &= \frac{5}{2}\left(\frac{77}{2}+\left(\frac{41}{64}\pi^2-\frac{125}{3}\right)\nu+\frac{7}{4}\nu^2\right)\\
a_2(\nu) &= \frac{105}{2}+\left(\frac{41}{64}\pi^2-\frac{218}{3}\right)\nu+\frac{45}{6}\nu^2\\
a_3(\nu) &= \frac{1}{4}(5-5\nu+4\nu^2)\\
a_4(\nu) &= \frac{5}{128}(21-105\nu+15\nu^2+5\nu^3)
\end{aligned}
$$

ISCO determination

Dynamics: $H = H(\mathbf{p},\mathbf{r})$, $\quad p^2 = p_r^2 + j^2/r^2$, $\quad p_r = (\mathbf{p}\cdot\mathbf{r})/r$

circular orbits: $p_r = 0$, $\quad p^2 = j^2/r^2$, $\quad H = H(j,r)$

circular motion: $\frac{\partial}{\partial r} H(j,r) = 0 \to H(j)$

orbital frequency: $\omega = \frac{dH(j)}{dj} \to j(\omega) \to H(\omega)$

ISCO: $\frac{dH(\omega)}{d\omega} = 0$, $\quad \frac{dj(\omega)}{d\omega} = 0 \qquad$ [alternatively $\frac{\partial^2}{\partial r^2} H(j,r) = 0$]

SBH:
$$\begin{aligned} E(x) &= \frac{1-2x}{(1-3x)^{1/2}} - 1 \\ &= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots \end{aligned}$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

circular orbits:

$$\omega_{\text{circ}} = \omega_{\text{radial}} + \omega_{\text{periastron}} = 2\pi \frac{1+k}{P}, \quad x = \left(\frac{GM\omega_{\text{circ}}}{c^3} \right)^{2/3}$$

$$c^2 E_{3PN} \equiv \hat{H}_N + \hat{H}_{[1PN]} + \hat{H}_{[2PN]} + \hat{H}_{[3PN]}$$

$$\begin{aligned} E_{3PN}(x) \quad = & \quad -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu \right) x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2 \right) x^3 \\ + & \quad \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2 \right) \nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3 \right) x^4 \end{aligned}$$

ISCO: $dE_{3PN}/dx = 0$

In approximations, ISCO is representation dependent.

$$E_{SK} = -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 + \left(\frac{27}{16} + \frac{29}{16}\nu - \frac{17}{48}\nu^2\right)x^3$$

$$+ \left(\frac{675}{128} + \frac{8585}{384}\nu - \frac{7985}{192}\nu^2 + \frac{1115}{10368}\nu^3\right)x^4$$

$$+ \sum_{i=5}^{11} e_i x^i + O(x^{12})$$

Faye/Jaranowski/S. (2004

$$E_{WM} = -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 + \left(\frac{27}{16} - \frac{39}{16}\nu - \frac{17}{48}\nu^2\right)x^3$$

$$+ O(x^4)$$

$$E_{GR} = -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right)x^3$$

$$+ \left(\frac{675}{128} + \left(\frac{205}{192}\pi^2 - \frac{34445}{1152}\right)\nu + \frac{155}{192}\nu^2\right.$$

$$\left. + \frac{35}{10368}\nu^3\right)x^4 + O(x^5)$$

Damour/Jaranowski/S. (200

$$E_{TB} = \frac{1-2x}{\sqrt{1-3x}} - 1 \qquad (\nu = 0)$$

Last stable circular orbit

$$x \equiv \left(\frac{GM\omega}{c^3}\right)^{2/3} \quad , \quad c = 1$$

$$E(x;\nu) = -\tfrac{1}{2}x\left[1 + E_1(\nu)x + E_2(\nu)x^2 + E_3(\nu)x^3 + \dots\right]$$

test body in Schwarzschild field:

$$E(x;\nu=0) = \frac{1-2x}{\sqrt{1-3x}} - 1 = -\tfrac{1}{2}x\left(1 - \tfrac{3}{4}x - \tfrac{27}{8}x^2 - \tfrac{675}{64}x^3 + \dots\right)$$

$$\text{LSO}: \quad \left(\frac{dE(x)}{dx}\right)_{x_{LSO}} = 0$$

$x = \frac{1}{3}$: light ring

$x = \frac{1}{6}$: LSO for Schw.

$$1 + e(x) \equiv \left(\frac{(E^R)^2 - m_1^2 - m_2^2}{2m_1m_2}\right)^2 \quad , \quad E^R = m_1 + m_2 + \mu E$$

$$e(x;\nu=0) = -x\,\frac{1-4x}{1-3x}$$ meromorphic function

$$e(x;\nu) = -x\left[1 + e_1(\nu)x + e_2(\nu)x^2 + e_3(\nu)x^3 + \dots\right]$$

$$j^2(x;\nu=0) = \frac{1}{x(1-3x)}$$ meromorphic function

$$j^2(x;\nu) = \frac{1}{x}\left[1 + j_1(\nu)x + j_2(\nu)x^2 + j_3(\nu)x^3 + \dots\right]$$

$$\text{LSO}: \quad \frac{dE}{dx} = x^{3/2}\frac{dj}{dx} = 0 \quad , \quad \frac{de}{dx} = 0$$

Padé approximants

$$e_{P_n}(x) \equiv P_\ell^k\left[T_n[e(x)]\right]$$

$$k+\ell=n$$

$$j^2_{P_n}(x) \equiv P_\ell^k\left[T_n[j^2(x)]\right]$$

$$P_\ell^k[T_n[f_p(x)]] \equiv x^p P_\ell^k[T_n[x^{-p} f_p(x$$

$$P_\ell^k(x) = \frac{N_k(x)}{D_\ell(x)}$$

i.g. best for $|k-\ell| \ll \ell, k$ (near "diagonal")

$$e_1(\nu) = -\left(1+\frac{1}{3}\nu\right)$$
$$e_2(\nu) = -\left(3-\frac{35}{12}\nu\right)$$
$$e_3(\nu) = -\frac{10}{3}\left(w_2(\nu) - \underline{\omega_{static}}\,\nu\right)$$

$$w_2(\nu) \equiv \frac{27}{10} + \frac{1}{16}\left(\frac{41}{4}\pi^2 - \frac{4309}{15}\right)\nu + \frac{103}{120}\nu^2 - \frac{1}{270}$$

$$e_{P_2}(x) \equiv P_1^1\left[T_2[e(x)]\right] = -x\,\frac{1+\frac{1}{3}\nu - \left(4-\frac{9}{4}\nu+\frac{1}{9}\nu^2\right)x}{1+\frac{1}{3}\nu - \left(3-\frac{35}{12}\nu\right)x}$$

$$e_{P_3}(x) \equiv P_1^2\left[T_3[e(x)]\right] = -x\,\frac{1-\left(1+\frac{1}{3}\nu+w_3(\nu)\right)x - \left(3-\frac{35}{12}\nu-\left(1+\frac{1}{3}\nu\right)w_3(\nu)\right)x}{1-w_3(\nu)x}$$

$$w_3(\nu) \equiv \frac{40}{36-35\nu}\left(w_2(\nu) - \underline{\omega_{static}}\,\nu\right)$$

Effective one-body dynamics

$$\hat{H}^R_{eff}(\vec{q}', \vec{p}') = \left(A(q') \left[1 + \vec{p}'^2 + \left(\frac{A(q')}{D(q')} - 1 \right) (\vec{n}' \cdot \vec{p}')^2 + \frac{z}{q'^2} (\vec{n}' \cdot \vec{p}')^4 \right] \right)^{1/2}$$

$$A(q') = 1 - \frac{2}{q'} + \frac{2\nu}{q'^3} + \frac{\nu}{q'^4} \left(\frac{94}{3} - \frac{41}{32} \boxed{\pi^2} + 2 \underline{\omega_{static}} \right)$$

$$D(q') = 1 - \frac{6\nu}{q'^2} + \frac{2\nu}{q'^3} (3\nu - 26)$$

$$z = 2\nu(4 - 3\nu)$$

$$H^R_{real} = M \sqrt{1 + 2\nu (\hat{H}^R_{eff} - 1)}$$

$$0 = m_0^2 + g^{\alpha\beta}_{eff}(x) p_\alpha p_\beta + A^{\alpha\beta\gamma\delta}(x) p_\alpha p_\beta p_\gamma p_\delta$$

$$S = -m_0 \int ds_{eff} \left[1 + A_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta \right], \quad u^\alpha = dx^\alpha / ds_{eff}$$

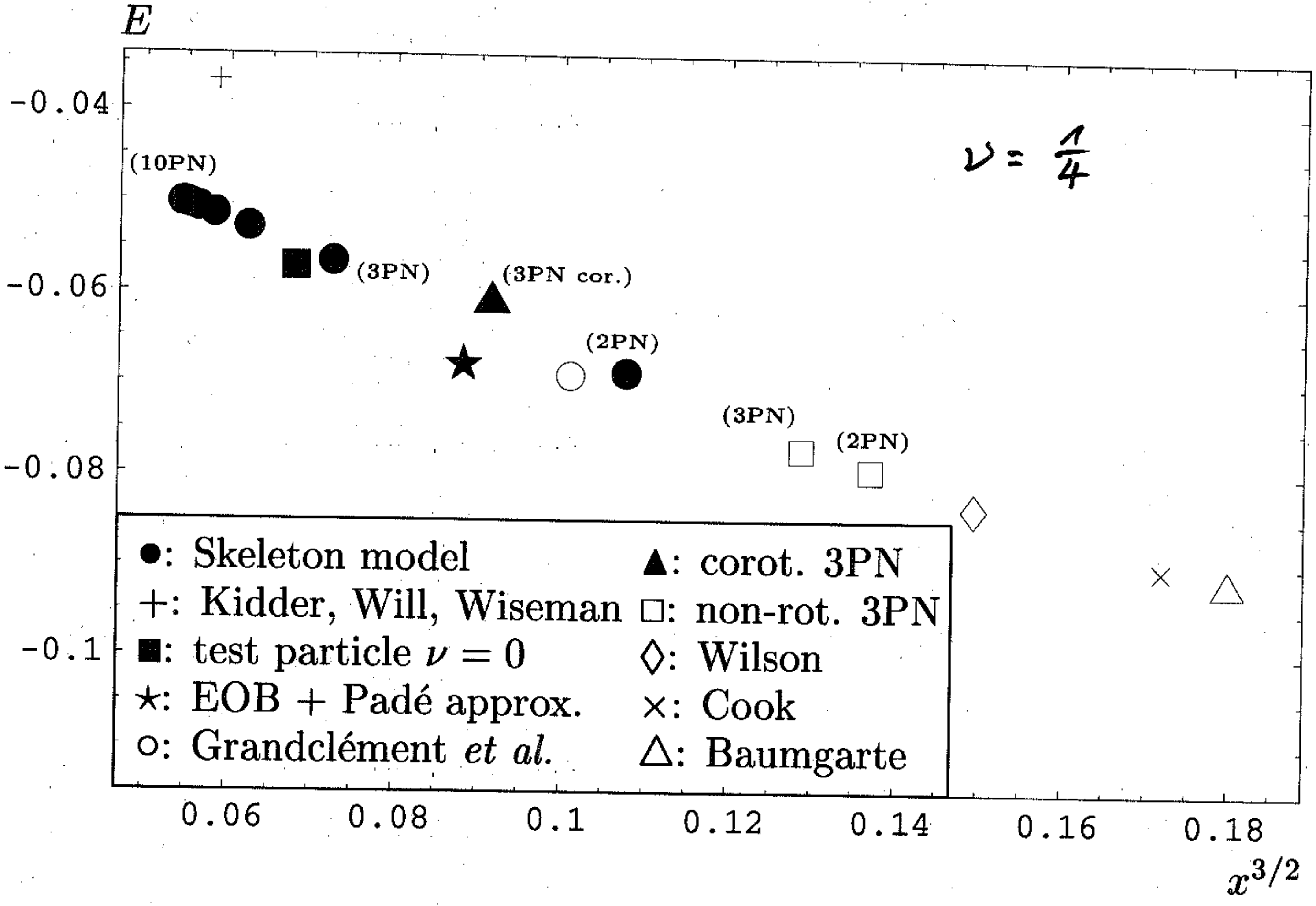
circular orbits : $(\vec{n}' \cdot \vec{p}') = p'_r = 0$

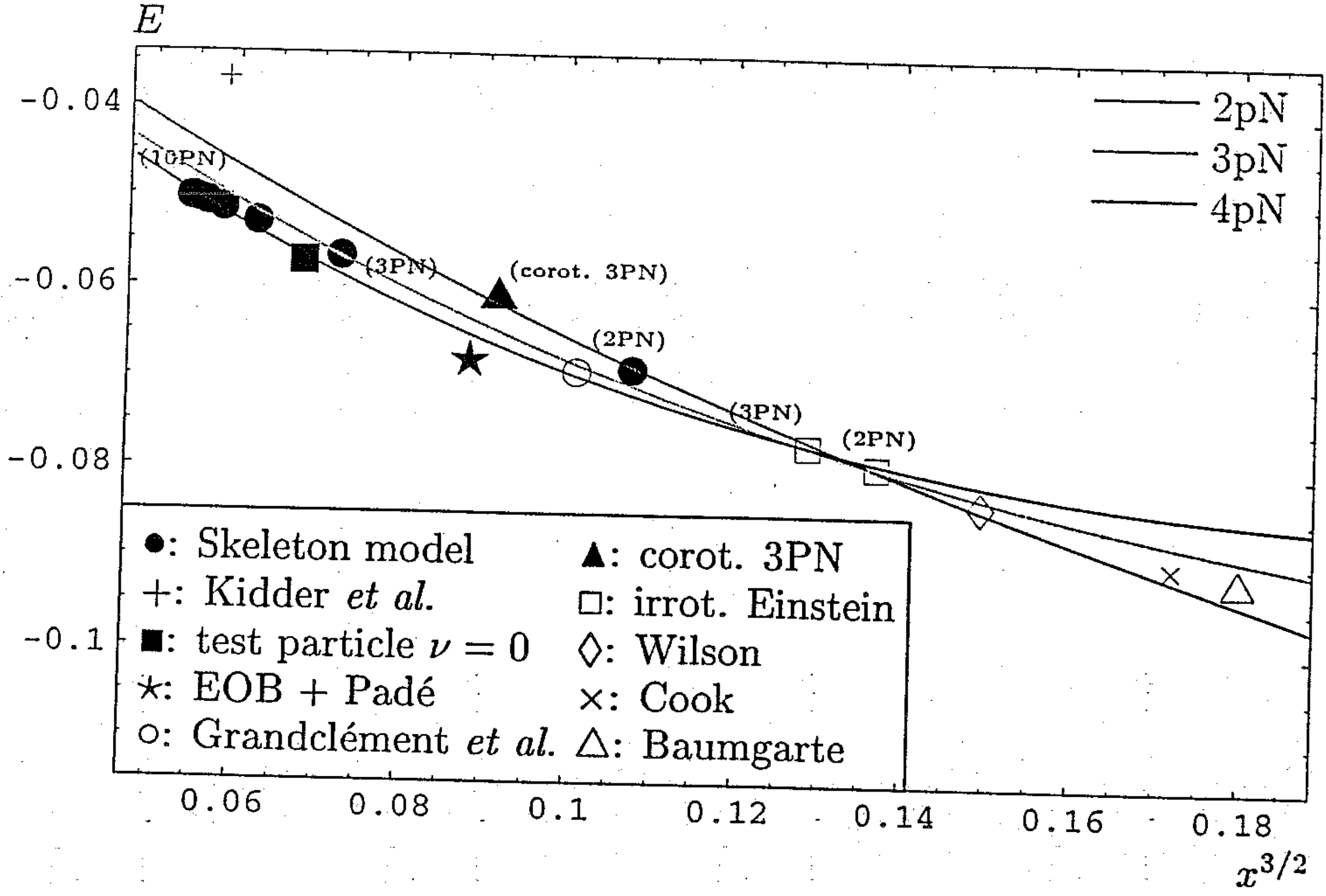
$$W_j(q') \equiv A(q') \left(1 + \frac{j^2}{q'^2} \right)$$

LSO : $\dfrac{\partial W_j(q')}{\partial q'} = 0 = \dfrac{\partial^2 W_j(q')}{\partial q'^2}$

$$W_j^{P_n}(q') = A_{P_n}(q') \left(1 + \frac{j^2}{q'^2} \right)$$

gr-qc/0311018 (Faye/Jaranowski/S)




E
-0.04
-0.06
-0.08
-0.1
2pN
3pN
4pN
(10PN)
(3PN)
(corot. 3PN)
(2PN)
(3PN)
(2PN)
●: Skeleton model
+: Kidder et al.
■: test particle $\nu = 0$
★: EOB + Padé
○: Grandclément et al.
▲: corot. 3PN
□: irrot. Einstein
◇: Wilson
×: Cook
△: Baumgarte
0.06
0.08
0.1
0.12
0.14
0.16
0.18
$x^{3/2}$

Local Solution of EOM

$$\hat{t}(r) = \int \frac{\partial p_r}{\partial E}\, dr = \int \frac{\sum_{n=0}^{5} a_n s^n}{s^2 \sqrt{(s-s_-)(s_+-s)}}\, ds$$

$$\varphi(r) = -\int \frac{\partial p_r}{\partial j}\, dr = \int \frac{\sum_{n=0}^{5} b_n s^n}{\sqrt{(s-s_-)(s_+-s)}}\, ds$$

$s = 1/r$

parametrization with eccentric anomaly u:

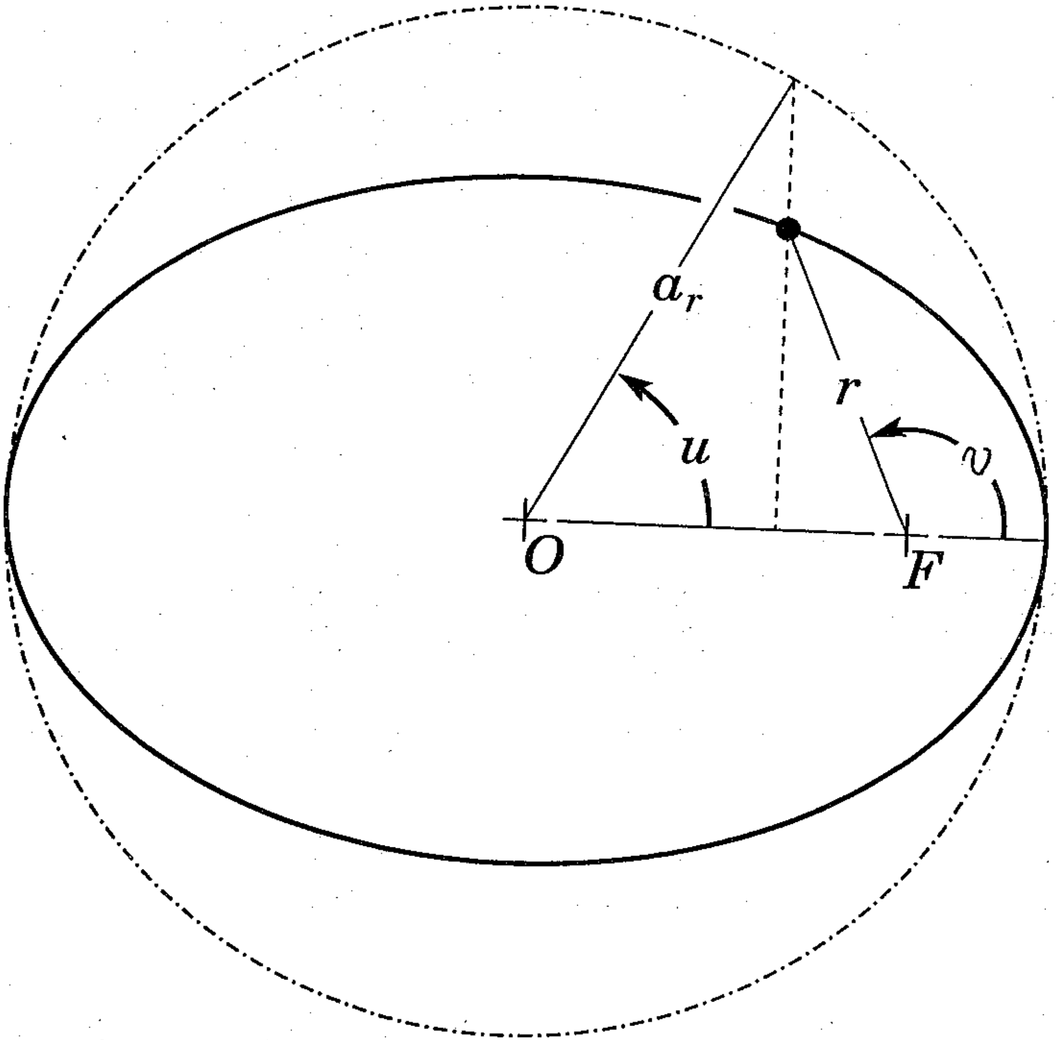
$$\boxed{r = \frac{1}{s} = a_r(1 - e_r \cos u)}$$

$$a_r \equiv \frac{s_+ + s_-}{2 s_+ s_-} \quad , \quad e_r \equiv \frac{s_+ - s_-}{s_+ + s_-}$$

$$v \equiv 2 \arctan\left(\sqrt{\frac{1+e_\varphi}{1-e_\varphi}} \tan\frac{u}{2}\right)$$

$\uparrow$

$e_r \to e_\varphi$: simplification


a_r
u
r
v
O
F

- **3PN accurate generalized quasi-Keplerian parametrization for non-spinning compact binaries**

$$\begin{aligned} r &= a_r\,(1 - e_r\,\cos u) \\ l \equiv n\,(t - t_0) &= u - e_t\,\sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6} \right)(v - u) \\ &\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6} \right) \sin v + \frac{i_{6t}}{c^6}\,\sin 2\,v + \frac{h_{6t}}{c^6}\,\sin 3\,v \\ (\phi - \phi_0) &= (1 + k)\,v + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6} \right) \sin 2v \\ &\quad + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6} \right) \sin 3v + \frac{i_{6\phi}}{c^6}\,\sin 4v + \frac{h_{6\phi}}{c^6}\,\sin 5v \end{aligned}$$

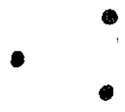
where $v = 2\arctan\left[\left(\frac{1+e_\phi}{1-e_\phi} \right)^{1/2} \tan\frac{u}{2} \right]$

- PN accurate expressions for orbital elements like a_r, e_r^2, n... and orbital functions like g_{4t}, g_{6t}, f_{4t}... in terms of conserved energy E, angular momentum h and mass ratio ν

$$
\begin{aligned}
a_r \;=\; & \frac{1}{(-2\,E)}\left\{1+\frac{(-2\,E)}{4\,c^2}\,(-7+\nu)+\frac{(-2\,E)^2}{16c^4}\left[(1+10\,\nu+\nu^2)\right.\right.\\
& +\frac{1}{(-2\,E\,h^2)}(-68+44\,\nu)\Bigg]+\frac{(-2\,E)^3}{192\,c^6}\left[3-9\,\nu-6\,\nu^2\right.\\
& +3\,\nu^3+\frac{1}{(-2\,E\,h^2)}\left(864+(-3\,\pi^2-2212)\,\nu+432\,\nu^2\right)\\
& \left.\left.+\frac{1}{(-2\,E\,h^2)^2}\left(-6432+(13488-240\,\pi^2)\,\nu-768\,\nu^2\right)\right]\right\}
\end{aligned}
$$

Post-Newtonian approach to radiation reaction

System of n point masses interacting gravitationally



• $m_2\,\delta(\vec{x}-\vec{x}_2)$, $\vec{p}_2\,\delta(\vec{x}-\vec{x}_2)$

• $m_1\,\delta(\vec{x}-\vec{x}_1)$, $\vec{p}_1\,\delta(\vec{x}-\vec{x}_1)$

$$H = \sum_{a=1}^{n} m_a c^2 + \underset{\text{n point masses}}{H_N(\vec{x}_a,\vec{p}_a)} + \frac{1}{c^2}\underset{\text{n point masses}}{H_{1PN}(\vec{x}_a,\vec{p}_a)}$$

$$+\frac{1}{c^4}\underset{\text{2 a. 3 point masses (Sch. '87)}}{H_{2PN}(\vec{x}_a,\vec{p}_a)} + \underset{\text{n point masses}}{\underline{\frac{1}{c^5}H_{2.5PN}(\vec{x}_a,\vec{p}_a;\vec{x}'_b,\vec{p}'_b)}}$$

$$+\frac{1}{c^6}\underset{\text{2 point masses}}{H_{3PN}(\vec{x}_a,\vec{p}_a)} + \underset{\text{n point masses}}{\underline{\frac{1}{c^7}H_{3.5PN}(\vec{x}_a,\vec{p}_a;\vec{x}'_b,\vec{p}')}}$$

gr-qc/0003051
gr-qc/0105038
(Damour/Jaranowski/Sch.)

(Jaranowski/Sch. '97)

EOM : $$\dot{\vec{p}}_a = - \left.\frac{\partial H(\vec{x}_a, \vec{p}_a ; \vec{x}_b', \vec{p}_b')}{\partial \vec{x}_a}\right|_{\substack{\vec{x}_a = \vec{x}_a' \\ \vec{p}_a = \vec{p}_a'}}$$

$$\dot{\vec{x}}_a = \left.\frac{\partial H(\vec{x}_a, \vec{p}_a ; \vec{x}_b', \vec{p}_b')}{\partial \vec{p}_a}\right|_{\substack{\vec{x}_a = \vec{x}_a' \\ \vec{p}_a = \vec{p}_a'}}$$

Energy Loss :

$$\frac{dH}{dt} = \sum_b \left[\frac{\partial H}{\partial \vec{x}_b'} \dot{\vec{x}}_b' + \frac{\partial H}{\partial \vec{p}_b'} \dot{\vec{p}}_b' \right]\Bigg|_{\substack{\vec{x}_a = \vec{x}_a' \\ \vec{p}_a = \vec{p}_a'}}$$

$$= \left.\frac{\partial H}{\partial t}\right|_{\substack{\vec{x}_a = \vec{x}_a' \\ \vec{p}_a = \vec{p}_a'}}$$

$$\left\langle \frac{dH}{dt} \right\rangle_t = \text{negative definite}$$

Dissipative Hamiltonian $H^{\text{int}}_{2.5\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$H^{\text{int}}_{2.5\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) = 5\pi\, \dot{\chi}_{(4)ij}(t)\, \chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)$$

with

$$\chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) := \frac{1}{60\pi}\left[\sum_a \frac{2}{m_a}(\boldsymbol{p}_a^2 \delta_{ij} - 3p_{ai}p_{aj}) + \frac{1}{16\pi}\sum_a \sum_{b\neq a} \frac{m_a m_b}{r_{ab}}(3n^i_{ab}n^j_{ab} - \delta_{ij})\right]$$

Dissipative Hamiltonian $H^{\rm int}_{3.5{\rm PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$\begin{aligned}
H^{\rm int}_{3.5{\rm PN}}\,(\boldsymbol{x}_a, \boldsymbol{p}_a, t) = {} & 5\pi\, \chi_{(4)ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)\left[\dot{\Pi}_{1ij}(t) + \dot{\Pi}_{2ij}(t) + \ddot{\Pi}_{3ij}(t)\right] \\
& + 5\pi\, \dot{\chi}_{(4)ij}(t)\left[\Pi_{1ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) + \widetilde{\Pi}_{2ij}(\boldsymbol{x}_a, t)\right] \\
& - 5\pi\, \ddot{\chi}_{(4)ij}(t)\Pi_{3ij}(\boldsymbol{x}_a, \boldsymbol{p}_a) \\
& + \dot{\chi}_{(4)ij}(t)\left[{Q'}_{ij}(\boldsymbol{x}_a, \boldsymbol{p}_a, t) + {Q''}_{ij}(\boldsymbol{x}_a, t)\right] \\
& + \frac{\partial^3}{\partial t^3}\left[R'(\boldsymbol{x}_a, \boldsymbol{p}_a, t) + R''(\boldsymbol{x}_a, t)\right]
\end{aligned}$$

$\Pi_{2ij}(\boldsymbol{x}_a, \boldsymbol{p}_a)$ in $H^{\text{int}}_{3.5\text{PN}}(\boldsymbol{x}_a, \boldsymbol{p}_a, t)$

$$\begin{aligned}
\Pi_{2ij}\left(\boldsymbol{x}_a, \boldsymbol{p}_a\right) := {} & \frac{1}{5}\left(\frac{1}{16\pi}\right)^2 \sum_a \sum_{b\neq a} \frac{m_b}{m_a r_{ab}} \left\{ \left[5(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)^2 - \boldsymbol{p}_a^2\right]\delta_{ij} - 2p_{ai}p_{aj} \right. \\
& \left. + \left[5\boldsymbol{p}_a^2 - 3(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)^2\right] n_{ab}^i n_{ab}^j - 6(\boldsymbol{n}_{ab}\cdot\boldsymbol{p}_a)(n_{ab}^i p_{aj} + n_{ab}^j p_{ai}) \right\} \\
& + \frac{6}{5}\left(\frac{1}{16\pi}\right)^3 \sum_a \sum_{b\neq a} \frac{m_a^2 m_b}{r_{ab}^2}\left(3n_{ab}^i n_{ab}^j - \delta_{ij}\right) \\
& + \frac{1}{10}\left(\frac{1}{16\pi}\right)^3 \sum_a \sum_{b\neq a} \sum_{c\neq a,b} m_a m_b m_c \left\{ \left[5\frac{r_{ca}}{r_{ab}^3}\left(1-\frac{r_{ca}}{r_{bc}}\right) + \frac{13}{r_{ab}r_{ca}} - \frac{40}{r_{ab}s_{abc}}\right]\delta_{ij} \right. \\
& + \left[3\frac{r_{ab}}{r_{ca}^3} + \frac{r_{bc}^2}{r_{ab}r_{ca}^3} - \frac{5}{r_{ab}r_{ca}} + \frac{40}{s_{abc}}\left(\frac{1}{r_{ab}} + \frac{1}{s_{abc}}\right)\right] n_{ab}^i n_{ab}^j \\
& \left. + \left[2\frac{(r_{ab}+r_{ca})}{r_{bc}^3} - 16\left(\frac{1}{r_{ab}^2} + \frac{1}{r_{ca}^2}\right) + \frac{88}{s_{abc}^2}\right] n_{ab}^i n_{ca}^j \right\},
\end{aligned}$$

with $s_{abc} := r_{ab} + r_{bc} + r_{ca}$,

Rel. acceleration associated with $H^{\text{int}}_{2.5\text{PN}}$ and $H^{\text{int}}_{3.5\text{PN}}$

$$
\begin{aligned}
\boldsymbol{a} = & -\frac{GM}{r^2}\boldsymbol{n} + \frac{1}{c^2}\left\{\frac{GM}{r^2}\left\{\left[-(1+3\nu)\,\boldsymbol{v}^2 + \frac{3}{2}\dot{r}^2\,\nu\right]\boldsymbol{n} + (4\dot{r} - 2\dot{r}\nu)\,\boldsymbol{v}\right\} + \frac{G^2M^2}{r^3}\,(4+2\nu)\,\boldsymbol{n}\right\} \\
& + \frac{1}{c^5}\left\{\frac{G^2M^2}{r^3}\left[\left(-24\dot{r}^3\nu + \frac{96}{5}\dot{r}\boldsymbol{v}^2\nu\right)\boldsymbol{n} + \left(\frac{64}{5}\dot{r}^2\nu - \frac{88}{15}\boldsymbol{v}^2\nu\right)\boldsymbol{v}\right] + \frac{G^3M^3}{r^4}\left(\frac{16}{5}\dot{r}\nu\boldsymbol{n} - \frac{8}{15}\nu\boldsymbol{v}\right)\right\} \\
& + \frac{1}{c^7}\left\{\frac{G^2M^2}{r^3}\left\{\left[-46\dot{r}^5\nu + 24\dot{r}^5\nu^2 + \boldsymbol{v}^4\left(-\frac{138}{35}\dot{r}\nu - \frac{516}{35}\dot{r}\nu^2\right) + \boldsymbol{v}^2\left(56\dot{r}^3\nu - \frac{4}{7}\dot{r}^3\nu^2\right)\right]\boldsymbol{n}\right.\right. \\
& \quad + \left.\left[\frac{334}{7}\dot{r}^4\nu - \frac{268}{7}\dot{r}^4\nu^2 + \boldsymbol{v}^4\left(\frac{1006}{105}\nu - \frac{64}{105}\nu^2\right) + \boldsymbol{v}^2\left(-\frac{2356}{35}\dot{r}^2\nu + \frac{148}{5}\dot{r}^2\nu^2\right)\right]\boldsymbol{v}\right\} \\
& \quad + \frac{G^3M^3}{r^4}\left\{\left[\frac{10188}{35}\dot{r}^3\nu + \frac{324}{7}\dot{r}^3\nu^2 + \boldsymbol{v}^2\left(-\frac{18656}{105}\dot{r}\nu - \frac{1116}{35}\dot{r}\nu^2\right)\right]\boldsymbol{n} + \left[-\frac{17308}{105}\dot{r}^2\nu - \frac{244}{21}\dot{r}^2\nu^2\right.\right. \\
& \quad \left.\left. + \boldsymbol{v}^2\left(\frac{4394}{105}\nu - \frac{16}{35}\nu^2\right)\right]\boldsymbol{v}\right\} + \left.\frac{G^4M^4}{r^5}\left[\left(-\frac{152}{15}\dot{r}\nu - \frac{632}{105}\dot{r}\nu^2\right)\boldsymbol{n} - \left(\frac{386}{105}\nu + \frac{16}{15}\nu^2\right)\boldsymbol{v}\right]\right\}
\end{aligned}
$$

Effects of spinning objects

$$\mathbf{P}_1\delta(\mathbf{x}-\mathbf{x}_1) \to (\mathbf{P}_1 + \tfrac{1}{2}\mathbf{S}_1 \times \boldsymbol{\nabla}_1)\delta(\mathbf{x}-\mathbf{x}_1)$$

$$\boxed{H_{\mathrm{SO}} = \tfrac{2G}{c^2R^3}(\mathbf{S}\cdot\mathbf{L}) + \tfrac{3GM_1M_2}{2c^2R^3}(\hat{\mathbf{S}}\cdot\mathbf{L})}$$

$$\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2, \quad \hat{\mathbf{S}} \equiv \tfrac{\mathbf{S}_1}{M_1^2} + \tfrac{\mathbf{S}_2}{M_2^2}, \quad \mathbf{L} \equiv \mathbf{R} \times \mathbf{P}, \quad \mathbf{P} \equiv \mathbf{P}_1 = -\mathbf{P}_2$$

$$\boxed{H_{\mathrm{SS}} = \tfrac{G}{c^2R^3}\left(\tfrac{3(\mathbf{S}_1\cdot\mathbf{R})(\mathbf{S}_2\cdot\mathbf{R})}{R^2} - (\mathbf{S}_1\cdot\mathbf{S}_2)\right)}$$

$$H_{\mathrm{SS}}^{\mathrm{Kerr}} = \tfrac{GM_1M_2}{2c^2R^3}\left(\tfrac{3(\tilde{\mathbf{S}}\cdot\mathbf{R})(\tilde{\mathbf{S}}\cdot\mathbf{R})}{R^2} - (\tilde{\mathbf{S}}\cdot\tilde{\mathbf{S}})\right), \qquad \tilde{\mathbf{S}} \equiv \tfrac{\mathbf{S}_1}{M_1} + \tfrac{\mathbf{S}_2}{M_2}$$

gr-qc/0407116 (Schäfer)

Effects of spinning objects

Lense-Thirring effect:

RLLL-vector: $\mathbf{A} = \mathbf{P} \times \mathbf{L} - GM\mu^2 \frac{\mathbf{R}}{R}$

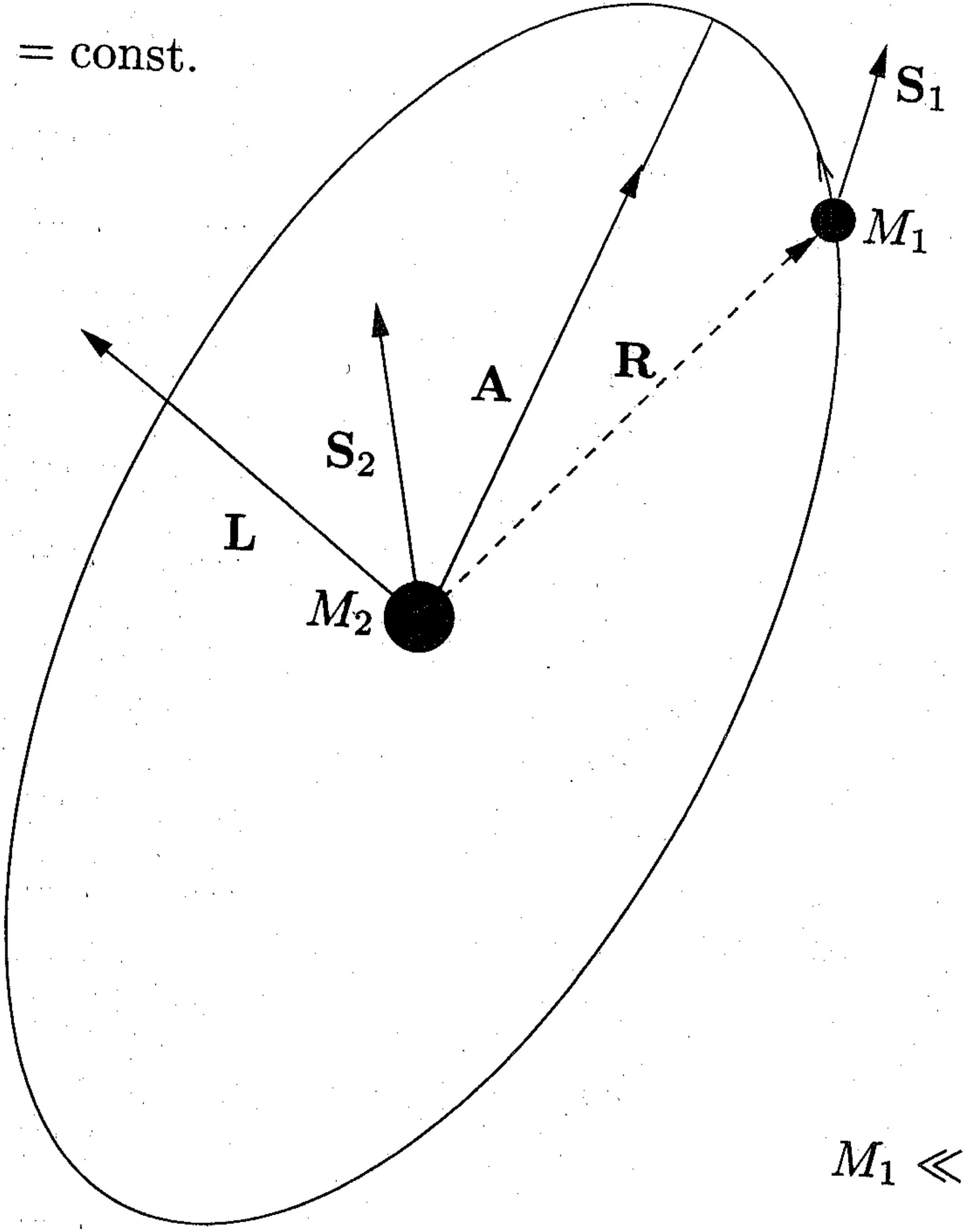
$$< (\tfrac{d\mathbf{L}}{dt})^{\mathrm{SO}} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{L}$$

$$< (\tfrac{d\mathbf{A}}{dt})^{\mathrm{SO}} >_t = \mathbf{\Omega}_{\mathrm{SO}} \times \mathbf{A}$$

$$\mathbf{\Omega}_{\mathrm{SO}} = \tfrac{2G}{c^2} < \tfrac{1}{R^3} >_t (\mathbf{S}_{\mathrm{eff}} - 3\tfrac{(\mathbf{L}\cdot\mathbf{S}_{\mathrm{eff}})\mathbf{L}}{L^2}), \quad \mathbf{S}_{\mathrm{eff}} \equiv \mathbf{S} + \tfrac{3}{4} M_1 M_2 \hat{\mathbf{S}}$$

LAGEOS: 31 mas/yr

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{const.}$$



$$M_1 \ll M_2$$

Effects of spinning objects

Schiff effect (Lense-Thirring for spin, frame dragging):

$$\left(\frac{d\mathbf{S}_1}{dt}\right)^{\mathrm{SS}} = \mathbf{\Omega}_{\mathrm{SS}} \times \mathbf{S}_1, \qquad \mathbf{\Omega}_{\mathrm{SS}} = \frac{G}{c^2R^3}\left(3\frac{(\mathbf{R}\cdot\mathbf{S}_2)\mathbf{R}}{R^2} - \mathbf{S}_2\right)$$

GP-B: 42 mas/yr

de Sitter effect (Fokker effect, geodetic precession):

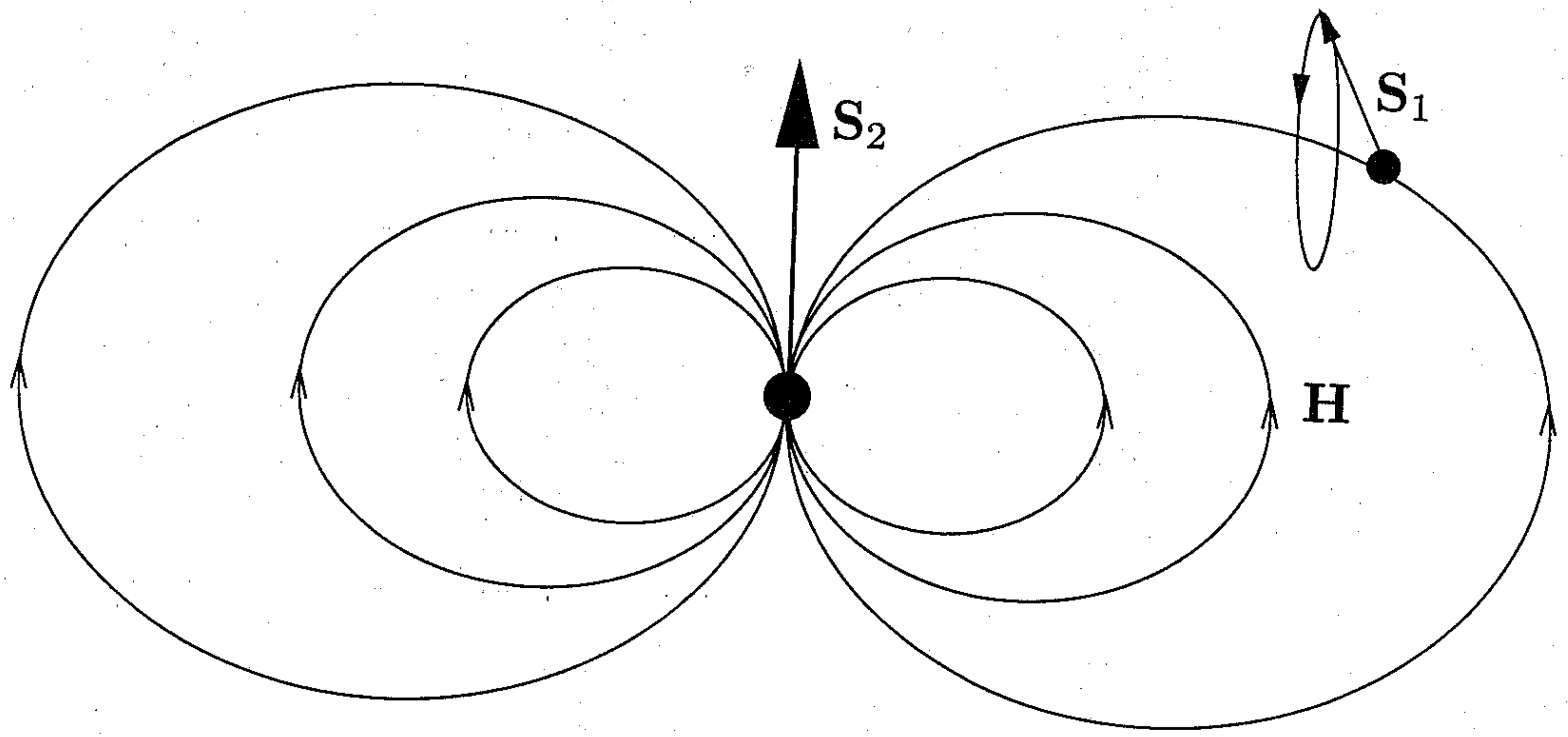
$$\left(\frac{d\mathbf{S}_1}{dt}\right)^{\mathrm{SO}} = \mathbf{\Omega}^{\mathrm{s}}_{\mathrm{SO}} \times \mathbf{S}_1, \qquad \mathbf{\Omega}^{\mathrm{s}}_{\mathrm{SO}} = \frac{2G}{c^2R^3}\left(1 + \frac{3M_2}{4M_1}\right)\mathbf{L}$$

$$\mathbf{L} = \frac{M_1M_2}{M_1+M_2}\mathbf{R} \times \mathbf{V}$$

Earth-Moon: 19 mas/yr, GP-B: 6,600 mas/yr

$$\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{N}c$$

$$\boldsymbol{\Omega}_{\mathrm{SS}} = -\tfrac{1}{2}\mathbf{H}$$

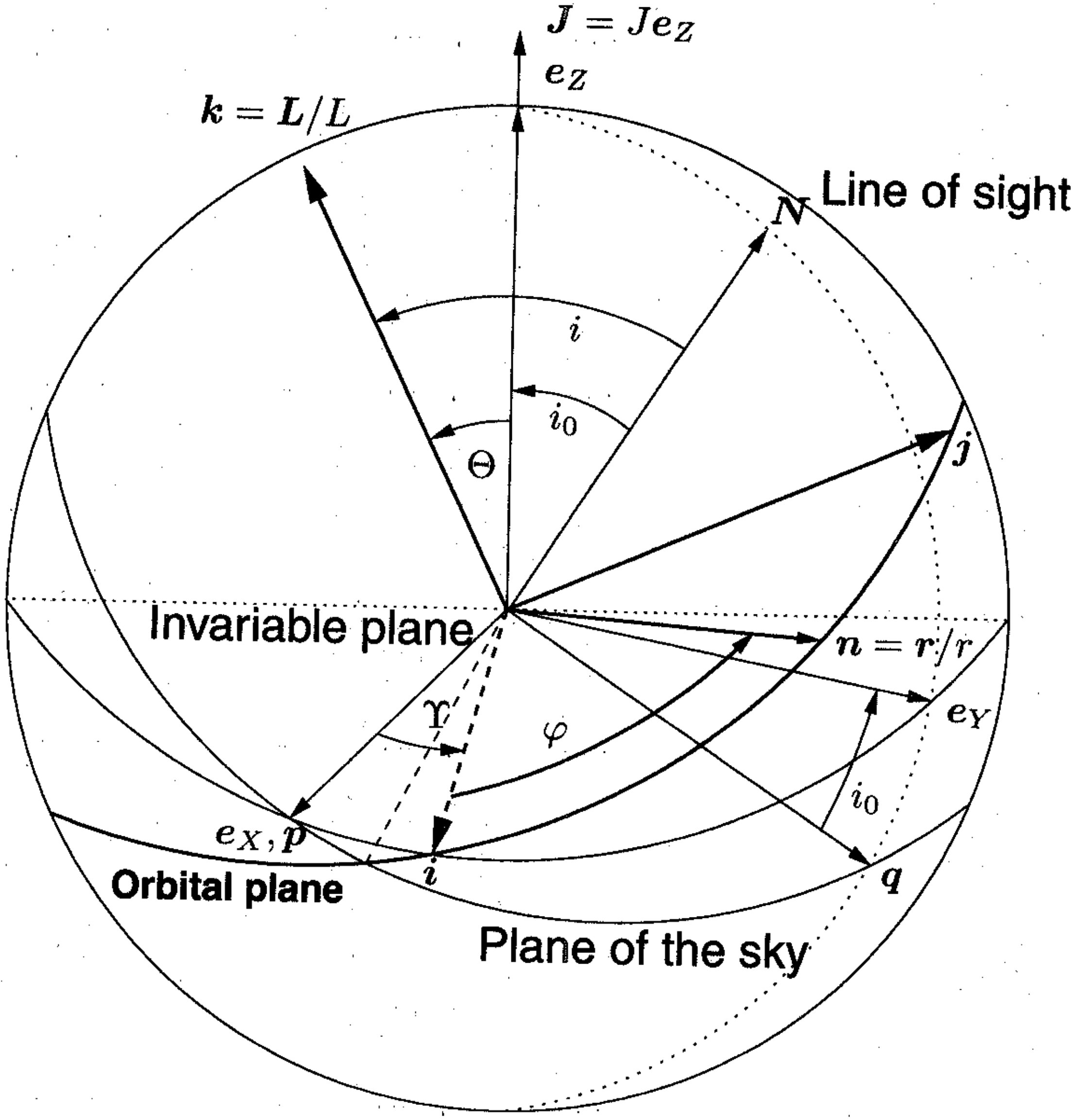


spin-orbit coupling and motion

Königsdörffer/Gopakumar

gr-qc/0501011
gr-qc/0509012

Binary geometry and orbital plane precession



- A Keplerian type parametric solution to PN accurate dynamics of compact binaries with leading order relativistic spin-orbit interactions exists *only* when
(i) $m_1 \neq m_2$, $\mathbf{S_1} \neq \mathbf{0}$ *or* $\mathbf{S_2} \neq \mathbf{0}$ (Single spin case)
(ii) $\mathbf{S_1}$ and $\mathbf{S_2}$ are arbitrary but $m_1 = m_2$.

$$\begin{aligned}
\mathbf{r}(t) &= r(t)\cos\varphi(t)\,\mathbf{i}(t) + r(t)\sin\varphi(t)\,\mathbf{j}(t)\,, \\
\mathbf{L}(t) &= L\,\mathbf{k}(t)\,, \\
\mathbf{S}(t) &= J\,\mathbf{e}_Z - L\,\mathbf{k}(t)\,,
\end{aligned}$$

$$\begin{aligned}
\mathbf{i}(t) &= \cos\Upsilon(t)\,\mathbf{e}_X + \sin\Upsilon(t)\,\mathbf{e}_Y\,, \\
\mathbf{j}(t) &= -\cos\Theta\sin\Upsilon(t)\,\mathbf{e}_X + \cos\Theta\cos\Upsilon(t)\,\mathbf{e}_Y + \sin\Theta\,\mathbf{e}_Z\,, \\
\mathbf{k}(t) &= \sin\Theta\sin\Upsilon(t)\,\mathbf{e}_X - \sin\Theta\cos\Upsilon(t)\,\mathbf{e}_Y + \cos\Theta\,\mathbf{e}_Z
\end{aligned}$$

Θ the precessional angle of $\mathbf{L}$ around $\mathbf{J}$ &
$J \equiv |\mathbf{J}| = (L^2 + S^2 + 2LS\cos\alpha)^{1/2}$

$$\begin{aligned}
r &= a_r\,(1 - e_r \cos u) \\
l \equiv n\,(t - t_0) &= u - e_t \sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) \\
&\quad + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v \\
\varphi - \varphi_0 &= (1 + k)v + \left(\frac{f_{4\varphi}}{c^4} + \frac{f_{6\varphi}}{c^6}\right)\sin 2v \\
&\quad + \left(\frac{g_{4\varphi}}{c^4} + \frac{g_{6\varphi}}{c^6}\right)\sin 3v + \frac{i_{6\varphi}}{c^6}\sin 4v + \frac{h_{6\varphi}}{c^6}\sin 5v \\
\Upsilon - \Upsilon_0 &= \frac{\chi_{\rm so} J}{c^2 L^3}(v + e \sin v)
\end{aligned}$$

$$v = 2\arctan\left[\left(\tfrac{1+e_\varphi}{1-e_\varphi}\right)^{1/2}\tan\tfrac{u}{2}\right]$$

⋆ The orbital elements & functions are expressible in terms of E, L, S, m_1, m_2 & α

Dynamics in (r, φ, Υ) in $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$

- the associated parameters in terms of E, L, S, η and $\alpha = \angle(\boldsymbol{L}, \boldsymbol{S})$:

$$
\begin{aligned}
a_r &= -\frac{1}{2E}\left(1 - 2\chi_{\text{so}} \cos\alpha \frac{S}{L}\frac{E}{c^2}\right), \\
e_r^2 &= 1 + 2EL^2 + 8(1 + EL^2)\chi_{\text{so}} \cos\alpha \frac{S}{L}\frac{E}{c^2}, \\
n &= (-2E)^{3/2}, \\
e_t^2 &= 1 + 2EL^2 + 4\chi_{\text{so}} \cos\alpha \frac{S}{L}\frac{E}{c^2}, \\
k &= \frac{1}{c^2 L^2}\left(\chi_{\text{so}} - 3\chi_{\text{so}} \cos\alpha \frac{S}{L}\right), \\
e_\varphi^2 &= 1 + 2EL^2 - 4(1 + 2EL^2)\chi_{\text{so}} \frac{E}{c^2} \\
&\quad + 4(3 + 4EL^2)\chi_{\text{so}} \cos\alpha \frac{S}{L}\frac{E}{c^2}.
\end{aligned}
$$