Derivatives ~ Cartan eqs

Geometry of Lie groups

- Left & right invariant fields
- Cartan formulas
- Invariant metric
- Homogeneous cosmologies

Geometry of Coset spaces

- Group action
- Reductive spaces
- Cartan formulas
- Symmetric spaces

Real Lie algebras

- Split and compact real forms
- SL(2,ℂ) as an example
-
- Main theorem
- Iwasawa & Cartan decompositions
- Cayley transformations
- Vogan diagrams
- Tits-Satake diagrams

Some references

A.W. Knapp : Lie groups, Beyond an introduction

S. Helgason: Differential geometry, Lie groups, and symmetric spaces

M. Henneaux, M. Leston, D. Persson, PL-S.
"Symmetries in gravity" (in preparation)

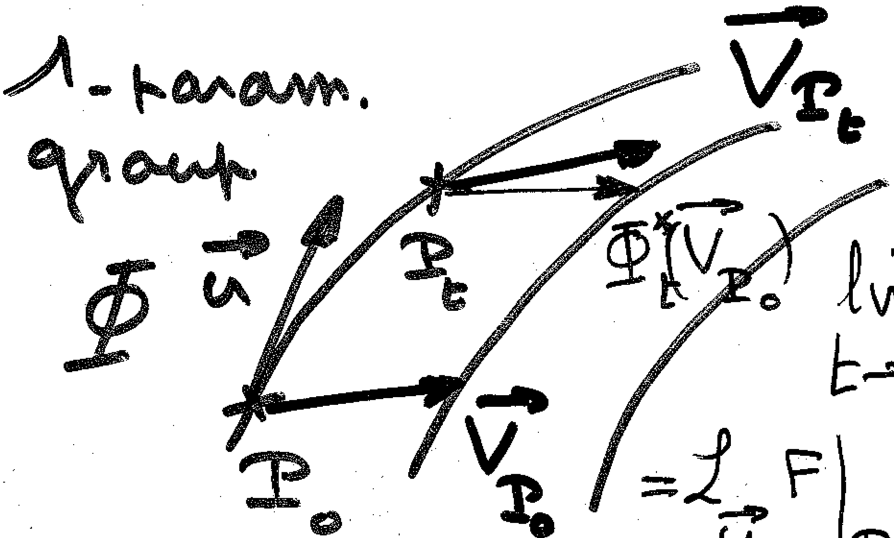
S. Araki : J. Math Osaka Univ 13 (1962) 1-34

J. Tits : Proc. Symp. Pure Math 9 (1966) 336

E. Cartan : (see refs in Helgason)

DERIVATIVES

Lie derivative



Fonction

$$\lim_{t\to 0}\frac{1}{t}\left(F[\Phi_t(P_0)]-F[P_0]\right)$$

$$=\mathcal{L}_{\vec{u}}F\Big|_{P_0}=\vec{u}(F)\Big|_{P_0}=\frac{d}{dt}F[\Phi_t(P_0)]\Big|_{t=}$$

Invariant vector field

$$\vec{V}_{\Phi_t(P_0)}=\Phi_t^*(\vec{V}_{P_0})$$

Lie derivative of a vector fields

$$\mathcal{L}_{\vec{u}}\vec{V}\Big|_{P_0}=\lim_{t\to 0}\frac{1}{t}\left(\vec{V}_{\Phi_t(P_0)}-\Phi_t^*(\vec{V}_{P_0})\right)$$

$$\left(\mathcal{L}_{\vec{u}}\vec{V}\right)^\alpha=u^\beta\partial_\beta V^\alpha-V^\beta\partial_\beta u^\alpha$$

Lie derivative of a 1-form

$$\mathcal{L}_{\vec{u}} \underline{\Sigma}\Big|_{P_0} = \lim_{t\to 0} \frac{1}{t}\left(\Phi_{t*}\left(\underline{\Sigma}_{\Phi_t(P_0)}\right) - \underline{\Sigma}_{P_0}\right)$$

$$\left(\mathcal{L}_{\vec{u}} \underline{\Sigma}\right)_\alpha = u^\beta \partial_\beta \Sigma_\alpha + \Sigma_\beta \partial_\alpha u^\beta$$

Covariant derivative

$\{\vec{e}_\alpha\}$ moving frame

$$\underline{\nabla} \vec{e}_\alpha = \vec{e}_\beta \otimes \underline{\omega}^\beta{}_\alpha$$

differential **LINEAR CONNECTION 1-FORM**

derivative

$$\nabla_{\vec{u}} \vec{e}_\alpha = \vec{e}_\beta \, \underline{\omega}^\beta{}_\alpha(\vec{u})$$

$$= \gamma^\beta{}_{\alpha\rho} \, u^\rho \, \vec{e}_\beta$$

Ricci connection coefficients

Natural basis

$$\vec{e}_\alpha = \frac{\partial}{\partial x^\alpha} \qquad \gamma^\beta{}_{\alpha\rho} \to \Gamma^\beta{}_{\alpha\rho}$$ Christoffel symbols

Dual frame

$$\{ \underline{\theta}^\alpha \mid \underline{\theta}^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta \}$$

Torsion $\quad \underline{\nabla}\, \underline{\theta}^\alpha = - \underline{\omega}^\alpha{}_\beta \otimes \underline{\theta}^\beta$

$$\underline{\underline{\Theta}}^\alpha = \underline{d}\, \underline{\theta}^\alpha + \underline{\omega}^\alpha{}_\beta \wedge \underline{\theta}^\beta$$

$$\Theta = \underline{\underline{\Theta}}^\alpha \otimes \vec{e}_\alpha$$ **TORSION TENSEUR**

$$= \frac{1}{2} T^\alpha{}_{\beta\gamma}\, \underline{\theta}^\beta \wedge \underline{\theta}^\gamma \otimes \vec{e}_\alpha$$

Curvature

$$\underline{\underline{\Omega}}^\alpha{}_\beta = \underline{d}\, \underline{\omega}^\alpha{}_\beta + \underline{\omega}^\alpha{}_\gamma \wedge \underline{\omega}^\gamma{}_\beta$$

$$\Omega = \underline{\underline{\Omega}}^\alpha{}_\beta \otimes \vec{e}_\alpha \otimes \underline{\theta}^\beta$$ **RIEMANN TENSEUR**

$$= \frac{1}{2} R^\alpha{}_{\beta\gamma\delta}\, \vec{e}_\alpha \otimes \underline{\theta}^\beta \otimes \underline{\theta}^\gamma \wedge \underline{\theta}^\delta$$

Metric $\quad \underline{\underline{g}} = g_{\alpha\beta}\, \underline{\theta}^\alpha \otimes \underline{\theta}^\beta$

Levi-Civita connection

$$\nabla \underline{\underline{g}} = 0 \iff dg_{\alpha\beta} = g_{\alpha\mu} \underline{\omega}^\mu{}_\beta + g_{\mu\beta} \underline{\omega}^\mu{}_\alpha$$

LIE GROUP GEOMETRY

Left translation: $G \to G$

$$L_h \quad g \to hg$$

Right translation: $G \to G$

$$R_h \quad g \to gh$$

$$L_h \circ R_{h'} = R_{h'} \circ L_h$$

Left invariant vector fields

$$\vec{l}_g [f] = \frac{d}{dt} f(g\, h(t)) \Big|_{t=0}$$

$$L_{k*}(\vec{l})\big|_g [f] = \vec{l}_g [L_k^*(f)] = \frac{d}{dt} f[k g h(t)]\Big|_{t=0}$$

$$= \vec{l}_{kg} [f]$$

- $\mathcal{G}$: Lie algebra of G

$\{T_A\}$ basis of $\mathcal{G}$ $\boxed{[T_A, T_B] = c_{AB}^{\;C}\, T_C}$

$\exp_G t T_A$ 1-parameter subgroups

$\{\vec{\ell}_A\}$ left invariant vector fields

$\{\vec{\ell}_A g\}$ basis of TG_g

- $$[\vec{\ell}_A, \vec{\ell}_B] = Q_{AB}^{\;C}\, \vec{\ell}_C$$

$$[L_{k*}\vec{\ell}_A, L_{k*}\vec{\ell}_B] = L_{k*}[\vec{\ell}_A, \vec{\ell}_B]$$

$$Q_{AB}^{\;C}(kg) = Q_{AB}^{\;C}(g) : \text{constants}$$

- $\exp tX \exp sY = \exp sY \exp tX \exp st[X,Y]$

$$\vec{\ell}_B[\vec{\ell}_A[f]] = \vec{\ell}_A[\vec{\ell}_B[f]] + c_{AB}^{\;C}\, \vec{\ell}_C[f]$$

$$\boxed{[\vec{\ell}_A, \vec{\ell}_B] = -c_{AB}^{\;C}\, \vec{\ell}_C}$$

- $\left\{ \underline{\theta}^A \;\middle|\; \underline{\theta}^A(\vec{l}_B) = \delta^A_B \right\}$ left invariant 1-forms

- Formulas

$$\vec{l}_A = l_A^{\ B}\, \partial_B \,, \quad \underline{\theta}^A = \theta^A_{\ B}\, dx^B$$

$$l_A^{\ K}\, \theta^B_{\ K} = \delta^B_A = l^B_{\ K}\, \theta^K_{\ A}$$

$\mathcal{M}$, T_A : matrix rep. of $\left\{ \begin{array}{l} G \\ \mathfrak{g} \end{array} \right.$

$$\boxed{\begin{array}{l} l_A^{\ B} \dfrac{\partial}{\partial x^B} \mathcal{M} = \mathcal{M}\, T_A \\[2em] \mathcal{M}^{-1} \dfrac{\partial}{\partial x^B} \mathcal{M} = T_A\, \theta^A_{\ B} \end{array}}$$

$$\boxed{\mathcal{M}^{-1}\, d\mathcal{M} = T_A\, \underline{\theta}^A}$$

Right invariant vector fields
forms

$$\vec{r}_g[f] = \frac{d}{dt} f(h(t)g)\Big|_{t=0}$$

$\{\vec{r}_A\}$, $\{\underline{\sigma}^A\}$ invariant basis

$$\vec{l}_A{}_g = \Lambda^B_A(g)\,\vec{r}_B{}_g\ ,\quad \Lambda^B_A(Id) = \delta^B_A$$

$$[\vec{l}_A, \vec{r}_B] = 0\ ,\quad [\vec{l}_A, \vec{l}_B] = -C^C_{AB}\,\vec{l}_C$$

$$\vec{l}_A(\Lambda^C_B)\,\Lambda^{-1}{}^D_C = -C^D_{AB}$$

$$[\vec{r}_A, \vec{r}_B] = +\tilde{C}^C_{AB}\,\vec{r}_C$$

$$\Lambda^K_A\,\Lambda^L_B\,\Lambda^{-1}{}^C_M\,\tilde{C}^M_{KL} = +C^C_{AB} = \tilde{C}^C_{AB}$$

$$\boxed{r^B_A \frac{\partial}{\partial x^B} M_G = T_A M_G\ ,\quad T_A\,\underline{\sigma}^A = -M_G\, dM^{-1}}$$

$$d\underline{\theta}^A(\vec{\ell}_B, \vec{\ell}_C) = \vec{\ell}_B\left[\underline{\theta}^A(\vec{\ell}_C)\right] - \vec{\ell}_C\left[\underline{\theta}^A(\vec{\ell}_B)\right]$$

$$- \underline{\theta}^A\left([\vec{\ell}_B, \vec{\ell}_C]\right)$$

$$- C^K_{BC}\,\vec{\ell}_K$$

CARTAN FORMULA

$$\boxed{d\wedge\underline{\theta}^A - \frac{1}{2} C^A_{BC}\,\underline{\theta}^B \wedge \underline{\theta}^C = 0}$$

Left and right invariant global frames

$\Updownarrow$

Left and right parallelisms

$\Downarrow$

Left ($\overset{L}{\nabla}\underline{\theta}^A = 0$) and right connection

$$\boxed{\begin{array}{l} \Theta_R = \frac{1}{2}\,\vec{\ell}_A\, C^A_{BC}\,\underline{\theta}^B \wedge \underline{\theta}^C = -\Theta_L \\ \mathrm{Riem}_R = 0 \qquad\qquad = \mathrm{Riem}_R \end{array}}$$ $\triangle!$

Torsionless connection

$$\boxed{\overset{\circ}{\nabla} = \frac{1}{2}\left(\overset{L}{\nabla} + \overset{R}{\nabla}\right)}$$

$$\overset{\circ}{\nabla}_{\vec{\ell}_A} \vec{\ell}_B = \frac{1}{2} \overset{R}{\nabla}_{\vec{\ell}_A} \vec{\ell}_B$$

$$= \frac{1}{2} \overset{R}{\nabla}_{\vec{\ell}_A} \Lambda^B{}_A \vec{r}_B$$

$$= \frac{1}{2} \vec{\ell}_A \left(\Lambda^K{}_B\right) \vec{r}_K$$

$$= \frac{1}{2} C^K{}_{BA} \vec{\ell}_K$$

$$\overset{\circ}{\omega}{}^A{}_B = \frac{1}{2} C^A{}_{BC} \underline{\theta}^C$$

$$\boxed{\begin{aligned} \overset{\circ}{\Omega}{}^A{}_B &= \frac{1}{2} \overset{\circ}{R}{}^A{}_{BCD} \underline{\theta}^C \wedge \underline{\theta}^D \\ &= \frac{1}{4} C^A{}_{BK} C^K{}_{CD} \underline{\theta}^C \wedge \underline{\theta}^D \end{aligned}}$$

INVARIANT METRIC

$$[\vec{l}_A, \vec{l}_B] = -C_{AB}^{\ \ K}\, \vec{l}_K = \mathcal{L}_{\vec{l}_A} \vec{l}_B$$

$$\Rightarrow \mathcal{L}_{\vec{l}_A} \underline{\theta}^B = C_{AK}^{\ \ B}\, \underline{\theta}^K \ ; \ \mathcal{L}_{\vec{r}_A} \underline{\theta}^B = 0$$

Metric : $\underline{\underline{g}} = g_{AB}\, \underline{\theta}^A \otimes \underline{\theta}^B$

Left invariant : $\mathcal{L}_{\vec{r}_A} \underline{\underline{g}} = 0 \Leftrightarrow g_{AB} = C^{te}$

Right invariant : $\mathcal{L}_{\vec{l}_A} \underline{\underline{g}} = 0$

$$\vec{l}_A (g_{PQ}) + g_{PK}\, C_{AQ}^{\ \ K} + g_{KQ}\, C_{AP}^{\ \ K} = 0$$

Bi-invariant:

$$g_{PK}\, C_{AQ}^{\ \ K} + g_{KQ}\, C_{AP}^{\ \ K} = 0$$

$$\boxed{\overset{\circ}{\nabla} \underline{\underline{g}} = 0} \qquad C_{APQ} = C_{[APQ]}$$

$$C_{AP}^{\ \ P} \equiv 0$$

SEMISIMPLE GROUPS

$$G = G_1 \times G_2 \ \cdots \times G_N$$

$$\mathcal{B} = \lambda^1 \mathcal{B}_1 + \lambda^2 \mathcal{B}_2 \quad + \lambda^N \mathcal{B}_N$$

"Killing metric" $\mathcal{B}_{AB} = C^L_{AK}\, C^K_{BL}$

Einstein space (piecewise)

$$\overset{\circ}{R}_{AB} = \frac{1}{4}\, C^P_{AQ}\, C^Q_{PB} = -\frac{1}{4}\mathcal{B}_{AB}$$

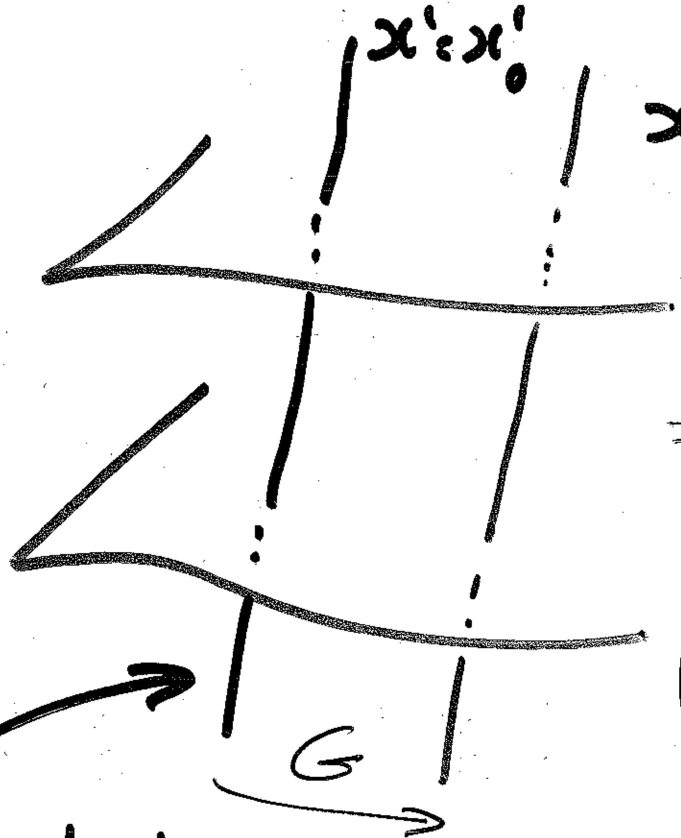
HOMOGENEOUS COSMOLOGIES

Canonical metric

$$ds^2 = -N^2 (dx^0)^2 + g_{kl}(N^k dx^0 + dx^k)(N^l dx^0 + dx^l)$$

Transiticity surfaces $\quad x^0 = C^{te}$

$$\mathcal{L}_{\vec{\xi}_{(i)}} ds^2 = 0 \quad \Rightarrow \begin{cases} g_{kl} \quad \text{invariant metric} \\ N_k := g_{0k} \left\{ \overset{(3)}{\mathcal{L}}_{\xi_{(i)}} N_k + g_{kl}\, \partial_0 \xi^l_{(i)} = 0 \right. \\ \partial_{x^i} N = 0 \Rightarrow N = \mathcal{N}(x^0) \end{cases}$$



Arbitrary curve

$$\Rightarrow \partial_0 \xi^l_{(i)} = 0$$

$$\Downarrow$$

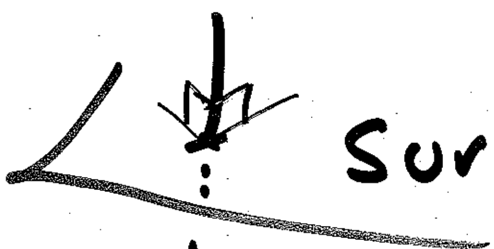
$$\left({}^{(3)}\mathcal{L}_{\vec{\xi}_{(i)}} N\right)_k = 0$$

$$\Downarrow$$

$$N = \nu_k(x^0)\,\frac{\theta^k}{4}$$

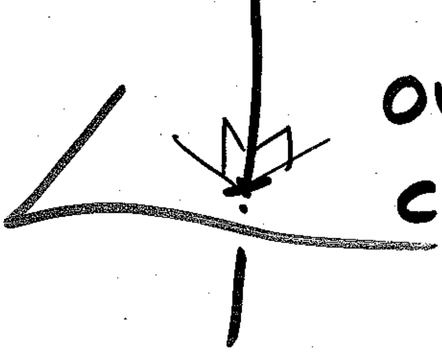
G-invariant forms

$$ds^2 = -\mathcal{N}^2(x^0)(dx^0)^2 + \gamma_{kl}(x^0)\left(\nu^k(x^0)\,dx^0 + \underline{\theta}^k\right)\left(\nu^l(x^0)\,dx^0 + \underline{\theta}^l\right)$$



Surface

$$dt = \mathcal{N}(x^0)\,dx^0$$



orthogonal curve

$$\boxed{ds^2 = -dt^2 + \gamma_{kl}(t)\,\underline{\theta}^k\underline{\theta}^l}$$

BIANCHI 3-D REAL LIE ALGEBRA

$$[\vec{\xi}_1, \vec{\xi}_2] = n^3 \vec{\xi}_3 + a \vec{\xi}_2$$

$$[\vec{\xi}_2, \vec{\xi}_3] = n^1 \vec{\xi}_1$$

$$[\vec{\xi}_3, \vec{\xi}_1] = n^2 \vec{\xi}_2 - a \vec{\xi}_3$$

TYPE	a	n^1	n^2	n^3		
					CLASS A	
I	0	0	0	0	Translation	sol
II	0	1	0	0	Galilean(*)	nil
VII_0	0	1	1	0	Euclidean	sol
VI_0	0	1	-1	0	Poincaré	sol
IX	0	1	1	1	Rotation	simple
VIII	0	1	1	-1	Lorentz	simpl
					CLASS B	
V	1	0	0	0		sol
IV	1	0	1	0		sol
VII_a	a>0	0	1	1		sol
III_1, $VI_{a\neq 1}$	a>0	0	1	-1		sol

(*) Also called Heisenberg Lie algebra

EXAMPLE Type III

$$[\vec{\xi}_1, \vec{\xi}_2] = \vec{\xi}_2 - \vec{\xi}_3 \;, \quad [\vec{\xi}_2, \vec{\xi}_3] = 0, \quad [\vec{\xi}_3, \vec{\xi}_1] = \vec{\xi}_2 - \vec{\xi}_3$$

$$\Downarrow$$

$$\vec{\xi}_2 = \partial_2 \qquad \vec{\xi}_3 = \partial_3$$

$$\vec{\xi}_1 = a\,\partial_1 + b\,\partial_2 + c\,\partial_3$$

$$-a_{,2}\partial_1 - b_{,2}\partial_2 - c_{,2}\partial_3 = \partial_2 - \partial_3$$

$$a_{,3}\partial_1 + b_{,3}\partial_2 + c_{,3}\partial_3 = \partial_2 - \partial_3$$

$$a = A(x^1) \;, \quad b = B(x^1) - x^2 + x^3, \quad c = C(x^1) + x^2 - x^3$$

Extra coordinate freedom: **leftgenerators**

$$\tilde{x}^1 = f(x^1) \qquad \tilde{x}^2 = x^2 + g(x^1) \qquad \tilde{x}^3 = x^3 + h(x^1)$$

$$* \quad \vec{\xi}_2 = \partial_{\tilde{x}^2} \qquad \vec{\xi}_3 = \partial_{\tilde{x}^3}$$

$$\vec{\xi}_1 = A\left[f'\,\partial_{\tilde{x}^1} + g'\,\partial_{\tilde{x}^2} + h'\,\partial_{\tilde{x}^3} \right] + \left[B + g - \tilde{x}^2 - h + \tilde{x}^3 \right]\partial_{\tilde{x}^2} + \left[C - g + \tilde{x}^2 + h - \tilde{x}^3 \right]\partial_{\tilde{x}^3}$$

$$A f' = 1, \quad A h' + B + g - h = 0, \quad A h' + C - g + h = 0$$

$$* \quad \vec{\xi}_1 = \partial_{\tilde{x}^1} + (x^3 - x^2)\left(\partial_{\tilde{x}^2} - \partial_{\tilde{x}^3}\right)$$

Remaining coordinate freedom

$$x'^1 = x^1 + \alpha$$

$$x'^2 = \beta e^{-2x^1} + \gamma$$

$$x'^3 = -\beta e^{-2x^1} + \gamma$$

3 - parameter group

$$\vec{Y}_\alpha = \partial_1, \quad \vec{Y}_\beta = e^{-2x^1}(\partial_2 - \partial_3), \quad \vec{Y}_\gamma = \partial_2 + \partial_3$$

$$[\vec{Y}_{(i)}, \vec{\xi}_{(j)}] = 0$$

INVARIANT FRAME
(defined up to linear transfor.)

At $x^i = 0$: $\vec{Y}_{(i)} = \vec{\xi}_{(i)}$

$$\vec{X}_{(1)} = \vec{Y}_\alpha$$

$$[\vec{X}_A, \vec{X}_B] = -C^C_{AB}\vec{X}_C$$

$$\vec{X}_{(2)} = \frac{1}{2}(\vec{Y}_\beta + \vec{Y}_\gamma)$$

Right generators

$$\vec{X}_{(3)} = \frac{1}{2}(\vec{Y}_\gamma - \vec{Y}_\beta)$$

COSET SPACES GEOMETRY

- G n-dim. Lie group

  H p-dim closed subgroup of G

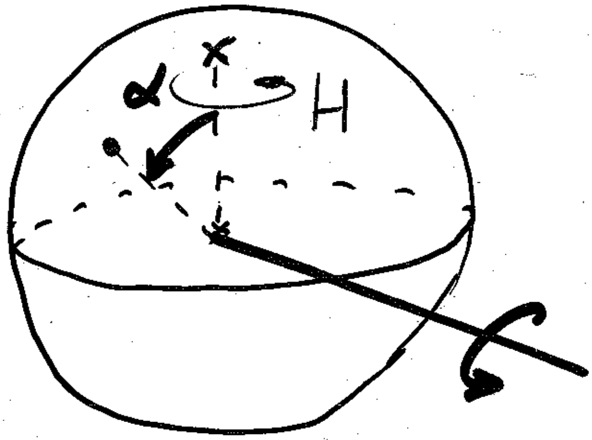
$$G/H = \{ [g] \mid [g] = \{ g' \in G \mid g = g'h, h \in H \} \}$$

- Example $SO(3)/SO(2) = S^2$
- Gauge choice V: neighborhood of $[Id]$

  $V \ni [g] \ni L_g$ Select an element in the class

$$\left. \begin{aligned} g &= L_g h_g \\ g' &= L_{g'} h_{g'} \end{aligned} \right\} \quad [g] = [g'] \text{ iff } L_g = L_{g'}$$

$$\left(g = g' \text{ iff } \begin{aligned} L_g &= L_{g'} \\ h_g &= h_{g'} \end{aligned} \right)$$



Coordinates

On G : x^A_g } $x^A_{g_1 * g_2} = \varphi^A[x_{g_1}, x_{g_2}]$

$\{L\}$ submanifold of dimension $n-p$ of G.

- $\exists\ Y^A{}_{A=1,\dots,n}$, y^α $\alpha = 1,\dots, n-p$
  ("embedding" $L \to G$) (coord. on L)

  $x^A = Y^A[y^\alpha] \Rightarrow \exists\ L$ } $x^A = x^A_L$

  corresponding generators T_α

- $\exists\ y^\alpha$ } $Y^A[y^\alpha(x_g)] = x^A_{L_g}$ $\quad \overset{\Phi}{\mathcal{P}}$
  ("projection" $G \to L$)

- $\exists\ U^A$ } $U^A[x_g] = x^A_{h_g}$

On H : ξ^i_h $\quad i = 1,\dots, p$ $\quad$ Liealg

$$\xi^i_{h_1 * h_2} = \psi^i[\xi_{h_1}, \xi_{h_2}] \Rightarrow \begin{matrix}\mathcal{H} \\ \cup \\ T_i\end{matrix}$$

$$\mathcal{G} = \mathcal{P} \oplus \mathcal{H}$$

Action of G on G/H

$$G \times G/H \longrightarrow G$$

$$(\bar{g}, [g]) \longmapsto [\bar{g}g]$$

$$\bar{g}\, L[y] = L[y']\, \bar{h}[y]$$

Infinitesimal action

$$\bar{g} = \mathrm{Id} + \epsilon^A T_A$$

$$y'^\alpha = y^\alpha + \epsilon^A K_A^\alpha(y)$$

$$\bar{h}[y] = \mathrm{Id} + \epsilon^A \Omega_A^i(y)\, T_i$$

$$\boxed{T_A L[y] = K_A^\alpha \frac{\partial L}{\partial y^\alpha}[y] + \Omega_A^i L[y] T_i}$$

$$|||$$

$$T_A L[y] = \vec{K}_A(L)[y] + \Omega_A^i L[y] T_i$$

$$\varepsilon^A \vec{K}_A = \vec{K}_\varepsilon \qquad \varepsilon^A \Omega_A^i T_i = \Omega_\varepsilon$$

Calculation

- $(\mathrm{Id} + \varepsilon^A T_A)\, L[y] = L[y + \vec{K}_\varepsilon(y)]\,(\mathrm{Id} + \Omega_\varepsilon(y))$

- $(\underset{0}{\mathrm{Id}} + \sigma^B T_B)(\underset{0}{\mathrm{Id}} + \varepsilon^A T_A)\, L[y] =$

$$L[y + \vec{K}_\sigma(y) + \vec{K}_\varepsilon(y + \vec{K}_\sigma(y))]$$

$$(\mathrm{Id} + \Omega_\sigma[y + \vec{K}_\varepsilon(y)])(\mathrm{Id} + \Omega_\varepsilon(y))$$

$$= \left(L + \vec{K}_\sigma(L) + \vec{K}_\varepsilon(L) + \vec{K}_\varepsilon(\vec{K}_\sigma(L))\right)$$

$$(\mathrm{Id} + \Omega_\sigma + \vec{K}_\varepsilon(\Omega_\sigma))(\mathrm{Id} + \Omega_\varepsilon)$$

$$= \left(L + \vec{K}_\sigma(L) + \vec{K}_\varepsilon(L) + \vec{K}_\varepsilon(\vec{K}_\sigma(L))\right)$$

$$(\mathrm{Id} + \Omega_\sigma + \Omega_\varepsilon + \Omega_\sigma \Omega_\varepsilon + \vec{K}_\varepsilon(\Omega_\sigma))$$

$$= \underset{0}{L} + \vec{K}_\sigma(L) + \vec{K}_\varepsilon(L) + \vec{K}_\varepsilon(\vec{K}_\sigma(L))$$
$$+ L(\Omega'_\sigma + \Omega'_\varepsilon) + (\vec{K}_\sigma(L) + \vec{K}_\varepsilon(L))(\Omega_\sigma + \Omega_\varepsilon)$$
$$+ L\, \vec{K}'_\varepsilon(\Omega'_\sigma) + L\, \Omega_\sigma \Omega_\varepsilon$$

$$[T_B, T_A]\, L = C^c_{BA}\big(\underbrace{\vec{K}_c(L)}_{\mathcal{D}} + \underbrace{L\,\Omega_c}_{\mathcal{H}}\big)$$

$$= \underbrace{[\vec{K}_A, \vec{K}_B](L)}_{\mathcal{D}} + \underbrace{L\,\vec{K}_A(\Omega_B) - L\,\vec{K}_B(\Omega_A)}_{\mathcal{H}\qquad\mathcal{H}}$$

$$+ \underbrace{L\,[\Omega_B, \Omega_A]}_{\mathcal{H}}$$

$$\boxed{\begin{aligned} &[\vec{K}_A, \vec{K}_B](L) = C^c_{BA}\,\vec{K}_c(L) \quad (\mathcal{D}) \\ &\vec{K}_A(\Omega^i_B) - \vec{K}_B(\Omega^i_A) + C^i_{jk}\,\Omega^j_B\,\Omega^k_A \\ &= C^c_{BA}\,\Omega^i_c \qquad (\mathcal{H}) \end{aligned}}$$

REDUCTIVE ALGEBRA

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{P}, \quad [\mathcal{H}, \mathcal{P}] \subset \mathcal{P}$$

$$\{T_A\} = \{T_i\} \cup \{T_\alpha\}$$

$$C_{\alpha i}^{\ \ j} = 0$$

- Let $g_{AB} := \langle T_A, T_B \rangle$ metric on $\mathcal{G}$
- $\mathcal{H}$-invariant metric

$$\langle [T_i, T_A], T_B \rangle + \langle T_A, [T_i, T_B] \rangle = 0$$

$$C_{iAB} + C_{iBA} = 0$$

$$\Rightarrow C_{iAB} = C_{[iAB]}$$

- If $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$ with $\mathcal{H} \perp \mathcal{P}$: $g_{\alpha i} = 0$

$g_{AB} = \begin{pmatrix} \ast & 0 \\ 0 & \ast \end{pmatrix}$: block diagonal

$$C_{\alpha i}^{\ \ j} = (g^{-1})^{jk} C_{\alpha i k} = (g^{-1})^{jk} C_{ik\alpha}$$

$$= (g^{-1})^{jk} g_{\alpha\beta} \underbrace{C_{ik}^{\ \ \beta}}_{=0} = 0$$

$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$

Moving frame (à la Cartan)

$$\underline{\Sigma}(y) = L^{-1}(y)\, dL(y)$$

$$L(y') = \bar{g}\, L(y)\, \bar{h}^{-1}(y)$$

$$\underline{\Sigma}(y') = \bar{h}(y)\, \underline{\Sigma}(y)\, \bar{h}^{-1}(y) + \bar{h}(y)\, d\bar{h}^{-1}(y)$$

Lie derivative $y' = y + \vec{K}_B(y)$

$$\boxed{\begin{aligned} \underline{\Sigma}(y') - \underline{\Sigma}(y) &= \mathcal{L}_{\vec{K}_A} \underline{\Sigma}(y) \\ &= -d\Omega_A + [\Omega_A, \underline{\Sigma}] \end{aligned}}$$

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$$

$$\underline{\Sigma} = \underline{\Sigma}^A T_A = \underset{\uparrow}{\underline{\theta}^\alpha} T_\alpha + \underline{\varpi}^i T_i$$

Reductive space moving frame

$$\boxed{\mathcal{L}_{\vec{K}_A} \underline{\theta}^\alpha = \Omega_A^i C_{i\beta}^\alpha \underline{\theta}^\beta, \quad \mathcal{L}_{\vec{K}_A} \underline{\varpi}_i = -d\Omega_A^i + C_{jk}^i \Omega_A^j \underline{\varpi}^k}$$

Let $\gamma_{\alpha\beta}$ the components of the Killing metric restricted on $\mathcal{P}$

$\Rightarrow C_{i\alpha\beta} = C_{[i\alpha\beta]}$

Metric : $\underline{\underline{\gamma}} = \gamma_{\alpha\beta} \, \underline{\theta}^\alpha \otimes \underline{\theta}^\beta$

$$\left(\mathcal{L}_{\vec{K}_A} \underline{\underline{\gamma}} \right)_{\alpha\beta} = \vec{K}_A(\gamma_{\alpha\beta}) + 2 \, \Omega_A^i \, C_{i(\alpha\beta)}$$

$$= 0 \qquad G\text{-invariant}$$

Symmetric spaces

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$$

$$[\mathcal{H},\mathcal{H}] \subset \mathcal{H} \qquad [\mathcal{H},\mathcal{P}] \subset \mathcal{P}$$

$[\mathcal{P},\mathcal{P}] \subset \mathcal{H}$: symmetric

θ : involutive automorphism

$$\theta|_{\mathcal{H}} = +1 \qquad \theta|_{\mathcal{P}} = -1$$

$$\underline{\theta}^{\alpha} T_\alpha = \frac{1}{2}\left(L^{-1}dL - \theta(L^{-1}dL)\right)$$

$$\underline{\omega}^{i} T_i = \frac{1}{2}\left(L^{-1}dL + \theta(L^{-1}dL)\right)$$

$$C_{ij}^{\ \ \alpha} = 0\,, \quad C_{i\alpha}^{\ \ j} = 0\,, \quad C_{\alpha\beta}^{\ \ \gamma} = 0$$

Connection $\quad \underline{\Sigma} = L^{-1} dL$

$$\underline{d}\wedge\underline{\Sigma} = -\underline{\Sigma}\wedge\underline{\Sigma}$$

$$d\underline{\theta}^\alpha = -\frac{1}{2} C^\alpha_{\beta\gamma}\, \underline{\theta}^\beta \wedge \underline{\theta}^\gamma - C^\alpha_{j\beta}\, \underline{\varpi}^j \wedge \underline{\theta}^\beta$$

$$d\underline{\omega}^i = -\frac{1}{2} C^i_{jk}\, \underline{\varpi}^j \wedge \underline{\varpi}^k - C^i_{j\beta}\, \underline{\varpi}^j \wedge \underline{\theta}^\beta - \frac{1}{2} C^i_{\alpha\beta}\, \underline{\theta}^\alpha \wedge \underline{\theta}^\beta$$

TORSIONLESS CONNECTION

$$\underline{\omega}^\alpha{}_\beta := -\frac{1}{2} C^\alpha_{\beta\gamma}\, \underline{\theta}^\gamma + C^\alpha_{j\beta}\, \underline{\varpi}^j$$

If $\gamma_{\mu\alpha}$ is $\mathcal{H}$-invariant: $C_{j\beta\alpha} = C_{j\beta\alpha}$

If $\mathfrak{g} = \mathcal{H} \oplus \mathcal{P}$ is reductive: $C^\tau_{\beta\gamma} = 0$

$$\underline{\omega}_{\alpha\beta} := \gamma_{\alpha\mu}\, \underline{\omega}^\mu{}_\beta = C_{j\beta\alpha}\, \underline{\varpi}^j$$

$$= -\underline{\omega}_{\beta\alpha}$$

LEVI-CIVITA CONNECTION

Curvature

$$\Omega^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$$
$$= \frac{1}{2} R^\alpha{}_{\beta\mu\nu} \underline{\theta}^\mu \wedge \underline{\theta}^\nu$$

$\underline{\omega}^i \wedge \underline{\omega}^j$: vanish in virtue of the Jacobi identity and $C^\alpha_{ij} = 0$

$\underline{\omega}^\rho \wedge \underline{\theta}^\mu$: $\sim \frac{1}{2}\left(C^\alpha_{i\beta} C^i_{\mu\rho} + C^\alpha_{i\mu} C^i_{\beta\rho}\right)$

vanish on a reductive space ($C^\delta_{\beta i} = 0$)

$\underline{\theta}^\mu \wedge \underline{\theta}^\nu$

$$\boxed{\frac{1}{8}\left(C^\alpha_{\gamma\mu} C^\gamma_{\beta\nu} - C^\alpha_{\gamma\nu} C^\gamma_{\beta\mu} - 2 C^\alpha_{\gamma\beta} C^\gamma_{\mu\nu} - 4 C^\alpha_{i\beta} C^i_{\mu\nu}\right)}$$

$$C^\alpha_{ij}=0\ ,\quad C^j_{i\alpha}=0\ ,\quad C^\gamma_{\alpha\beta}=0$$

$$\underline{\omega}^\alpha{}_\beta = C^\alpha_{\gamma\beta}\,\underline{\varpi}^\gamma$$

$$R^\alpha{}_{\beta\mu\nu} = -\tfrac{1}{2}\, C_i{}^\alpha{}_\beta\, C^i_{\mu\nu}$$

$$(\underline{\nabla}\,\mathrm{Riem})^\alpha{}_{\beta\mu\nu\rho} = R^\alpha{}_{\beta\mu\nu;\rho}$$

$$= \Big(C^\alpha_{\gamma\sigma}R^\sigma{}_{\beta\mu\nu} - R^\alpha{}_{\sigma\mu\nu}C^\sigma_{\gamma\beta} - R^\alpha{}_{\beta\sigma\nu}C^\sigma_{\gamma\mu} - R^\alpha{}_{\beta\mu\sigma}C^\sigma_{\gamma\nu}\Big)\,\varpi^\gamma{}_\rho$$

$$= -\tfrac{1}{2}\Big(C^\alpha_{\gamma\sigma}C^\sigma_{i\beta}\underbrace{C^i_{\mu\nu}} - C^\alpha_{i\sigma}\underbrace{C^i_{\mu\nu}}C^\sigma_{\gamma\beta} - \underbrace{C^\alpha_{i\beta}}\,\underbrace{C^i_{\sigma\nu}C^\sigma_{\gamma\mu}} - \underbrace{C^\alpha_{i\beta}}\,\underbrace{C^i_{\mu\sigma}C^\sigma_{\gamma\nu}}\Big)\,\varpi^\gamma{}_\rho$$

$$C^\alpha_{I\beta}\,C^I_{S\gamma}\,\underbrace{C^S_{\mu\nu}} \qquad \text{Jacobi } (\nu\gamma\mu)$$

$$= -\tfrac{1}{2}\Big(C^\alpha_{\gamma\sigma}C^\sigma_{I\beta} + C^\alpha_{I\sigma}C^\sigma_{\beta\gamma} + C^\alpha_{S\beta}C^S_{I\gamma}\Big)\underbrace{C^{I\varpi}_{\mu\nu}}$$

$$= 0 \quad (\text{Jacobi})$$

$$\boxed{\nabla\,\mathrm{Riem} = 0}$$

Remarks

1°) $\nabla \text{Riem} = 0$ is obvious

If $T_{\alpha_1 \dots \alpha_{2n+1}}$ is an invariant tensor

it is equal to zero!

2°) $R^{\alpha}{}_{\beta\mu\nu} = -\frac{1}{2} C^{\alpha}{}_{i\beta} C^{i}{}_{\mu\nu}$

$R^{\alpha}{}_{\beta\alpha\nu} = -\frac{1}{2} C^{\alpha}{}_{i\beta} C^{i}{}_{\alpha\nu}$

$= -\frac{1}{2} C^{\alpha}{}_{I\beta} C^{I}{}_{\alpha\nu}$

Killing metric: $\gamma_{\alpha\beta} = - C^{P}{}_{\alpha Q} C^{Q}{}_{\beta P}$

$= - C^{j}{}_{\alpha\rho} C^{\rho}{}_{\beta j} - C^{\mu}{}_{\alpha j} C^{j}{}_{\beta\mu} = -2 C^{\mu}{}_{\alpha j} C^{j}{}_{\beta\mu}$

$R_{\beta\nu} = \frac{1}{4} \gamma_{\beta\nu}$

GEODESIC EQUATIONS

$$\nabla_{\vec{u}} \vec{u} = \vec{0}$$

$$\vec{u} = \dot{y}^{\alpha} \partial_{\alpha} = \dot{y}^{\alpha} \theta_{\alpha}^{\hat{\rho}} \vec{e}_{\hat{\rho}}$$

$$\hookrightarrow \underline{\theta}^{\hat{\rho}} = \theta^{\hat{\rho}}{}_{\alpha} dy^{\alpha}$$

$$\nabla \vec{e}_{\hat{\rho}} = \vec{e}_{\hat{\beta}} \otimes \underline{\omega}^{\hat{\beta}}{}_{\hat{\rho}} = \vec{e}_{\hat{\beta}} C^{\hat{\beta}}_{j\hat{\rho}} \underline{\varpi}^{j}$$

$$\frac{d}{dt}\left(\dot{y}^{\alpha} \theta_{\alpha}^{\hat{\rho}}\right) + \underbrace{\omega^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}} \theta^{\hat{\mu}}_{\alpha} \theta^{\hat{\nu}}_{\beta}}_{C^{\hat{\rho}}_{i\hat{\mu}} \theta^{\hat{\mu}}_{\alpha} \varpi^{i}_{\beta}} \dot{y}^{\alpha} \dot{y}^{\beta} = 0$$

$$\hookrightarrow \underline{\varpi}^{i} = \varpi^{i}_{\hat{\nu}} \underline{\theta}^{\hat{\nu}}$$

$$= \varpi^{i}_{\hat{\nu}} \theta^{\hat{\nu}}_{\beta} dy^{\beta}$$

$$= \varpi^{i}{}_{\beta} dy^{\beta}$$

$$\mathcal{P} = \theta^{\hat{a}}_{\mu} \dot{y}^{\mu} T_{\hat{a}} \qquad \mathcal{Q} = \varpi^{i}{}_{\mu} \dot{y}^{\mu} T_{i}$$

$$\boxed{\frac{d}{dt}\mathcal{P} + [\mathcal{Q}, \mathcal{P}] = 0}$$

SYMMETRIC SPACES
REAL LIE ALGEBRAS

Cartan-Weyl basis

$$[H, E_\alpha] = \alpha(H)\, E_\alpha$$ Cartan subalgebra $\mathcal{H}$

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}\, E_{\alpha+\beta} & \alpha+\beta \in \Delta \\ H_\alpha & \alpha+\beta = 0 \\ 0 & \alpha+\beta \notin \Delta \end{cases}$$

Killing form $\quad B(X,Y) = \mathrm{Tr}(\mathrm{ad}X\, \mathrm{ad}Y)$

- $\forall H \in \mathcal{H} \qquad \alpha(H) = B(H_\alpha, H)$

- $B(E_\alpha, E_\beta) = \delta_{\alpha+\beta,0}$

- $(\alpha|\beta) = B(H_\alpha, H_\beta)$

$$= \sum_{\gamma\in\Delta} (\alpha|\gamma)(\gamma|\beta) \in \mathbb{Q}$$

- $N_{\alpha,\beta} = -N_{\beta,\alpha} = -N_{-\alpha,-\beta} = N_{\beta,-\alpha-\beta}$

$$N^2_{\alpha,\beta} = \frac{1}{2}\, q\,(p+1)\,(\alpha|\alpha) \qquad p, q \in \mathbb{N}$$

$$\boxed{N_{\alpha,\beta} \in \mathbb{R}}$$

$\dim_{\mathbb{C}} \mathcal{G} = \frac{1}{2} \dim_{\mathbb{R}} (\mathcal{G})^{\mathbb{R}}$ realification

$\mathcal{G}_0$ real Lie algebra

- $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_0 \otimes_{\mathbb{R}} \mathbb{C}$ complexification
- $\mathcal{G}^{\mathbb{R}} = \mathcal{G}_0 \oplus i\,\mathcal{G}_0$ real form

Split real form

$$\mathcal{S}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R} H_\alpha \bigoplus_{\alpha\in\Delta} \mathbb{R} E_\alpha$$

NONCOMPACT GENERATORS

$$\{ H_\alpha,\ E_\alpha + E_{-\alpha} \mid \alpha\in\Delta \}$$

$\frac{1}{2}(\dim \mathcal{S}_0 + \operatorname{rank} \mathcal{S}_0)$

Killing form signature:

$\left(\frac{1}{2}(\dim \mathcal{S}_0 + \operatorname{rank} \mathcal{S}_0)_+,\ \frac{1}{2}(\dim \mathcal{S}_0 - \operatorname{rank} \mathcal{S}_0)_-\right)$

Compact (max.) real form

$$\mathfrak{U}_0 = \bigoplus_{\alpha\in\Delta} \mathbb{R}(iH_\alpha) \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}(E_\alpha - E_{-\alpha})$$

$$\oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}i(E_\alpha + E_{-\alpha})$$

Killing form negative definite

Example: $sl(2,\mathbb{C})$

$sl(2,\mathbb{R})$

- $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$su(2)$

- $\tau^x = i(e+f) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\tau^y = (e-f) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\tau^z = ih = \begin{pmatrix} i & 0 \\ 0 & - \end{pmatrix}$

Basis: $\{e, f, h\}$ of $sl(2,\mathbb{R})$

$$\mathrm{ad}\, h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathbb{R}h \text{ Cartan subalg.}$$

$\cancel{\mathbb{R}}\ sl(2,\mathbb{R})$

$$\mathrm{ad}(e-f) = \begin{pmatrix} 0 & 1 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \Rightarrow \mathbb{R}\tau^y \text{ Cartan subalg}$$

$\left\{\exp th = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\right\} = A$ noncompact

subgroups

$\left\{\exp t\tau^y = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\right\} = K$ compact

$sl(2,\mathbb{R})$ $\qquad\qquad$ $su(2)$

$t = e - f = \tau^y$ $\qquad\qquad$ $\tau^x = i x$

$x = e + f = -i\tau^x$ $\qquad\qquad$ $\tau^y = t$

$h = -i\tau^z$ $\qquad\qquad$ $\tau^z = i h$

noncompact generators of $sl(2,\mathbb{R})$
are imaginary in terms of those of $su(2)$

CONJUGATION

$\sigma(X) = \bar{X}$ $\qquad\qquad$ $\tau(X) = -X^{+}$

pointwise

fixes $sl(2,\mathbb{R})$ $\qquad\qquad$ fixes $su(2)$

$$[\sigma, \tau] = 0$$

globally

fixes $su(2)$ $\qquad\qquad$ fixes $sl(2,\mathbb{R})$

$\mathcal{K}_0 = \mathbb{R}t$ subalg. $\qquad\qquad$ $[t, x \pm i h] = \pm 2i(x \pm i h)$

$\mathcal{P}_0 = \mathbb{R}x + \mathbb{R}h$ $\qquad\qquad$ imaginary noncompact root

$\tau|_{\mathcal{K}} = +1$ $\qquad\qquad$ $[(x+ih),(x-ih)] = 4i\,t$

$\tau|_{\mathcal{P}} = -1$

HERMITIAN FORM ON $sl(2,\mathbb{C})$

$$X \bullet Y = -\mathrm{Tr}(Y\,\tau(X))$$

CARTAN INVOLUTION

$$\theta = \sigma\tau$$

IWASAWA DECOMPOSITION

$$\boxed{\underset{K}{\{\operatorname{Exp}(\theta t)}\ \underset{A}{\operatorname{Exp}(a h)}\ \underset{N}{\operatorname{Exp}(n e)\}} = SL(2,\mathbb{R})}$$

$$= \left\{ \begin{pmatrix} e^{a}\cos\theta & n e^{a}\cos\theta + e^{-a}\sin\theta \\ -e^{a}\sin\theta & e^{-a}\cos\theta - n e^{a}\sin\theta \end{pmatrix} \right\}$$

CARTAN DECOMPOSITION

$$\mathfrak{P} = \{ \underset{\in \mathbb{R}}{\rho} (\cos\alpha\, h + \sin\alpha\, x) \} = \{ \rho \operatorname{Exp}(\tfrac{\alpha}{2} t)\, h \operatorname{Exp}(-\tfrac{\alpha}{2} t) \}$$

$$= \operatorname{Ad}(K)\,\mathfrak{a} \qquad (\mathfrak{a} = \mathbb{R} h)$$

$$\boxed{SL(2,\mathbb{R}) = K A K}$$

Conjugation ~ Real form

$$\mathcal{G} = \mathcal{G}_0 + i\,\mathcal{G}_0$$

$$\sigma : \overset{\Psi}{Z} = X_0 + i\,Y_0 \longmapsto \overline{Z} = X_0 - i\,Y_0$$

MAIN THEOREM

- $\begin{cases} \mathcal{U}_0 & \text{a maximally compact real form} \\ \tau & \text{the conjugation it defines} \end{cases}$

- $\begin{cases} \mathcal{G}_0 & \text{a real form} \\ \sigma & \text{the conjugation it defines} \end{cases}$

- $\begin{cases} \lambda = \sigma\tau & \text{an automorphism of } \mathcal{G} \\ B(\lambda Z, \lambda Z') = B(Z, Z') \end{cases}$

- $B^\tau(Z, Z') = -B_{\underset{= \mathcal{U}_0 \oplus i\mathcal{U}_0}{\mathcal{G}^{\mathbb{R}}}}(Z, \tau Z')$ HERMITIAN FORM

$$= -2\left(B_{\mathcal{U}_0}(X_0, X_0') + B_{\mathcal{U}_0}(Y_0, Y_0')\right) \geq 0$$

- $B^\tau(\lambda Z, Z') = -B(\sigma\tau Z, \tau Z') = -B(Z, \tau\sigma\tau Z')$

$= B^\tau(Z, \lambda Z')$: λ IS SYMMETRIC

- $\lambda^2 = \rho$ IS POSITIVE

- ρ^t ONE-PARAMETER AUTOMORPHISM GROUP OF $\mathcal{G}_0$

- $\rho^t \tau = \tau \rho^{-t}$
  - $\rho\lambda = \lambda\rho$ λ diagonalle
  - $\rho\tau = \lambda^2\tau = \sigma\tau\sigma\tau\underbrace{\tau}_{=1}$
    $= \tau\tau\sigma\tau\sigma = \tau\rho^{-1}$
  - $\rho_{ii}^t \tau_{ij} = \tau_{ij} \rho_{jj}^{-1t}$

- $\rho^{1/4} \tau \rho^{-1/4}\sigma = \rho^{1/2} \tau \sigma = \rho^{-1/2} \rho\tau\sigma$

$= \rho^{-1/2} \underbrace{\sigma\tau}_{\lambda} = \sigma\tau\rho^{-1/2} = \sigma\rho^{1/4}\tau\rho^{-1/4}$

$\tilde{\tau} = \rho^{1/4}\tau\rho^{-1/4}$ commutes with σ

$\tilde{\tau}$ defines $\rho^{1/4}[\mathcal{U}_0] = \tilde{\mathcal{U}}_0$

- $\tilde{\tau}$ fixes $\mathcal{G}_0$
  - $\sigma X = X$ iff $X \in \mathcal{G}_0$
  - $\forall X \in \mathcal{G}_0$ $\tilde{\tau} X = \tilde{\tau}\sigma X = \sigma\tilde{\tau} X$

$$\boxed{\text{$\mathcal{G}_0$ is always aligned with a maximally compact real form $\mathcal{U}_0$}}$$

then: $\theta = \sigma\tilde{\tau}$ is an involutive automorphism

Cartan decomposition

$$X = \frac{1}{2}\left(X + \theta(X)\right) + \frac{1}{2}\left(X - \theta(X)\right)$$

$$\begin{array}{cccc} \mathfrak{g}_0 = & \underset{\text{compact}}{\mathfrak{K}_0} & \oplus & \underset{\text{non-compact}}{\mathfrak{P}_0} \\ \theta & +1 & & -1 \end{array}$$

$$\mathfrak{u}_0 = \mathfrak{K}_0 \oplus i\mathfrak{P}_0$$

$[\mathfrak{K}_0, \mathfrak{K}_0] \subset \mathfrak{K}_0$ ad $\mathfrak{K}_0$ ad $\mathfrak{P}_0$:

$[\mathfrak{K}_0, \mathfrak{P}_0] \subset \mathfrak{P}_0$ $\mathfrak{K}_0 \to \mathfrak{P}_0$, $\mathfrak{P}_0 \to \mathfrak{K}_0$

$[\mathfrak{P}_0, \mathfrak{P}_0] \subset \mathfrak{K}_0$ $B(\mathfrak{K}_0, \mathfrak{P}_0) = 0$

$$\boxed{\begin{array}{c} \mathfrak{K}_0 \perp \mathfrak{P}_0 \\[2em] \mathrm{ad}(\theta X) = -(\mathrm{ad}\, X)^t \end{array}}$$

Proof: $B^\theta(\mathrm{ad}(\theta X)Y, Z) = -B([\theta X, Y], \theta Z) = B(Y, [\theta X, \theta Z])$
$= B(Y, \theta[X, Z]) = -B^\theta(Y, [X, Z]) = -B^\theta(Y, \mathrm{ad}\, X\, Z)$

Iwasawa decomposition

Choose $\mathcal{A}_0$ maximally abelian in $\mathcal{P}_0$

ad $\mathcal{A}_0$ · SYMMETRIC TRASFORMATIONS
· SIMULTANEOUSLY DIAGONA-BLE ON $\mathbb{R}$

$\mathcal{G}_0 = \oplus g_\lambda \qquad g_\lambda = \{ X \in \mathcal{G}_0 \mid \forall H \in \mathcal{A}_0 :$

$\mathrm{ad}\, H(X) = \lambda(H)\, X \}$

· RESTRICTED ROOT SPACE

Let $\mathcal{N}_6$ center of $\mathcal{A}_0$ in $\mathcal{K}_0$

Choose $\mathcal{T}_0$ maximally abelian in $\mathcal{N}_6$

$$\boxed{\mathcal{H}_0 = \mathcal{T}_0 \oplus \mathcal{A}_0}$$ Cartan subalgebra

Σ restricted root space

Σ^+ positive roots

$\mathcal{N} = \bigoplus_{\lambda \in \Sigma^+} g_\lambda$ nilpotent subalgebra

$[g_\lambda, g_\mu] \subset g_{\lambda+\mu}$

$\mathfrak{a}_0 \oplus \mathcal{N}$ solvable algebra

$$\boxed{\mathcal{G}_0 = \mathcal{K}_0 \oplus \mathfrak{a}_0 \oplus \mathcal{N}}$$

$$\begin{array}{ccccc} X = & X_0 & + \sum_{\lambda \in \Sigma^+} (X_{-\lambda} + \theta X_{-\lambda}) & + \sum_{\lambda \in \Sigma^+} (X_\lambda - \theta X_{-\lambda}) \\ & \cap & \cap & \cap \\ & \mathfrak{a}_0 \oplus \mathfrak{M} & \mathcal{K}_0 & \mathcal{N} \\ & \cap & & \\ & \mathcal{K}_0 & & \end{array}$$

- $\forall A \in \mathfrak{a}_0 \quad \theta A = -A$

- $\forall X \in g_\lambda \quad [A, X] = \lambda(A) X$

$\theta[A, X] = [\theta A, \theta X] = -[A, \theta X] = \lambda(A) \theta X$

$\theta g_\lambda = g_{-\lambda}$

Cayley transformations

$$H \in \mathcal{A}_0 \Rightarrow \operatorname{ad} H = +(\operatorname{ad} H)^t$$
$$H \in \mathcal{T}_0 \Rightarrow \operatorname{ad} H = -(\operatorname{ad} H)^t$$

Roots: real on $\mathcal{A}_0$

imaginary on $\mathcal{T}_0$

α a root : $\alpha \in \mathcal{H}^*$

$$\tau^* : \mathcal{H}^* \to \mathcal{H}^* : \alpha \mapsto \tau^*[\alpha]$$

$$\forall H \in \mathcal{H} : \tau^*[\alpha] H = \overline{\alpha(\tau H)}$$

$$\theta^* = \sigma^* \tau^* = \tau^* \sigma^*$$

$$\forall H \in \mathcal{H} : \theta^*[\alpha] H = \alpha(\theta H)$$

$$[H, \theta E_\alpha] = \theta[\theta H, E_\alpha] = \alpha(\theta H) \theta E_\alpha$$

$$\boxed{\theta \mathfrak{g}_\alpha = \mathfrak{g}_{\theta^*[\alpha]}}$$

Imaginary roots

$\forall H \in \mathcal{H}_0 \quad \theta^*[\alpha](H) \in i\,\mathbb{R}$

$\Rightarrow \theta^*[\alpha](\mathcal{H}_0) = \theta^*[\alpha](\mathcal{T}_0) = \alpha(\mathcal{T}_0)$

$= \alpha(\mathcal{H}_0)$

$\Rightarrow \theta^*[\alpha] = \alpha$

$\Rightarrow \theta E_\alpha = \pm E_\alpha$ (root spaces are 1-dimensional)

$E_\alpha = X_\alpha + i Y_\alpha$

$\theta E_\alpha = + E_\alpha \Rightarrow X_\alpha, Y_\alpha \in \mathcal{K}_0$ compact root

$\theta E_\alpha = - E_\alpha \Rightarrow X_\alpha, Y_\alpha \in \mathcal{P}_0$ non compact root

10) $\underline{\beta}$: Imaginary noncompact root

- $$E_\beta \in g_\beta \subset \mathcal{P} = (\mathcal{P}_0 + i\mathcal{P}_0)$$
$$\begin{array}{c} \quad\quad\quad\quad\quad \Psi \quad\; \Psi \\ E_\beta = X_\beta + i\, Y_\beta \end{array}$$

- $\sigma E_\beta = X_\beta - i\, Y_\beta = \overline{E}_\beta$

- $[H, \sigma E_\beta] = \sigma[\sigma H, E_\beta] = \overline{\beta(\sigma H)}\, \sigma E_\beta$
$\overline{\beta(H)} \sigma E_\beta = -\beta(H)\, \overline{E}_\beta$

- $\theta E_\beta = -E_\beta \qquad \theta \sigma E_\beta = -\sigma E_\beta$

- $B(E_\beta, \sigma E_\beta) = -B(E_\beta, \theta\sigma E_\beta)$
$= B^\theta(E_\beta, \overline{E}_\beta) > 0$

~~A~~fter normalization of E_β:

$$sl(2,\mathbb{R}) \begin{cases} [E_\beta, \overline{E}_\beta] = H'_\beta & \in i\mathcal{K}_0 \\ [H'_\beta, E_\beta] = 2 E_\beta \\ [H'_\beta, \overline{E}_\beta] = -2\, \overline{E}_\beta \end{cases}$$

$$h = E_\beta + \bar{E}_\beta$$

$$\sigma h = h \qquad \theta h = -h$$

$$h \in \mathcal{P}_0$$

$$\mathcal{H}'_0 = \ker(\beta|\mathcal{H}_0) \oplus \mathbb{R}h$$

CARTAN SUBALGEBRA

$$\dim(\mathcal{H}'_0 \cap \mathcal{P}_0) = \dim(\mathcal{H}_0 \cap \mathcal{P}_0) + 1$$

2°) β : Real root : $\beta(\mathcal{T}_0) = 0$

- $\beta(\mathcal{H}_0) = \beta(A_0) = -\beta(\theta\mathcal{H}_0) = -\theta^*[\beta](\mathcal{H}_0)$

  $\theta^*[\beta] = -\beta$

- $E_\beta \in g_\beta$, $[H, \sigma E_\beta] = \sigma[H, E_\beta]$

  $= \beta(H)\, \sigma E_\beta$

$E_\beta = \sigma E_\beta$ by fixing a phase

- $\theta E_\beta = \tau E_\beta = E_{-\beta}$, $B(E_\beta, \theta E_\beta) < 0$

$$\begin{cases} [E_\beta, \theta E_\beta] = -H'_\beta \in \mathcal{P}_0 \\ [H'_\beta, E_{\pm\beta}] = \pm 2 E_{\pm\beta} \end{cases}$$

$$t = E_\beta + \theta E_\beta$$

$$\sigma t = t \qquad \theta t = t$$

$$t \in \mathcal{H}_0$$

$$\mathcal{H}_0' = \ker(\beta|\mathcal{H}_0) \oplus \mathbb{R}t$$

CARTAN SUBALGEBRA

$$\dim(\mathcal{H}_0' \cap \mathcal{P}_0) = \dim(\mathcal{H}_0 \cap \mathcal{P}_0) - 1$$

Cayley transformations

noncompact imaginary root $\Rightarrow$ increase dim $\mathcal{T}_0$

real root $\Rightarrow$ increase dim $\mathcal{T}_0$

VOGAN DIAGRAMS

Ingredients:

$\mathfrak{g}_0$ real semisimple Lie algebra

$\mathfrak{g}$ the complexification of $\mathfrak{g}_0$

θ Cartan involution

$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ Cartan decomposition

$\mathfrak{h}_0$ Cartan θ-stable subalgebra

by using Cayley transformations

$\mathfrak{h}_0 = \mathfrak{a}_0 \oplus \mathfrak{t}_0$ maximally compact

Δ root lattice of $\mathfrak{g} \div \mathfrak{h}$

NO REAL ROOT

Ordering: $\{e_\alpha, e_a\}$ $\theta|_{\mathfrak{t}_0} = +1$

$i\mathfrak{t}_0$ A_0

$\boxed{\theta\Delta^+ = \Delta^+}$

θ
- permutes the simple roots
- fixes the imaginary roots
- exchanges in 2-cycles the complex roots

Vogan diagram

Dynkin diagram
plus
decorations

↔	●	○
2-cycle	noncompact root	compact root
complex roots	imaginary roots	

Illustration with $sl(5,\mathbb{C}) \sim A_4$

$$sl(5,\mathbb{C}) = \{ X \mid X \in \mathbb{C}^{5\times 5},\ \operatorname{Tr} X = 0 \}$$

$$\dim sl(5,\mathbb{C}) = 24$$

$$B(X,Y) = \operatorname{Tr}(\operatorname{ad}X\, \operatorname{ad}Y) = 10\operatorname{Tr}(X\,Y)$$

$$\Delta = \{ \vec{e}_i - \vec{e}_j \mid \vec{e}_k \in E^5,\ i \neq j \}$$

$$\Delta \perp \vec{e}_1 + \vec{e}_2 + \vec{e}_3 + \vec{e}_4 + \vec{e}_5$$

$$\Delta^+ = \{ \vec{e}_i - \vec{e}_j \mid i < j \}$$

Cartan-Weyl basis

$$K^p_q = \begin{array}{cc} \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 \cdots 0 & 1 & 0 \cdots 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix} & \rightarrow p \\ \uparrow q & \end{array} \qquad (K^p_q)^\alpha{}_\beta = \delta^{p\alpha}\delta_{q\beta}$$

$$\left.\begin{aligned} H_1 &= \operatorname{diag}(1,-1,0,0,0) \\ H_2 &= \operatorname{diag}(0,1,-1,0,0) \\ H_3 &= \operatorname{diag}(0,0,1,-1,0) \\ H_4 &= \operatorname{diag}(0,0,0,1,-1) \end{aligned}\right\} \text{Cartan subalgebra}$$

Root space

$$\mathcal{H} = \{ \mathrm{diag}(d_1, \dots, d_s) \mid \sum_{i=1}^{s} d_i = 0 \}$$

$\epsilon_p(d) = d_p$ linear form acting on diagonal matrices

$$\mathcal{H}^* = \{ A^p \epsilon_p \mid \sum_{p=1}^{s} A^p = 0 \}$$

$$[H_k, K^p_q] = (\epsilon_p(H_k) - \epsilon_q(H_k)) K^p_q$$

K^p_q root vector

$\epsilon_p - \epsilon_q$ root

<u>$sl(S, \mathbb{R})$</u> $= \mathrm{vect}\{ K^p_q, H_k \}_{\mathbb{R}}$ split form

$\sigma(X) = \overline{X}$ conjugation

<u>$su(S)$</u> $= \mathrm{vect}\{ K^p_q - K^q_p, i(K^p_q + K^q_p), iH_k \}_{\mathbb{R}}$

compact form

$\tau(X) = -X^+$ conjugation

$$\theta = \sigma\tau, \quad \theta(X) = -X^t$$

Other real subalgebras (aligned with $su(5)$)

$$su(p,q) = \left\{ \begin{pmatrix} A & C \\ C^{+} & B \end{pmatrix} \middle| \begin{array}{l} A = -A^{+} \in \mathbb{C}^{p\times p} \\ B = -B^{+} \in \mathbb{C}^{q\times q} \\ \mathrm{Tr}\, A + \mathrm{Tr}\, B = 0 \end{array} \right\}$$

$$I_{p,q} = \begin{pmatrix} \mathbb{1}_{p\times p} & \\ & -\mathbb{1}_{q\times q} \end{pmatrix}$$

$$\sigma_{p,q}(X) = -I_{p,q} X^{+} I_{p,q} \qquad (p,q) = \begin{array}{c} (1,4) \\ (2,3) \end{array}$$

Dynkin A_4 

$su(2,3)$

CSA $\{ i H_k \}$

All roots imaginary

Cartan involution $\theta(X) = I_{p,q} X I_{p,q}$

compact elements $\left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\}$

non compact elements $\left\{ \begin{pmatrix} 0 & C \\ C^{+} & 0 \end{pmatrix} \right\}$

Natural ordering $\epsilon_1 > \epsilon_2 \ldots > \epsilon_5$

Simple roots $\alpha_1 = \epsilon_1 - \epsilon_2$ compact

$\alpha_2 = \epsilon_2 - \epsilon_3$ compact

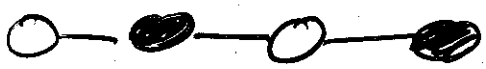
$\alpha_3 = \epsilon_3 - \epsilon_4$ noncompact

$\alpha_4 = \epsilon_4 - \epsilon_5$ compact

Another ordering $\epsilon_1 > \epsilon_2 > \epsilon_4 > \epsilon_5 > \epsilon_3$

Simple roots $\tilde{\alpha}_1 = \epsilon_1 - \epsilon_2$ compact

$\tilde{\alpha}_2 = \epsilon_2 - \epsilon_4$ noncompact

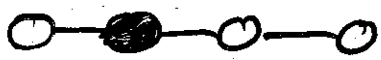


$\tilde{\alpha}_3 = \epsilon_4 - \epsilon_5$ compact

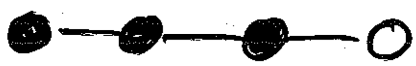
$\tilde{\alpha}_4 = \epsilon_5 - \epsilon_3$ noncompact

or

 $\epsilon_1 > \epsilon_5 > \epsilon_3 > \epsilon_4 > \epsilon_2$

 $\epsilon_5 > \epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1$

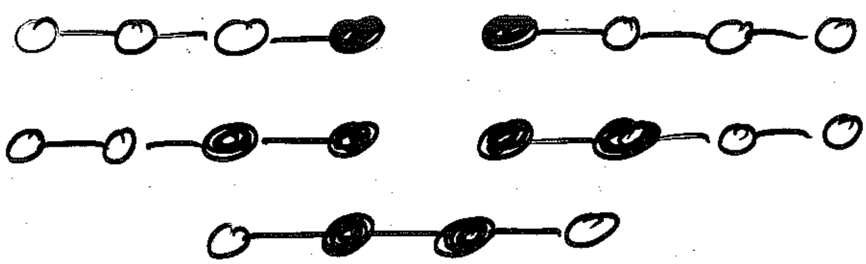
$5!$  $\epsilon_3 > \epsilon_5 > \epsilon_4 > \epsilon_2 > \epsilon_1$

$= 120$  $\epsilon_1 > \epsilon_5 > \epsilon_2 > \epsilon_4 > \epsilon_3$

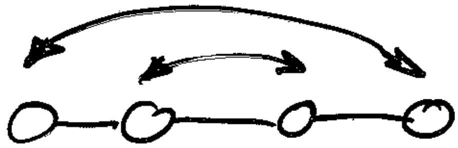
$= 10\ 3!\,2!$ 

 $\epsilon_4 > \epsilon_1 > \epsilon_2 > \epsilon_5 > \epsilon_3$

 $\epsilon_5 > \epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4$

$su(4,1)$ 

$su(5)$ 

$sl(5,\mathbb{R})$ 

Borel & de Siebenthal

There always exists an ordering of the simple roots such that:

- at most one simple root is noncompact (i.e. at most one ●)
- If $Aut(Vog) = Id$, let $\{\alpha_p\}$ a simple root basis, $\{\omega_q\}$ the dual basis: $(\omega_q | \alpha_p) = \delta_{pq}$, then the single painted root α_p may be choosen such that there is no q such that $(\omega_p - \omega_q | \omega_q) > 0$

G_2  $\alpha_2 \quad \alpha_1$

$\alpha_1 = \vec{e}_1 - \vec{e}_2$

$\alpha_2 = -2\vec{e}_1 + \vec{e}_2 + \vec{e}_3$

$\Delta \perp \vec{e}_1 + \vec{e}_2 + \vec{e}_3$

$$\bar{\omega} = a\,\vec{e}_1 + b\,\vec{e}_2 - (a+b)\,\vec{e}_3$$

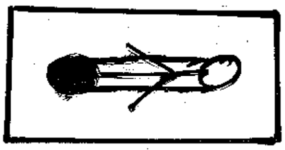
$$\omega_1 \qquad \left.\begin{array}{l} a - b = 1 \\ -2a + b - (a+b) = 0 \end{array}\right\} \ \omega_1 = -\vec{e}_2 + \vec{e}_3$$

$$\omega_2 \qquad \left.\begin{array}{l} a - b = 0 \\ -2a + b - (a-b) = 1 \end{array}\right\} \ \omega_2 = -\frac{1}{3}\vec{e}_1 - \frac{1}{3}e_2 + \frac{2}{3}\vec{e}_3$$

$$\omega_1 - \omega_2 = \frac{1}{3}\vec{e}_1 - \frac{2}{3}\vec{e}_2 + \frac{1}{3}\vec{e}_3$$

$$(\omega_1 - \omega_2 \mid \omega_2) = -\frac{1}{9} + \frac{2}{9} + \frac{2}{9} = \frac{1}{3}$$

$$(\omega_2 - \omega_1 \mid \omega_1) = \frac{1}{9} - \frac{2}{9} - \frac{2}{9} = -\frac{1}{3}$$



Reconstruction

- We know θ^* on Δ
- $\theta H_\alpha = H_{\theta^*[\alpha]}$ $\qquad \theta \mathcal{U}_0 = \mathcal{U}_0$
- $\theta E_\alpha \doteq E_{\theta^*[\alpha]}$

On simple root

$$\theta E_\alpha = \begin{cases} E_\alpha & \circ \\ -E_\alpha & \bullet \\ E_{\theta^*[\alpha]} & \circ \leftrightarrow \circ \end{cases}$$

In general $\theta E_\mu = a_\mu E_{\theta^*[\mu]}$

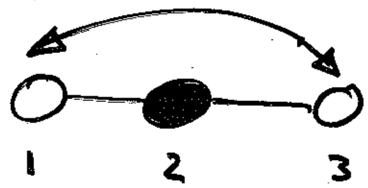
$$\theta [E_\beta , E_\gamma] = N_{\beta,\gamma}\, \theta E_{\beta+\gamma} = N_{\beta,\gamma} a_{\beta+\gamma} E_{\theta^*[\beta+\gamma]}$$

$$a_\beta a_\gamma [E_{\theta^*[\beta]}, E_{\theta^*[\gamma]}] = N_{\theta^*[\beta]\theta^*[\gamma]} a_\beta a_\gamma E_{\theta^*[\beta+\gamma]}$$

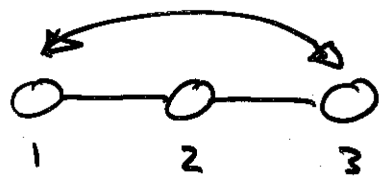
$$N_{\theta^*[\beta]\theta^*[\gamma]} a_\beta a_\gamma = N_{\beta\gamma} a_{\beta+\gamma}$$

$$\boxed{a_{\beta+\gamma} = \pm a_\beta a_\gamma = \pm 1}$$

Illustration

- 

$sl(4,\mathbb{R})$



$sl(2,\mathbb{H})$

- $$\begin{cases} \theta H_{\alpha_1} = H_{\alpha_3} & \theta H_{\alpha_2} = H_{\alpha_2} & \theta H_{\alpha_3} = H_{\alpha_1} \\ \theta E_{\alpha_1} = E_{\alpha_3} & \theta E_{\alpha_2} = \mp E_{\alpha_2} & \theta E_{\alpha_3} = E_{\alpha_1} \end{cases}$$

- $[E_{\alpha_1}, E_{\alpha_2}] = E_{\alpha_1+\alpha_2}$ $\quad [E_{\alpha_2}, E_{\alpha_3}] = E_{\alpha_2+\alpha_3}$

$$[E_{\alpha_1+\alpha_2}, E_{\alpha_3}] = E_{\alpha_1+\alpha_2+\alpha_3} = [E_{\alpha_1}, E_{\alpha_2+\alpha_3}]$$

- $\theta E_{\alpha_1+\alpha_2} = [E_{\alpha_3}, \mp E_{\alpha_2}] = \pm E_{\alpha_2+\alpha_3}$

$\theta E_{\alpha_2+\alpha_3} = \pm E_{\alpha_1+\alpha_2}$

$\theta E_{\alpha_1+\alpha_2+\alpha_3} = \mp E_{\alpha_1+\alpha_2+\alpha_3}$

$sl(4,\mathbb{R})$

$$\mathcal{K} = \mathrm{span}\{H_{\alpha_1}+H_{\alpha_3}, H_{\alpha_2}, E_{\alpha_1}+E_{\alpha_3}, E_{-\alpha_1}+E_{-\alpha_3}, E_{\alpha_1+\alpha_2}+E_{\alpha_2+\alpha_3}, E_{-\alpha_1-\alpha_2}+E_{-\alpha_2-\alpha_3}\}_{\mathbb{C}}$$

$$\mathcal{P} = \mathrm{span}\{ H_{\alpha_1} - H_{\alpha_3}, E_{\pm\alpha_2}, E_{\pm\alpha_1} - E_{\pm\alpha_3}$$

$$E_{\pm(\alpha_1+\alpha_2)} - E_{\pm(\alpha_2+\alpha_3)}, E_{\pm(\alpha_1+\alpha_2+\alpha_3)}\}_{\mathbb{C}}$$

$$\mathcal{K}_0 = \mathcal{U}_0 \cap \mathcal{K} = so(4,\mathbb{R}) = so(3,\mathbb{R}) \oplus so(3,\mathbb{R})$$

$$= \mathrm{span}\{ i(H_{\alpha_1} + H_{\alpha_3}), E_{\alpha_1} + E_{\alpha_3} - E_{-\alpha_1} - E_{-\alpha_3},$$

$$i(E_{\alpha_1} + E_{\alpha_3} + E_{-\alpha_1} + E_{-\alpha_3})\}_{\mathbb{R}} \oplus$$

$$\mathrm{span}\{ i(H_{\alpha_1} + H_{\alpha_3} + 2H_{\alpha_2}), (E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3}$$

$$- E_{-\alpha_1-\alpha_2} - E_{-\alpha_2-\alpha_3}), i(E_{\alpha_1+\alpha_2} + E_{\alpha_2+\alpha_3}$$

$$+ E_{-\alpha_1-\alpha_2} + E_{-\alpha_2-\alpha_3})\}_{\mathbb{R}}$$

Tits-(Satake) Diagrams

Maximally compact CSA $\Rightarrow$ Vogan

Maximally noncompact CSA $\Rightarrow$ Iwasawa

[Classification of algebraic semisimple groups, in Algebraic Groups and Discontinuous Groups, Boulder (1965), Proc. Symp. Pure Math 9 (1966)

Witt indices]

Examples: $su(3,2)$

VOGAN

$\alpha_3 = \epsilon_3 - \epsilon_4$ imaginary noncompact root

$E_{\alpha_3} = K^3_4$, $\overline{E}_{\alpha_3} = \sigma K^3_4 = K^4_3$, iH_3

$CSA_D = \mathrm{span}\{iH_k\}$

$\ker(\alpha_3 | CSA_D) = \mathrm{span}\{iH_1, i(2H_2 + H_3), i(2H_4 + H_3)\}$

$i(1,-1,0,0,0)$

$i(0,2,-1,-1,0)$ $\quad i(0,0,1,1,-2)$

noncompact

$$H' = (K^3_4 + K^4_3), \quad Tr(H'H') = 2$$

$$B(H'H') = 20$$

$$\boxed{CSA' = \ker(\alpha_3 | CSA_D) \oplus \mathbb{R} H'}$$

ROOTS:

12 complex

$\pm(i, -3i, i, \pm 1), \pm(0, i, -3i, \pm 1), \pm(i, i, -i, \pm 1$

6 imaginary

$\pm(2, -2, 0, 0), \pm i(1, -2, -2, 0), \pm i(1, 0, 2, 0)$

2 real

$\pm(0, 0, 0, 2)$

Real root CT

$$E_{(0,0,0,2)} = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \frac{i}{2}\left(H_3 + K^4_3 - K^3_4\right.,$$

$$\sigma_{3,2} E_{(0,0,0,2)} = - I_{3,2} E^+_{(0,0,0,2)} I_{3,2}$$

$$= + \frac{i}{2}\left(H_3 - K^4_3 + K^3_4\right)$$

$$h = i H_3 = \left(E_{(0,0,0,2)} + \sigma_{3,2} E_{(0,0,0,2)}\right)$$

Imaginary root CT

$$E_{i(1,-2,-2,0)} = K^5_2$$

$$H'' = K^5_2 + K^2_5$$

$$\ker(\beta \mid CSA') = \mathrm{span}\left\{ i(2H_1 + 2H_2 + H_3),\ i(2H_1 + H_3 + 2H_4),\ K^3_4 + K^4_3 \right\}$$

$$CSA'' = \mathrm{span}\left\{ i\,\mathrm{diag}(2,0,-1,-1,0),\ i\,\mathrm{diag}(2,-2,1,1,-2),\ K^3_4 + K^4_3,\ K^5_2 + K^2_5 \right\}_{\mathbb{R}}$$

Diagonalisation

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \end{pmatrix}$$

$$\tilde{I}_{3,2} = S^T I_{3,2} S$$

$$\sigma(X) = -\tilde{I}_{3,2} X^+ \tilde{I}_{3,2}$$

$$\overline{CSA}'' = \mathrm{span}\{\underbrace{H_3, H_2+H_3+H_4}_{A}, \underbrace{i(2H_1+2H_2+H_3), i(2H_1+H_2+H_3+H_4)}_{\mathcal{T}}\}$$

$$:= \mathrm{span}\{h_1, h_2, h_3, h_4\}$$

Computation of θ^*

$\mathcal{D} = \mathrm{span}\{H_1, H_2, H_3, H_4\} = \{\text{diag mat}\}$

2 - basis on $\mathcal{D}^*$

1°) $\{F_1, F_2, F_3, F_3\} = \mathrm{dual}\{H_1, H_2, H_3, H_4\}$

2°) $\{f_1, f_2, f_3, f_4\} = \mathrm{dual}\{h_1, h_2, -ih_3, -ih_4\}$

$\theta^*\{f_1, f_2, f_3, f_4\} = \{-f_1, -f_2, f_3, f_4\}$

$\alpha_1 = \vec{e_1} - \vec{e_2} = 2F_1 - F_2 = -f_2 + 2f_3 + 3f_4$

$\alpha_2 = -f_1 + f_2 + f_3 - f_4$

$\alpha_3 = 2f_1$

$\alpha_4 = -f_1 + f_2 - f_3 + f_4$

$\theta^*[\alpha_1] = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\theta^*[\alpha_2] = \alpha_4$, $\theta^*[\alpha_3] = -\alpha_3$, $\theta^*[\alpha_4] = -$

Remark: $\theta E_{\alpha_1} = -\tilde{I}_{3,2} K'_2 \tilde{I}_{3,2}$
$= E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$

f_i- ordering:

$\alpha_1 < 0$ $\quad -f_2 \cdots$ $\qquad \alpha_2 < 0$ $\quad -f_1 \cdots$ $\qquad \alpha_3 > 0$ $\quad 2f_1$ $\qquad \alpha_4 < 0$ $\quad -f_1 \cdots$

$\alpha_1+\alpha_2 < 0$ $\quad -f_1 \cdots$ $\qquad \alpha_2+\alpha_3 > 0$ $\quad f_1 \cdots$ $\qquad \alpha_3+\alpha_4 > 0$ $\quad f_1 \cdots$

$\alpha_1+\alpha_2+\alpha_3 > 0$ $\quad f_1 \cdots$ $\qquad \alpha_2+\alpha_3+\alpha_4 > 0$ $\quad 2f_2 \cdots$

$\alpha_1+\alpha_2+\alpha_3+\alpha_4 > 0$ $\quad f_2 \cdots$

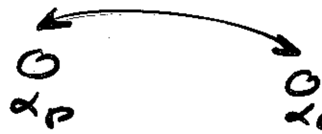
Simple roots

$$\tilde{\alpha}_1 = -\alpha_4 = f_1 - f_2 + f_3 - f_4$$
$$\tilde{\alpha}_2 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 = f_2 + 2f_3 + 3f_4$$
$$\tilde{\alpha}_3 = -\alpha_1 = f_2 - 2f_3 - 3f_4$$
$$\tilde{\alpha}_4 = -\alpha_2 = f_1 - f_2 - f_3 + f_4$$

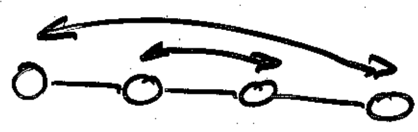
$$\boxed{\theta^*[\tilde{\alpha}_1] = -\tilde{\alpha}_4, \quad \theta^*[\tilde{\alpha}_2] = -\tilde{\alpha}_3, \quad \theta^*[\tilde{\alpha}_3] = -\tilde{\alpha}_2, \quad \theta^*[\tilde{\alpha}_4] = -\tilde{\alpha}_1}$$

Tits-Satake Diagrams

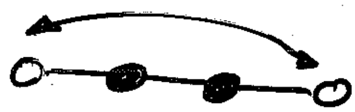
- Dynkin diagram
- $\theta(\bullet) = \bullet \;,\; \circ$
- $\theta[\alpha_p] + \theta[\alpha_q] \in \Delta_0$



$$\Delta_0 = \{\alpha \in \Delta \mid \theta^*[\alpha] = \alpha\} \cup \{\vec{0}\}$$



$su(3,2)$



$su(4,1)$

Formal Considerations

- $\mathcal{G}_\sigma$ real form of $\mathcal{G}$
- $\mathcal{U}_\tau$ aligned with $\mathcal{G}_\sigma$ $\quad [\sigma, \tau] = 0$
- $\mathcal{H}$ CSA of $\mathcal{U}_\tau$ } $\sigma(\mathcal{H}^{\mathbb{C}}) = \mathcal{H}^{\mathbb{C}}$
- θ Cartan involution, $\theta^*: \Delta \to \Delta$
- $\mathcal{G}_\sigma = \mathcal{K}_0 \oplus \mathcal{P}_0 \quad \mathcal{U}_\tau = \mathcal{K}_0 \oplus i\mathcal{P}_0$

- $\mathcal{P} \supset \mathcal{A}$ maximally abelian noncompact SA.
- $\mathcal{H} = \mathcal{A} \oplus \mathcal{T}$ CSA, $\mathcal{T} \subset \mathcal{K}$
- $\mathcal{H}^{\mathbb{R}} = i\,\mathcal{T}_0 \oplus \mathcal{P}_0 = \sum_{\alpha\in\Delta} \mathbb{R} H_\alpha$
- $\theta(E_\alpha) \underset{(*)}{=} \rho_\alpha E_{\theta^*[\alpha]}$, $\rho_\alpha \rho_{\theta^*[\alpha]} \underset{(*)}{=} 1$

$$\rho_\alpha \rho_\beta N_{\theta^*[\alpha],\theta^*[\beta]} = \rho_{\alpha+\beta} N_{\alpha,\beta}$$

$$\theta(H_\alpha) = H_{\theta^*[\alpha]}$$

$$\rho_\alpha \rho_{-\alpha} = 1$$

- If $\alpha \in \Delta_0 = \{\alpha \mid \theta^*[\alpha] = \alpha\}$: $\rho_\alpha \underset{(*)}{=} \pm 1$
- $\Delta_{0,\mp} = \{\alpha \mid \rho_\alpha = \mp 1\}$ $\underset{(*)}{}$ $\alpha \in \Delta_{0,-}$ $E_\alpha \in \mathcal{P}$, $\alpha \in \Delta_{0,+}$ $E_\alpha \in \mathcal{K}$
- $\alpha \in \Delta \setminus \Delta_0$ $E_\alpha + \theta(E_\alpha) \in \mathcal{K}$

$$E_\alpha - \theta(E_\alpha) \in \mathcal{P}$$

- $$\mathfrak{U}_\tau = \bigoplus_{\alpha\in\Delta} \mathbb{R}\, i H_\alpha \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}(E_\alpha - E_{-\alpha}) \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}\, i(E_\alpha + E_{-\alpha}$$

$$\Rightarrow \tau H_\alpha = -H_\alpha$$

$$\tau E_\alpha = -E_{-\alpha}$$

$$\theta = \sigma\tau = \tau\sigma \qquad \theta H_\alpha = H_{\theta^*[\alpha]}$$

$$\sigma(H_\alpha) = -H_{\sigma^*[\alpha]}$$

$$\theta E_\alpha = \rho_\alpha E_{\theta^*[\alpha]}$$

$$\sigma(E_\alpha) = \tau\theta E_\alpha = -\overline{\rho}_\alpha E_{-\theta^*[\alpha]}$$

$$= \kappa_\alpha E_{\sigma^*[\alpha]}$$

$$\sigma^*[\alpha] = -\theta^*[\alpha]$$

$$\kappa_\alpha = -\overline{\rho}_\alpha$$

$$\forall \alpha\ \Delta_0 : \quad \kappa_\alpha = -\rho_\alpha = \pm 1$$

- $$\mathcal{H} = \mathcal{T} \oplus \bigoplus_{\alpha\in\Delta_{0,+}} \mathbb{C}E_\alpha \oplus \bigoplus_{\alpha\in\Delta\setminus\Delta_0} \mathbb{C}(E_\alpha+\theta(E_\alpha))$$

$$\mathcal{P} = \mathcal{A} \oplus \bigoplus_{\alpha\in\Delta_{0,-}} \mathbb{C}E_\alpha \oplus \bigoplus_{\alpha\in\Delta\setminus\Delta_0} \mathbb{C}(E_\alpha-\theta(E_\alpha))$$

- If $H\in\mathcal{A}$, $\mu\in\Delta_{0,-}$: $[H,E_\mu]=\mu(H)E_\mu$

$$= [\theta H, \theta E_\mu] = \theta[H,E_\mu] = -\mu(H)E_\mu$$

$\mathcal{A}$ maximal $\Longrightarrow \Delta_{0,-}=\emptyset$

- $\beta = \{\text{basis of } \mathcal{H}\} = \{\text{basis } \underset{\theta=-1}{\mathcal{A}}, \text{ basis } \underset{\theta=+1}{\mathcal{T}}\}$

$\mathcal{A}$-$\mathcal{T}$ ordering

$\mathcal{B}$ = dual basis of β

$$\alpha\notin\Delta_0 \quad \alpha>0 \Longrightarrow \theta^*[\alpha]<0$$

- $\mathcal{B}=\{\alpha_1,\dots,\alpha_l\}$, $\mathcal{B}_0=\mathcal{B}\cap\Delta_0=\{\alpha_{r+1},\dots,\alpha_l\}$

- $\mathcal{B}_0=\mathcal{B}\cap\Delta_0$ is a basis for Δ_0

If $\beta = \sum b^k\alpha_k > 0$ $(b^k\geq 0)$ and $\beta\in\Delta_0$

$$0=\beta-\theta^*[\beta]=\sum_{k=1}^{r} b^k(\alpha_k-\theta^*[\alpha_k]) \Rightarrow b^k=0 \quad k\leq r$$

- Let $\alpha_i \in B \setminus B_0$ $(i \leq r)$

$$-\theta^*[\alpha_i] = \sum_{j \leq r} \underset{\in \mathbb{N}_0}{p_i^j}\, \alpha_j + \sum_{j \geq r+1} q_i^j\, \alpha_j$$

$$\alpha_i = (-\theta^*)^2[\alpha_i] = \sum_{\substack{j \leq r \\ k \leq r}} \underbrace{p_i^j\, p_j^k}_{\delta_i^k}\, \alpha_k$$

$$+ \sum_{\substack{j \leq r \\ k \geq r+1}} p_i^j q_j^k \alpha_k - \sum_{k \geq r+1} q_i^k\, \alpha_k$$

$\{p_i^j\} = \{0, 1\}$ $\quad (p_i^j)$ is a permutation matrix

$$\theta^*[\alpha_i] = -\alpha_{\pi(i)} \mod \Delta_0$$

- $\forall \alpha \in \Delta : \theta^*[\alpha] + \alpha \notin \Delta$

If $\alpha \in \Delta_0 \Rightarrow 2\alpha \notin \Delta$

If $\alpha \in \Delta / \Delta_0 \quad \theta^*[\alpha] + \alpha \in \Delta$

$\Rightarrow \theta^*[\alpha] + \alpha \in \Delta_0 = \Delta_{0,+}$ (A max.)

$$[\sigma(E_\alpha), E_{-\alpha}] = -\bar{\rho}_\alpha N_{\sigma^*[\alpha], -\alpha} E_{\sigma^*[\alpha]-\alpha}$$

$$[E_\alpha, \sigma(E_{-\alpha})] = -\bar{\rho}_{-\alpha} N_{\alpha, -\sigma^*[\alpha]} E_{\alpha - \sigma^*[\alpha]}$$

$$\sigma(\underbrace{E_{\sigma^*[\alpha]-\alpha}}_{\substack{-\theta^*[\alpha]-\alpha \\ \cap \\ \Delta_{0,+}}}) = -E_{\alpha-\sigma^*[\alpha]}$$

$$[E_\alpha, \sigma(E_{-\alpha})] = \sigma[\sigma(E_\alpha), E_{-\alpha}]$$

$$= (-\rho_\alpha) N_{\sigma^*[\alpha],-\alpha}^{(-)} E_{\alpha-\sigma^*[\alpha]}$$

$$= -\bar{\rho}_{-\alpha} N_{\alpha,-\sigma^*[\alpha]} E_{\alpha-\sigma^*[\alpha]}$$

$$\Longrightarrow \rho_\alpha N_{\sigma^*[\alpha],-\alpha} = -\bar{\rho}_{-\alpha} N_{\alpha,-\sigma^*[\alpha]}$$

$\rho_\alpha = -\bar{\rho}_{-\alpha}$ but $\rho_\alpha \rho_{-\alpha} = +1$

$$\Longrightarrow \theta^*[\alpha] + \alpha \notin \Delta$$

- $\forall \alpha_q \in \mathcal{B}_0 \quad \theta^*[\alpha_q] = \alpha_q$

$$(\theta[\alpha_k] + \alpha_{\pi(k)} | \alpha_q) = \sum_{j \geq r+1} a_k^j \underbrace{(\alpha_j | \alpha_q)}$$

Killing metric on $\mathcal{B}_0$

- $\{\bullet\}$ basis of B_0

  Dynkin diag of $\mathcal{M}$ $*$

- Rank $\mathcal{P}$ = real rank of $\mathcal{G}$
  = number of cycles of π

  $$\operatorname{Rank} \mathcal{M} = \operatorname{Rank} \mathcal{G} - \operatorname{Rank} \mathcal{P} *$$

- $*$ and $*$ $\Rightarrow$ $\dim \mathcal{M}$

- $$\dim \mathcal{P} = \frac{1}{2}\left(\dim \mathcal{G} - \dim \mathcal{M} + \operatorname{Rank} \mathcal{P}\right)$$

- Restricted root space:

  $$\Delta \to \overline{\Delta} : \alpha \mapsto \overline{\alpha} = \frac{1}{2}\left(\alpha - \theta^*_{[\alpha]}\right)$$

Illustration: F_4

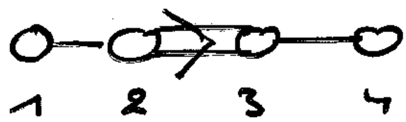
$\dim F_4 = 52 \qquad \mathrm{Rank}\, F_4 = 4$

$$E^4 \supset \Delta_{F_4} = \{\pm\vec{e}_i \pm \vec{e}_j \mid i<j\} \cup \{\pm \vec{e}_i\} \cup \{\tfrac{1}{2}(\pm\vec{e}_1 \pm \vec{e}_2 \pm \vec{e}_3 \pm \vec{e}_4)\}$$

$$48 = 24 + 8 + 16$$

Simple roots

$$\alpha_1 = \vec{e}_2 - \vec{e}_3,\ \alpha_2 = \vec{e}_3 - \vec{e}_4,\ \alpha_3 = \vec{e}_4,\ \alpha_4 = \tfrac{1}{2}(\vec{e}_1 - \vec{e}_2 - \vec{e}_3 - \vec{e}_4)$$



Tits-Satake diagram F_{II} $F_{4(-20)}$

$-\dim \mathfrak{K} + \dim \mathfrak{P}$

$-\dim \mathfrak{g} + 2\dim \mathfrak{P}$

$\mathrm{Rank}\, \mathfrak{P} = 1, \ \mathrm{Rank}\, \mathfrak{K} = 3$

$\mathfrak{K} = so(7) \qquad \dim \mathfrak{P} = \frac{1}{2}(52-21+1) = 16$

$$\theta^*[\alpha_4] = -\alpha_4 + a\alpha_1 + b\alpha_2 + c\alpha_3$$

$$\left.\begin{aligned} (\theta^*[\alpha_4]+\alpha_4 \mid \alpha_1) &= 0 = 2a-b \\ (\theta^*(\alpha_4)+\alpha_4 \mid \alpha_2) &= 0 = -a+2b-c \\ (\theta^*(\alpha_4)+\alpha_4 \mid \alpha_3) &= -1 = -b+c \end{aligned}\right\} \quad \begin{aligned} a &= -1 \\ b &= -2 \\ c &= -3 \end{aligned}$$

$$\theta^*[\vec{e}_1] = -\vec{e}_1 \quad , \quad \theta^*[e_k] = e_k \qquad k=2,3,4$$

$$\Sigma = \bar{\Delta}\Big|_{\mathcal{A}} = \left\{\pm\tfrac{1}{2}\vec{e}_1, \pm\vec{e}_1\right\}$$

$$\text{mult } \tfrac{1}{2}\vec{e}_1 = 8 \; : \; \tfrac{1}{2}(\vec{e}_1 \pm \vec{e}_2 \pm \vec{e}_3 \pm \vec{e}_4) \mapsto \tfrac{1}{2}\vec{e}_1$$

$$\text{mult } \vec{e}_1 = 7 \; : \; \vec{e}_1, \{\vec{e}_1 \pm \vec{e}_k \mid k=2,3,4\} \mapsto \vec{e}_1$$

Restricted root diagram $(BC)_1$

⚠ Remark  ?

$$\Rightarrow \theta^*[\vec{e}_1] = -\vec{e}_2 \qquad \theta^*[\vec{e}_k] = \vec{e}_k \quad k=3,4$$

but $\alpha = \vec{e}_1 \in \Delta \Rightarrow \alpha + \theta^*[\alpha] = \vec{e}_1 - \vec{e}_2 \in \Delta$

IMPOSSIBLE