Lecture 2 - Post-Newtonian Limit in GR

1. Newtonian Gravity

$$\nabla^2 U = -4\pi\rho \qquad \text{Poisson}$$

$$\rho \frac{dv}{dt} = \rho \nabla U - \nabla p \qquad \text{Euler eq.} \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}$$

$$v^i = v^i(\underset{\sim}{x}, t)$$

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}\cdot(\rho \underset{\sim}{v}) = 0$$

$$p = p(\rho, T, Z, \dots)$$

$$U(x,t) = \int \frac{\rho(x', t)}{|x - x'|} d^3x'$$

Properties

$$\frac{1}{|x-x'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\hat{n}_<) Y_{\ell m}(\hat{n}_>) \qquad \hat{n} = \frac{\underset{\sim}{x}}{r}$$

or, if $|x'| < |x|$,

$$\frac{1}{|x-x'|} = \sum_{q=0}^{\infty} \frac{(-)^q}{q!} x'^Q \nabla_Q \left(\frac{1}{r}\right)$$

$$x^Q \equiv x^i x^j \dots x^{\ell_q} \equiv x^{ij\dots} \qquad \nabla_Q \equiv \nabla_i \nabla_j \dots \nabla_{\ell_q} \equiv \nabla_{ij\dots}$$

Can show

$$\nabla_Q \left(\frac{1}{r}\right) = (-)^q (2q-1)!! \frac{\hat{n}^{\langle Q \rangle}}{r^{q+1}}$$

$n^{\langle Q \rangle} = \text{STF}$

$$n^{\langle ij \rangle} = n^{ij} - \tfrac{1}{3}\delta^{ij}$$

$$n^{\langle ijk \rangle} = n^{ijk} - \tfrac{1}{5}(\delta^{ij} n^k + \delta^{ik} n^j + \delta^{jk} n^i)$$

link between $\hat{n}^{\langle q\rangle}$ and $Y_{\ell m}$

$$N^q n^{\langle q\rangle} = \frac{q!}{(2q-1)!!} P_q(\cos\theta)$$

recall

$$P_\ell(\hat{N}\cdot\hat{n}) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(N)\, Y_{\ell m}(n)$$

Prove equivalence of 2 forms of $1/|x-x'|$

$$\oint \hat{n}^q \, d\Omega = \begin{cases} 0 & q \text{ odd} \\ \frac{4\pi}{(2q+1)!!}\Big(\underbrace{\delta^{ij}\delta^{kl}\cdots}_{q/2} + \text{sym}\Big) \end{cases}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho v) = 0 \qquad \text{[assume no flow out of region]}$$

$$\frac{\partial}{\partial t}\int \rho(x',t) f(x,x',t)\, d^3x = \int \rho(x',t)\left[\frac{\partial f}{\partial t} + v'\cdot\nabla' f(x,x',t)\right] d^3x'$$

$$\frac{d}{dt}\int \rho(x',t) f(x,x',t)\, d^3x = \int \rho(x',t)\left[\frac{\partial f}{\partial t} + v\cdot\nabla f + v'\cdot\nabla' f\right] d^3x'$$

eg show $\boxed{PPT}$

$$\frac{d}{dt}\int \rho(x',t)\, x'^i\, d^3x' = \int \rho(x',t)\, v'^i\, d^3x'$$

$$\dot{U} = \frac{\partial U}{\partial t} = \int \rho(x',t)\, \frac{v'\cdot(x-x')}{|x-x'|^3}\, d^3x'$$

$$\frac{dU}{dt} = -\int \rho(x',t)\, \frac{(v-v')\cdot(x-x')}{|x-x'|^3}\, d^3x$$

Euler Equation

$$\rho \frac{dv}{dt} = \rho \nabla U - \nabla p \qquad v = v(x,t)$$

Integrate over all space

$$\int \rho \frac{dv^i}{dt} = \frac{d}{dt}\int \rho v^i d^3x = \frac{d^2}{dt^2}\int \rho v^i d^3x$$

$$\int \rho \nabla^i U d^3x = -\int \rho(x,t)\int \rho(x',t)\frac{(x-x')^i}{|\ \ |^3} d^3x\, d^3x' \equiv 0$$

$$\int \nabla^i p\, d^3x = \int p\, n^i R^2 d\Omega = 0$$

Define $M = \int \rho\, d^3x \quad X = \frac{1}{M}\int \rho x\ ,\ V = \frac{1}{M}\int \rho v$

$$\therefore \boxed{\frac{d}{dt}(MV^i) = \frac{d^2}{dt^2}(MX^i) = 0 \qquad \text{cons. of momentum}}$$

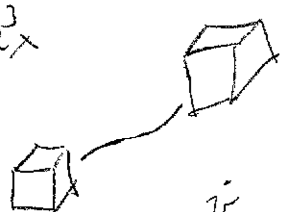
$$\int \underbrace{\rho\, v \cdot \frac{dv}{dt}}_{①} = \int \rho\, v \cdot \nabla U - \int v \cdot \nabla p$$

$$① = \frac{1}{2}\frac{d}{dt}\int \rho v^2 d^3x \equiv \frac{d}{dt}\mathcal{T}$$

$$② = -\int \rho \rho' \frac{v \cdot (x-x')}{|x-x'|^3} d^3x\, d^3x' = -\frac{1}{2}\int \rho\rho' \frac{(v-v')\cdot(x-x')}{|x-x'|^3}$$

$$= +\frac{1}{2}\frac{d}{dt}\int \frac{\rho\rho'}{|x-x'|} d^3x\, d^3x' \equiv -\frac{d}{dt}\Omega$$

$$③ = -\int v \cdot \nabla p = -\int \nabla \cdot (v p)\, d^3x + \int p\, \nabla \cdot v\, d^3x$$



$$= \int \frac{p\, d\mathcal{V}/dt}{\mathcal{V}} d^3x \qquad \nabla \cdot v = \frac{\dot{\mathcal{V}}}{\mathcal{V}} = -\frac{\dot{\rho}}{\rho}$$

$$= -\int \frac{1}{\ } \frac{d}{dt}(\rho \Pi \mathcal{V}) = -\int \rho \frac{d\Pi}{dt} = -\frac{d}{dt}\int \rho \Pi\, d^3x$$

Thus $\boxed{\mathcal{T} + \Omega + \mathcal{E} = \text{const}}$

Newtonian Stress Energy Tensor

$$T^{00} = \rho$$

$$T^{0j} = \rho v^j$$

$$T^{ij} = \rho v^i v^j + p\delta^{ij} + \frac{1}{4\pi}\left(U^{,i}U^{,j} - \frac{1}{2}\delta^{ij}U^{,k}U^{,k}\right)$$

Then

$$T^{0\nu}{}_{,\nu} = 0 \Rightarrow \text{continuity eq}^{n}$$

$$T^{j\nu}{}_{,\nu} = 0 \Rightarrow \text{Euler's eq}^{n}$$

$$\boxed{T^{\mu\nu}{}_{,\nu} = 0}$$

PPT

Equations of Motion

For a system of bodies, size R, separation r,

$$U_A = \sum_{q=0}^{\infty} \frac{(-)^q}{q!} I_A^{\varphi} \nabla_{\varphi}\left(\frac{1}{r}\right) \qquad r = |x - x_A|$$

$$I_A^{\varphi} = \int_A \rho' \bar{x}'^{\varphi} d^3x' \approx mR^{\varphi} \qquad \bar{x}' = |x' - x_A|$$

$$I_A^{0} = m \qquad I_A^{1} = 0$$

So for $R \ll r$, only the $\varphi = 0$ term counts

$$U_A = \frac{m_A}{|x - x_A|}, \qquad \int_A \nabla p \, d^3x = 0$$

$$\cancel{m_A} a_A^i = -\sum_{B \neq A} \frac{\cancel{m_A} m_B (x - x_B)^i}{|x - x_B|^3}$$

Two body motion

$$a_1^i = -\frac{m_2 x}{r^3} \qquad x = |x_1 - x_2|$$

$$a_2 = \frac{m_1 x}{r^3} \qquad v = v_1 - v_2$$

Relative equation of motion $\qquad a = a_1 - a_2$

$$\boxed{\underset{\sim}{a} = -\frac{m\underset{\sim}{x}}{r^3}} \qquad m = m_1 + m_2$$

$$L = \frac{1}{2} v^2 + U \qquad U = \frac{m}{r}$$

2. Post-Newtonian Theory

gravity weak ~ $U \sim m/r$ - 10^{-5} - 10^{-8} in SS

slow motion $v^2 \sim U$ -

pressure low $\frac{p}{\rho} \sim v^2 \sim U$ -

counting orders for dimensionless variables or functions

$$\varepsilon \sim v^2 \sim U \sim p/\rho \qquad \varepsilon^{1/2} \sim v \sim \frac{d/dt}{d/dx}$$

Geodesics

$$\delta S = 0$$

$$S = \int \underbrace{\sqrt{-g_{00} - 2g_{0i}v^i - g_{ij}v^iv^j}}_{L}\, dt$$

NGT ?

- recall $L = -\frac{1}{2}v^2 - U$

$$L_R = \left(1 - \delta g_{00} - 2g_{0i}v^i - v^2 - \delta g_{ij}v^iv^j\right)^{1/2}$$

$$\approx 1 - \frac{1}{2}\delta g_{00} - \frac{1}{2}v^2 + \cdots$$

If $\delta g_{00} = +2U$, get NGT

Note $U \sim \varepsilon \sim v^2$

What about next corrections

$$L_R = \Big(1 - \underset{\varepsilon}{2U} - \underset{\varepsilon^2}{\delta g^{(2)}_{00}} - 2\underset{\varepsilon^{3/2}}{\delta g^{(1.5)}_{0i}}\underset{\varepsilon^{1/2}}{v^i} - \underset{\varepsilon}{v^2} - \underset{\varepsilon}{\delta g^{(1)}_{ij}}\underset{\varepsilon}{v^iv^j}\Big)^{1/2}$$

Therefore need

$$g_{00} \text{ to } O(\varepsilon^2)$$
$$g_{0i} \text{ to } O(\varepsilon^{3/2})$$
$$g_{ij} \text{ to } O(\varepsilon)$$

What about light?

3. <u>PN Limit of GR</u>

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \qquad \left[\text{Bianchi identities} \quad G^{\alpha\beta}{}_{;\beta} = 0 \Rightarrow T^{\alpha\beta}{}_{;\beta} = 0 \right]$$

weak fields every where so that $g_{\mu\nu} \sim \eta_{\mu\nu}$

Define

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu} \qquad ; \qquad g = \det(g_{\mu\nu})$$

$$H^{\mu\alpha\nu\beta} \equiv \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta} \qquad \text{– symmetries of Riem}$$

Can prove

$$H^{\mu\alpha\nu\beta}{}_{,\mu\nu} = (-g)\left(2G^{\alpha\beta} + 16\pi t^{\alpha\beta}_{LL}\right)$$

$$= 16\pi(-g)T^{\alpha\beta} + 16\pi(-g)t^{\alpha\beta}_{LL}$$

$$t^{\alpha\beta}_{LL} \sim \left(\partial_\gamma \mathfrak{g}^{\alpha\beta}\right)^2$$

Now define $\quad \mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

Coordinate freedom: choose

$$h^{\mu\nu}{}_{,\nu} = 0 \qquad \text{Lorentz, deDonder, Harmonic coords}$$

$\left[\text{Ex: Prove } \Box X^\alpha = 0 \quad \Box = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\partial_\beta\right) \right]$

Then

$$H^{\mu\alpha\nu\beta}{}_{,\mu\nu} = -\underbrace{\Box_\eta}_{-\partial_t^2 + \nabla^2} h^{\alpha\beta} + h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu} - h^{\alpha\nu}{}_{,\mu}h^{\mu\beta}{}_{,\nu}$$

<u>Relaxed E. Eq.</u>

$$\Box_\eta h^{\alpha\beta} = -16\pi(-g)T^{\alpha\beta} - 16\pi(-g)t^{\alpha\beta}_{LL} + h^{\alpha\beta}{}_{,\mu\nu}h^{\mu\nu} - h^{\alpha\nu}{}_{,\mu}h^{\beta\mu}{}_{,\nu}$$

$$= -16\pi\tau^{\alpha\beta}$$

Note $\tau^{\alpha\beta}{}_{,\beta} = 0 \iff T^{\alpha\beta}{}_{;\beta} = 0$

Define

$$h^{00} = \underset{\varepsilon}{N} \qquad h^{0i} = \underset{\varepsilon^{3/2}}{K^i}, \quad h^{ij} = \underset{\varepsilon^2}{B^{ij}}, \quad h^{ii} = \underset{\varepsilon^2}{B}$$

Invert

$$g_{00} = -\left(1 - \tfrac{1}{2}N + \tfrac{3}{8}N^2\right) + \tfrac{1}{2}B + O(\varepsilon^3)$$

$$g_{0i} = -K^i + O(\varepsilon^{5/2})$$

$$g_{ij} = \delta^{ij}\left(1 + \tfrac{1}{2}N\right) + O(\varepsilon^2)$$

N to $O(\varepsilon)$

Recall: $T^{\alpha\beta} = (\rho + p)u^\alpha u^\beta + p g^{\alpha\beta}$

$$\Box_\eta N_0 = -16\pi\rho \qquad \boxed{N_0 = 4U}$$

$\Box_\eta N_0 = \nabla^2 N$

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \frac{dt}{d\tau}(1, \underset{\sim}{v})$$

B to $O(\varepsilon^2)$

$$\nabla^2 B = -16\pi \tau^{ii}$$

$$= -16\pi(\rho v^2 + 3p) + \underbrace{\tfrac{1}{8}\nabla N_0^2}_{t_{LL}}$$

$$= -16\pi(\rho v^2 + 3p) + 2(\nabla U)^2$$

$$= -16\pi(\rho v^2 + 3p) + \nabla^2(U^2) + 8\pi\rho U$$

$$\boxed{B = 4\Phi_1 + 12\Phi_4 + U^2 - 2\Phi_2} \qquad \boxed{\text{PPT}}$$

K^i to $O(\varepsilon^{3/2})$

$$\nabla^2 K^i = -16\pi \rho v^i$$

$$\boxed{K^i = 4V^i}$$

N to $O(\varepsilon^2)$

$$\Box_\eta N = -16\pi(-g)T^{00} + \tfrac{7}{8}|\nabla N_0|^2$$

$$(-g) \approx (1-\tfrac{1}{2}N_0)(1+\tfrac{1}{2}N_0)^3 = 1+4U$$

$$T^{00} = [\rho(1+\Pi)+p](u^0)^2 - p$$

$$= \rho(1+2U+v^2+\Pi)$$

$$|\nabla N_0|^2 = 16|\nabla U|^2 = 8\nabla^2(U^2) + 64\pi\rho U$$

$$\Box_\eta N = \nabla^2 N - \ddot{N}_0$$

$$= \nabla^2 N - 4\ddot{U} \qquad U = \tfrac{1}{2}\nabla^2 X$$

superpotential

$$= \nabla^2(N - 2\ddot{X}) \qquad \left(\nabla^2|x-x'| = \frac{2}{|x-x'|}\right)$$

$$\nabla^2(N-2\ddot{X}) = -16\pi\rho - 96\pi\rho U - 16\pi\rho v^2 - 16\pi\rho\Pi$$

$$+ 7\nabla^2 U^2 + 56\pi\rho U$$

$$\boxed{N = 4U + 10\Phi_2 + 4\Phi_1 + 4\Phi_3 + 7U^2 + 2\ddot{X}}$$

4. Gauge transf$^{\underline{n}}$

- general covariance - any coords can be used
- 2 coord transf$^{\underline{n}}$ — PN gauge
  — Lorentz transf

$$g_{\bar\alpha\bar\beta}(\bar x, \bar t) = \frac{\partial x^\mu}{\partial x^{\bar\alpha}}\frac{\partial x^\nu}{\partial x^{\bar\beta}} g_{\mu\nu}(x,t)$$

$\cdot P$

$x^{\bar\alpha}(x^\mu)$

A. PN gauge

$$x^{\bar\mu} = x^\mu + \xi^\mu(x^\nu) \qquad \frac{|\xi|}{|x|} \ll 1 \qquad [\text{first order in } \xi]$$

$$\frac{\partial x^\mu}{\partial x^{\bar\alpha}} = \delta^\mu_\alpha - \xi^\mu{}_{,\bar\alpha}$$

$$\boxed{g_{\bar\alpha\bar\beta}(\bar x) = g_{\alpha\beta}(\bar x) - \xi_{\alpha;\beta} - \xi_{\beta;\alpha}}$$

$\xi_{(\alpha;\beta)} \sim$ PN potential, ξ_α units of distance

$\xi_{(\alpha;\beta)} \to 0$ at ∞ $\qquad \frac{|\xi^\alpha|}{|x^\alpha|} \to 0$ at ∞

$$\xi_{0;0} \sim \varepsilon^2 \qquad \xi_{0;j} \sim \varepsilon^{3/2}, \quad \xi_{i;j} \sim \varepsilon$$

$$\Rightarrow \quad \boxed{\xi_0 = \lambda_1 X_{,0}, \quad \xi_j = \lambda_2 X_{,j}}$$

$$g_{\bar i\bar j} = g_{ij} - 2\lambda_2 X_{,ij} + O(\varepsilon^2)$$

$$g_{\bar 0\bar j} = g_{0j} - (\lambda_1 + \lambda_2) X_{,0j} + O(\varepsilon^{5/2})$$

Also, in g_{00} we have

$$U(\bar{x},\bar{t}) = \int \frac{\rho(x',\bar{t})}{|\bar{x}-x'|} d^3x'$$

– need to convert x' to $\bar{x}'$

$$\rho(x',\bar{t}) = \rho(\bar{x},\bar{t}) \quad - \text{ invariant – measured in LFF}$$

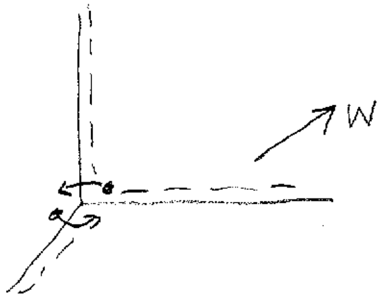
$$\sqrt{-g}\, u^0 d^3x = \text{invariant} \quad : \quad d^3x' \to d^3\bar{x}'(1+2\lambda_2 U)$$

$$\frac{1}{|\bar{x}-x'|} = \frac{1}{|\bar{x}-\bar{x}'|} + \underset{\sim}{\nabla}' \frac{1}{|\bar{x}-\bar{x}'|} \cdot \underset{\sim}{\xi}'$$

Final result $\boxed{PPT}$

$\boxed{PPT}$

B. Lorentz transf^n



$$X^\alpha = \Lambda^\alpha{}_\beta \, \xi^\beta$$

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & -\gamma W \\ -\gamma w & I + (\gamma - 1)\hat{W}^i \hat{W}^j \end{pmatrix}$$

for small w $\quad (w\tau \sim O(9))$

$$X = \underset{\sim}{\xi} + (1 + \tfrac{1}{2}w^2)\underset{\sim}{w}\tau + \tfrac{1}{2}(\xi \cdot w)\underset{\sim}{w} + O(\varepsilon^2)\xi$$

$$t = \tau(1 + \tfrac{1}{2}w^2 + \tfrac{3}{8}w^4) + (1 + \tfrac{1}{2}w)\underset{\sim}{w} \cdot \underset{\sim}{\xi} + O(\varepsilon^3)\tau$$

$$\bar{g}_{\mu\nu}(\underset{\sim}{\xi}, \tau) = \frac{\partial X^\alpha}{\partial \xi^\mu} \frac{\partial X^\rho}{\partial \xi^\nu} g_{\alpha\rho}(x, t)$$

$\bar{\rho} = \rho$ etc $\qquad v = \nu + w + O(\varepsilon^{3/2})$

$d^3x' \to d^3\xi'(1 - \nu' \cdot w - \tfrac{1}{2}w^2)$ — change in LF contraction

$$\frac{1}{\underset{\substack{\uparrow \\ \text{same } t}}{|x - x'|}} = \frac{1}{\underset{\substack{\uparrow \\ \text{same } \tau}}{|\xi - \xi'|}}\left(1 + \tfrac{1}{2}(w \cdot \hat{n})^2 + (w \cdot \hat{n})(\nu' \cdot \hat{n}) + \ldots\right)$$

Possible additional gauge transf^n

$$\bar{\tau} = \tau + \lambda_3 \underset{\sim}{w} \cdot \underline{\nabla} X$$