Cliff Will

Lecture 4 - Tests of PN Gravity

1. Deflection of light

$$\underset{\sim}{x}(t) = \underset{\sim}{x}_0 + \hat{\underset{\sim}{n}}(t-t_0) + \underset{\sim}{x}_p(t)$$

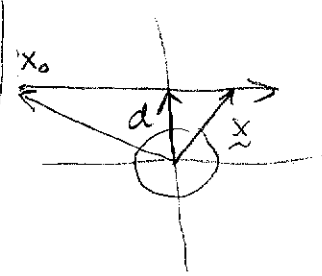
$$\frac{d^2 x_p}{dt^2} = (1+\gamma)\left(\nabla U - 2n\, n\cdot\nabla U\right) \qquad n\cdot\frac{dx_p}{dt} = -(1+\gamma)U$$

$$(x_p)_{\parallel} = n\cdot x_p$$

$$(x_p)_{\perp} = \underset{\sim}{x}_p - \underset{\sim}{n}\,(n\cdot x_p)$$

Then

$$\boxed{\frac{d^2 (x_p)_\perp}{dt^2} = (1+\gamma)\left[\underset{\sim}{\nabla U} - \underset{\sim}{n}\,(\underset{\sim}{n}\cdot\underset{\sim}{\nabla U})\right]}$$



$$U = \frac{m}{r} = \frac{m}{|x_0 + n(t-t_0)|}$$

$$d = n\times(x_0\times n)$$

$$= x_0 - n(n\cdot x_0)$$

$$\nabla U = -\frac{m(\underset{\sim}{x}_0 + \underset{\sim}{n}(t-t_0))}{|\quad|^3}$$

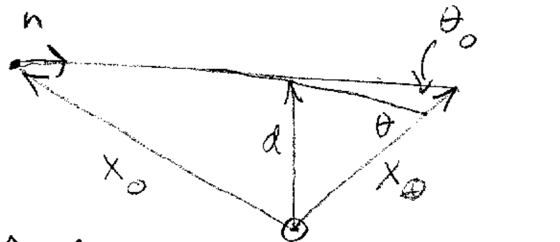
Integrate from t_0 to t

$$\frac{d(x_p)_\perp}{dt} = -(1+\gamma)\,\frac{m\underset{\sim}{d}}{d^2}\left(\frac{x(t)\cdot n}{r(t)} - \frac{x_0\cdot n}{r_0}\right)$$

$$\cos\theta_0 = \hat{n}\cdot\hat{x}_\oplus$$

$$\cos\theta = \left(n + \frac{d(x_p)_\perp}{dt}\right)\cdot\hat{x}_\oplus$$

$$= \cos\theta_0 + \frac{d(x_p)_\perp}{dt}\cdot\hat{x}_\oplus$$



$$\hat{d}\cdot\hat{x}_\oplus = \sin\theta_0 \qquad \theta = \theta_0 + \delta\theta$$

$$= \cos\theta_0 - \delta\theta\,\sin\theta_0$$

$$\delta\theta = \left(\frac{1+\gamma}{2}\right)\frac{2m}{d}\left(\underset{\cos\theta_0}{\underset{\uparrow}{\frac{x_\oplus\cdot n}{r_\oplus}}} - \underset{\to -1 \text{ for distant body}}{\underset{\uparrow}{\frac{x_0\cdot n}{r_0}}}\right)$$

$$= \left(\frac{1+\gamma}{2}\right)\frac{4m}{d}\left(\frac{1+\cos\theta_0}{2}\right)$$

or

$$\delta\theta = \frac{1+\gamma}{2}\,\frac{4m}{r_\oplus}\left(\frac{1+\cos\theta_0}{2\sin\theta_0}\right)$$

Cases

1. $d = R_\oplus$ $\quad m = 1.476$ km $\qquad \dfrac{4m}{d} = \dfrac{1.''7505}{(d/R_\odot)}$

$= 6.9\times10^5$ km

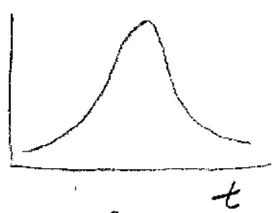
2 $\underline{90^\circ}$ $\qquad \dfrac{2m}{r_\oplus} = 4$ mas $\qquad$ PPT

PPT

2 Shapiro Time Delay

$$\frac{dx_{p\parallel}}{dt} = -(1+\gamma)U$$

$$(x_p)_\parallel = -(1+\gamma)\, m \ln\left(\frac{r(t) + x(t)\cdot\hat{n}}{r_0 + x_0\cdot n}\right)$$



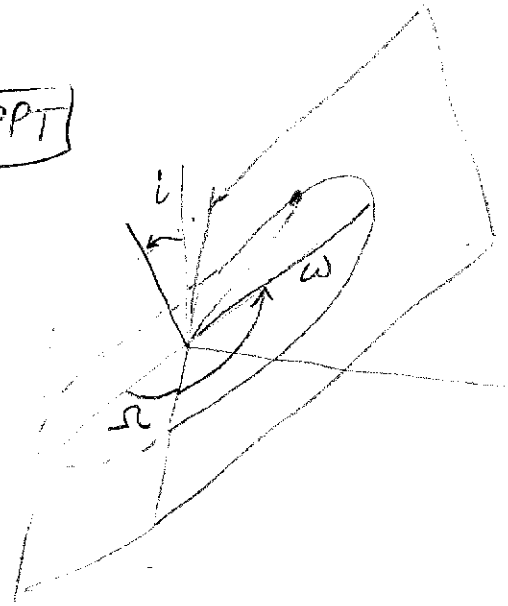
Then

$$|x(t) - x_0| = |n(t-t_e) + x_p(t)|$$

$$= (t-t_e) + n\cdot x_p(t) + \mathcal{O}(x_p)^2$$

3. Pericenter advance

Two body eom $(W=0)$ $\boxed{PPT}$



NGT

$$r = \frac{p}{1+e\cos(\varphi-\omega)}$$

$$p = a(1-e^2)$$

$$\left.\begin{matrix} a, e, \\ \omega, \Omega, i \end{matrix}\right\} - \text{constant}$$

$$\dot\varphi = \frac{\sqrt{mp}}{r^2}$$

Perturbation

$$\delta a = R e_r + S e_\lambda + W e_h$$

Then

$$\frac{d\tilde\omega}{dt} = -\frac{pR}{he}\cos\varphi + \frac{S(p+r)}{he}\sin\phi$$

$$\tilde\omega = \omega + \Omega\cos i$$

over one orbit

$$\boxed{\Delta\tilde\omega = \frac{6\pi m}{p}\left[\frac{2\gamma+2-\beta}{3} + \frac{1}{6}(2\alpha_1-\alpha_2+\alpha_3+J_2)\frac{\mu}{m}\right]}$$

Mercury $\quad a = 0.387$ au $\quad e = 0.2, \; P = 0.24$ yr

$$\frac{6\pi m}{p} = 42.98\ ''/c$$

$\boxed{PPT}$

$$\mu/m \sim m_☿/m_\odot \sim 2\times10^{-7}$$

- planets
- J_2

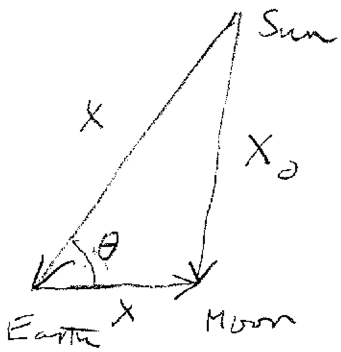
4. The Nordtvedt Effect

$$m_p = m_I\left(1 + \eta \frac{\Omega}{m}\right)$$

$$\eta = 4\beta - \gamma - 3 - \frac{10}{3}\xi - \alpha_1 + \frac{2}{3}\alpha_2 - \frac{2}{3}\zeta_1 - \frac{1}{3}\zeta_2$$

$$a^i_\oplus = -\left(\frac{m_p}{m_I}\right)_\oplus \left[\frac{(m_A)_\odot X^i}{R^3} - \frac{(m_A)_{\mathbb{C}} x^i}{r^3}\right]$$

$$a^i_{\mathbb{C}} = -\left(\frac{m_p}{m_I}\right)_{\mathbb{C}} \left[\frac{(m_A)_\odot X_0^i}{R_0^3} + \frac{(m_A)_\oplus x^i}{r^3}\right]$$



$$\boxed{\begin{aligned} a^i = a^i_{\mathbb{C}} - a^i_\oplus &= -\frac{m^* x^i}{r^3} + \eta\left[\left(\frac{\Omega}{m}\right)_\oplus - \left(\frac{\Omega}{m}\right)_{\mathbb{C}}\right]\frac{(m_A)_\odot X^i}{R^3} \\ &\quad + \left(\frac{m_p}{m}\right)_{\mathbb{C}} m_\odot \left[\frac{X^i}{R^3} - \frac{X_0^i}{R_0^3}\right] \end{aligned}}$$

$\to \sim x/R^4$ – tidal

$$\frac{d^2 r}{dt^2} = \frac{x \cdot a}{r} + \frac{\ell^2}{r^3} \qquad r = |x|, \quad \ell = x \times v$$

$$\frac{d\ell}{dt} = x \times a \qquad \delta a = \frac{m_\odot}{R^2}\eta\left[\quad\right]$$

$$\theta = \Lambda t \qquad \Lambda = \omega_0 - \omega_s$$

$$\ddot r = -\frac{m^*}{r^2} + \frac{\ell^2}{r^3} + \delta a \cos\Lambda t$$

$$\dot\ell = -r\,\delta a \sin\Lambda t$$

Let $r = r_0 + \delta r$ $\qquad \frac{m^*}{r_0^3} = \frac{\ell_0^2}{r_0^4} = \omega_0^2$

$\ell = \ell_0 + \delta\ell$

$$\delta \ell = +\frac{r}{\Lambda}\,\delta a \cos \Lambda t$$

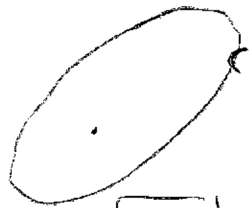
$$\boxed{\delta \ddot{r} + \omega_0^2\, \delta r = \delta a \left(1 + \frac{2\omega_0}{\Lambda}\right) \cos \Lambda t}$$

$$\delta r = \left(\frac{1 + 2\omega_0/\Lambda}{\omega_0^2 - \Lambda^2}\right) \delta a \cos \Lambda t$$

$$\frac{m_\odot}{R^2} \simeq 5.9 \times 10^{-6}\ \frac{\mathrm{km}}{\mathrm{s}^2} \qquad \omega_0 \sim 13.4\, \omega_s$$

$$\left(\frac{\Omega}{m}\right)_\oplus \simeq -4.6 \times 10^{-10} \qquad \left(\frac{\Omega}{m}\right)_☾ \simeq -0.2 \times 10^{-10} \qquad \odot$$

$$\boxed{\delta r \simeq (9.2\,\mathrm{m})\, \eta \cos(\omega_0 - \omega_s)t}$$



LLR ~ mm $\qquad \boxed{|\eta| < 9 \times 10^{-4}} \qquad \boxed{\mathrm{PPT}}$

5. Gravity Probe B

$$\boxed{\begin{aligned} \frac{dS}{dt} &= \Omega \times S \\ \Omega &= \tfrac{1}{4}(4\gamma+4+\alpha_1)\,\nabla\times V + (\gamma+\tfrac{1}{2})\, v\times\nabla U \end{aligned}}$$

$$V^i = \int_a \frac{\rho' v'^i d^3x'}{|x-x'|} = \int_a \rho' v'^i \left(\frac{1}{r} + \frac{\bar{x}'\cdot n}{r^2} + \dots\right) \qquad r = x - x_a$$

$$v = v_a + \bar{v}_a$$

$$= \frac{m_a v_a^i}{r} + \frac{n^j}{r^2}\int \rho' \bar{x}'^j \bar{v}'^i d^3x \qquad \bar{x}' = x' - x_a$$

But

$$\int \rho \bar{x}^j \bar{v}^i = \int \rho x^{(j} v^{i)} + \int \rho x^{[j} v^{i]}$$

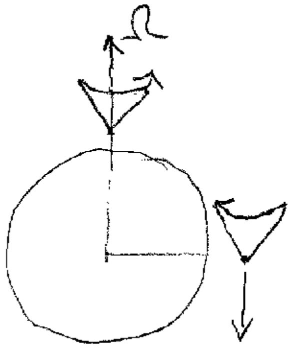
$$= \frac{1}{2}\underbrace{\frac{d}{dt}\int \rho \bar{x}^i \bar{x}^j}_{=0 \text{ stationary}} + \frac{1}{2}\varepsilon^{jik}\underbrace{\varepsilon^{klm}\int \rho \bar{x}^l \bar{v}^m}_{J^k}$$

at rest $=0$

$$\boxed{V^i = \frac{m_a V_a}{r} - \frac{1}{2}\frac{\mathbf{x}\times \mathbf{J}_a}{r^3}}$$

Then

$$\boxed{\Omega = (\gamma+\tfrac{1}{2})\,\mathbf{v}\times\nabla\left(\frac{m}{r}\right) - \frac{1}{2}(\gamma+1+\tfrac{1}{4}\alpha_1)\left(\frac{\mathbf{J} - 3\mathbf{n}(\mathbf{n}\cdot\mathbf{J})}{r^3}\right)}$$



6. Tests of SEP

Anisotropy in G

$$\delta G \sim \xi \frac{m_3}{R}(e \cdot N)^2 - \frac{1}{2}\alpha_2 (e \cdot w)^2$$

- Earth tides - gravimeter data

$$|\xi| < 10^{-3}$$

$$|\alpha_2| < 4 \times 10^{-4}$$

7. Tests of Conservation laws

Levi-civita barrier

$$\langle a_{self} \rangle = (\alpha_3 + \zeta_2) \frac{\pi m_1 m_2 \delta m}{P (mP)^{3/2}} \, e \, \underset{\sim}{e}_P$$

binary pulsar

$\underset{\sim}{a} \rightarrow dP/dt$ - but all pulsars have dP/dt

But e_p is changing $\sim 4°/yr$

$\dot{a} \rightarrow d^2P/dt^2$ - tight bound

$$|\alpha_3 + \zeta_2| < 4 \times 10^{-5}$$