Axion-dilaton cosmology
Ricci flows
and integrable structures

IHP

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based on work with

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Disclamer

- String theory embraces gravity in the UV

  → framework for addressing the issues of

  - initial singularity (big-bang)
  - inflation
  - cosmological constant

- Find time-dependent string backgrounds may help

  → either exact or at $\mathcal{O}(\alpha')$ – $\beta_{\text{background}} = 0$

- Limited progress

  - close to the big-bang exact solutions are required
  - after the inflation era a potential is expected to give a mass to the dilaton – not clear in perturbation

plus our universe is now far away from the UV with many phase transitions

Motivations

- Assume $d\Omega^2 = g_{ij}\, dx^i\, dx^j$ a string target space – homogeneous

  → can this be promoted to FRW

$$ds^2 = -dt^2 + e^{2\sigma(t)}\, d\Omega^2$$

with the usual "matter" content of string theory $(\Phi, B_{\mu\nu}, \dots)$?

- Straightforward in GR
  - assume g_{ij} Einstein and homogeneous $R_{ij} = \frac{R}{d}\, g_{ij}$
  - assume $T_{\mu\nu}$, Λ some matter plus light plus "vacuum energy" → FRWL eqs. for $\sigma(t)$
- More involved in string theory
  - * "matter" is not chosen arbitrarily : dilaton, axion, …
  - * there is an internal manifold
  - * there are two perturbation parameters α', g_{string} → keep under control

A first approach: the renormalization-group flow

- starting point: I-R fixed point of a σ-model with spatial target space
- perturbation by a relevant operator: R-G flow mimics time evolution $\log\frac{1}{\mu} \sim t$
- collection of I-R fixed points ≡ steady-state universes ['95 Carfora, ...]

→ ideas borrowed from Liouville theory in off-critical string ['88-'91 David, Kawai, Sen, Das, Wadia, ...]

→ popular but not a priori justified

- R-G flow eqs. are first-order
- genuine evolution eqs. are second-order

→ useful nevertheless for exploring the landscape around IR fixed points and the relaxation

- Ricci flows and integrable structures

A second approach : genuine time coordinate plus R-G flow dynamical promotion

- introduce an extra field t in the σ-model $\to$ time direction in the target space
- let the previously flowing couplings depend explicitly depend on t
- add a t-dependent dilaton to recover criticality

$$\to \mathcal{L}_{space}(c) \to -\partial t \bar{\partial} t + \mathcal{L}_{space}(c(t)) + R_{(2)} \Phi(t)$$

$$\equiv \mathcal{L}_{space-time}$$

$O(\alpha')$!

$\to$ homogeneous target space $\to$ stringy FRW-like target-space-time

$\to$ the R-G-flowing $c(\mu)$ is promoted to a genuine time-dependent $c(t)$

and we can address the question : $c(\mu) \sim c(t)$?

The R-G viewpoint – isotropic situation

- starting point : homogeneous Einstein space $\quad R_{ij} = \frac{R}{d} g_{ij}$

  plus Kolb - Ramond $\quad H = dB$

  → I-R fixed point at finite R

- breathing mode perturbation: $\quad \mathcal{L}(c) = \frac{1}{2}\left(c\, g_{ij} + B_{ij}\right) \partial X^i \bar{\partial} X^j$

- one-loop counterterm: $\quad \delta\mathcal{L}_{(1)} = \frac{\mu^{\varepsilon}}{4\pi\varepsilon} \bar{R}_{ij}\, \partial X^i \bar{\partial} X^j$

  [Alvarez-Gaumé, Freedman, Mukhi '81, Braaten, Curtright, Zachos '85, Friedan '85]

$$\bar{R}_{ij} = \left(1 - \frac{1}{c^2}\right) R_{ij}$$ is the curvature of $\bar{\Gamma}^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} - \frac{1}{2} H^{\mu}{}_{\nu\rho}$

- R-G flow equation: beta function $\quad \beta(c g_{ij}) = \frac{1}{2\pi} \bar{R}_{ij}$

- R-G-flow equation

$$\frac{d(c\, g_{ij})}{d\log\mu} = \frac{1}{2\pi} R^{-}{}_{ij} \quad \Leftrightarrow \quad \frac{dc}{d\log\mu} = \frac{R}{2\pi d}\left(1 - \frac{1}{c^2}\right)$$

- R-G time $t = -\log\mu$ → Ricci flow with torsion
- we can recast the equation as $(c \equiv \exp 2\sigma)$

$$4\pi \frac{d\sigma}{dt} = -V'(\sigma) \quad \text{with} \quad V(\sigma) = \frac{R}{6d}\, e^{-2\sigma}\left(e^{-4\sigma} - 3\right) + \text{cst.}$$

« velocity = force » → friction

- exact solution : $t = \frac{2\pi d}{R}\left(\operatorname{arctanh} e^{2\sigma} - e^{2\sigma}\right) + \text{cst.}$

at $t \to \infty$ → $\sigma \to 0$ or $c \to 1$: usual I-R fixed point (at this order of α')

The R-G viewpoint – anisotropic deformations

- Consider a conformal sigma model on a semi-simple group G

  → $ds^2 = \delta_{\alpha\beta} J^\alpha J^\beta$ $G \times G$ - invariant metric on G

  → $H = dB = f_{\alpha\beta\gamma} J^\alpha \wedge J^\beta \wedge J^\gamma$ parallelizing torsion

- anisotropic perturbation : $ds^2 = g_{\alpha\beta} J^\alpha J^\beta = \sum_\alpha \gamma_\alpha J^\alpha J^\beta$ & H unchanged

  → residual isometry : $G \times H$ $(H \subset G)$

  → off criticality : R-G flow at $\mathcal{O}(\alpha')$

$$\frac{d g_{\alpha\beta}}{d \log \mu} = \frac{1}{2\pi} R^{-}{}_{\alpha\beta} = \frac{1}{2\pi}\left(R_{\alpha\beta} - \frac{1}{4} H^2{}_{\alpha\beta}\right)$$

- Ricci flow with torsion for general group G

The three-sphere $\gamma_1, \gamma_2, \gamma_3 > 0$

- R-G time : $dt = -\dfrac{d \log \mu}{2\pi \gamma_1 \gamma_2 \gamma_3}$ growing towards the IR

- Ricci-flow equations :

$$\begin{cases} 2\dfrac{\dot{\gamma}_1}{\gamma_1} = (\gamma_2 - \gamma_3)^2 - \gamma_1^2 + 1 \\ 2\dfrac{\dot{\gamma}_2}{\gamma_2} = (\gamma_3 - \gamma_1)^2 - \gamma_2^2 + 1 \\ 2\dfrac{\dot{\gamma}_3}{\gamma_3} = (\gamma_1 - \gamma_2)^2 - \gamma_3^2 + 1 \end{cases}$$

[ ] : torsion

- In general Ricci flow is without torsion but normalised

$$\frac{d g_{ij}}{dt} = -\frac{1}{2\pi} R_{ij} + \frac{1}{6\pi} \langle R \rangle g_{ij}$$

Mathematical reminder in three dimensions

- Thurston's 3-dim geometrization conjecture:

  → any closed 3-manifold can be decomposed in pieces, each piece being locally homogeneous

  → Hamilton's programme: Ricci flows of homogeneous geometries and their sigularities, related to the manifold decomposition

  → Perelman's proof of Thurston's and Poincaré's conjectures

- Homogeneous geometries of simply connected manifolds fit into 9 classes labeled by the minimal isometry group and depending on continuous parameters

  → 6 Bianchi classes and 3 others

  → Example: $SU(2)$ class described by 3 continuous parameters and with 3 possible isometry groups, $SU(2)$, $SU(2)\times U(1)$, $SU(2)\times SU(2)$

- The above R-G-flow problem is the Ricci flow with torsion on the $SU(2)$ class

→ γ_1 γ_2 γ_3 are continuous parameters

→ the isometry group is

- $SU(2)$ if $\gamma_1 \neq \gamma_2 \neq \gamma_3$
- $SU(2) \times U(1)$ if $\gamma_1 \neq \gamma_2 = \gamma_3$
- $SU(2) \times SU(2)$ if $\gamma_1 = \gamma_2 = \gamma_3$

- The singularities and the large-t behaviour of the ordinary Ricci flows have been extensively analyzed for all 9 classes

→ for the $SU(2)$ class the flow always converges to the round sphere of radius zero

→ physically : the usual IR fixed point [Isenberg, Jackson '92, Chow, Knopf '04]

The full SU(2) Ricci flow with torsion

- The above system turns out to be integrable and related to several interesting problems in physics or mathematics
- Introduce a new time T : $t = \log (T + T_0)$

and $\Omega^1 = \dfrac{\gamma_2 \gamma_3}{T + T_0}$ and permutations

→ the equations now read

$$\begin{cases} \dot{\Omega}^1 = \Omega^2 \Omega^3 - \Omega^1 (\Omega^2 + \Omega^3) \\ \dot{\Omega}^2 = \Omega^3 \Omega^1 - \Omega^2 (\Omega^3 + \Omega^1) \\ \dot{\Omega}^3 = \Omega^1 \Omega^2 - \Omega^3 (\Omega^1 + \Omega^2) \end{cases}$$

→ known and integrable [also observed by K. Sfetsos]

- Note : the same equations appear without torsion provided

$t = T$ and $\Omega^1 = \gamma_2 \gamma_3$ and permutations

19^{th} century recreational mathematics

- If $\{\Omega^\alpha \mid \alpha = 1,2,3\}$ are viewed as coordinates of a three-manifold

  $$\rightarrow \frac{d\Omega^\alpha}{dt} = V^\alpha(\Omega)$$

  $\rightarrow$ search for integral lines of a vector field V

- This appears in Darboux's work [1878] on "triply orthogonal surfaces"
- In 1881 Halphen provided a general solution of this system in terms of elliptic functions
- Halphen's solution has been revisited, clarified and deeper understood over the past years in connexion with various integrability issues [Takhtajan '92, Maciejewski, Strelcyn '95, Chakravarty, Halburd, Ablowitz '03]