Ultraviolet properties of maximal supergravity theories

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based on some work done in collaboration with M.B. Green and J.Russo

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L-loop amplitudes

A L-loop four gravitons amplitude in D dimensions has the UV behaviour

$$[A_L^{(4)}] = \Lambda^{(D-2)L+2}$$

$N = 8$ higher-loop amplitudes factorize more momenta

$$[A_L^{(4)}] = \Lambda^{(D-2)L-6-2\beta_L}\,[D^{2\beta_L} R^4]$$

At one-loop one has $\beta_1 = 0$ and $\beta_2 = 2$ at two-loops Bern et al., D'Hoker/Phong, Berkovits

In this talk we will give arguments for the validity of

$$\beta_L = L \text{ for } L \geq 2$$

based on the works done with M.B. Green and J. Russo
hep-th/0610299, hep-th/0611273, hep-th/0612nnn, hep-th/07mmnnn

Outline

1. Systematics of four gravitons multiloop amplitudes
2. Multiloops amplitudes in eleven dimensions
3. Gravitational F-terms
4. Reduction to Four dimensions

UV behaviour of multiloop amplitudes

Higher-loop amplitudes factorize some higher power of momenta

$$[A_L^{(4)}] = \Lambda^{(D-2)L-6-2\beta_L}\,[D^{2\beta_L} R^4]$$

Starting from $D = 11$ a generic L-loop contribution to the operator $D^{2\beta_L}\, R^4$ after compactification on a circle of radius R_{11} is

$$S_L^{(\beta_L+v+3)} = \sum_{w=0}^{w_L} \ell_P^{9(L-1)}\, \frac{\Lambda^{9L-6-2\beta_L-w}}{R_{11}^{w-2}} (R_{11}^2 D^2)^v\, D^{2\beta_L} R^4$$

$$S_L^{(k+3)} = \sum_{q=\beta_L-k}^{q_L} \ell_P^{9(L-1)}\, \Lambda^{9L-6-2k-2q}\, R_{11}^{2(1-q)}\, D^{2k} R^4\,.$$

Relating string and M-theory parameters

The relation between the Type II string and M-theory parameters

$$
\begin{aligned}
R_{11}^3 &= (g_s^A)^2\,\ell_P^3 \\
ds^2_{M-th} &= \frac{\ell_P^3}{\ell_s^2}\,R_{11}^{-1}\,ds^2_{IIA} + R_{11}^2\,(dx^{11} - C_m dx^m)^2
\end{aligned}
$$

We will assume that these relations always hold and are not corrected by higher order corrections

$$
D^2 \to R_{11}\,\frac{\ell_P^3}{\ell_s^2}\,D^2
$$

UV behaviour of multiloop amplitudes

$$S^{(k+3)A}_{m,h} = \ell_s^{2k-2}\,\Lambda^{\delta} \int d^{10}x \sqrt{-g_A}\, c_h\, e^{2(h-1)\phi_A}\, D^{2k} R^4$$

$$\delta = 9L - 6 - 2\beta_L - w; \qquad h = 1 + k - \frac{w + 2\beta_L}{3}$$

Since w parametrizes the UV (sub-)divergences of the L-loop diagram

$$w = 3 + 3k - 3h - 2\beta_L \geq 0 \iff h \leq k + 1 - \frac{2\beta_L}{3}$$

Since at one-loop $\beta_1 = 0$ and the only diverging term is R^4
Since at higher-loop we have $\beta_L = 2$ for $L \geq 2$ then

$$h \leq k$$

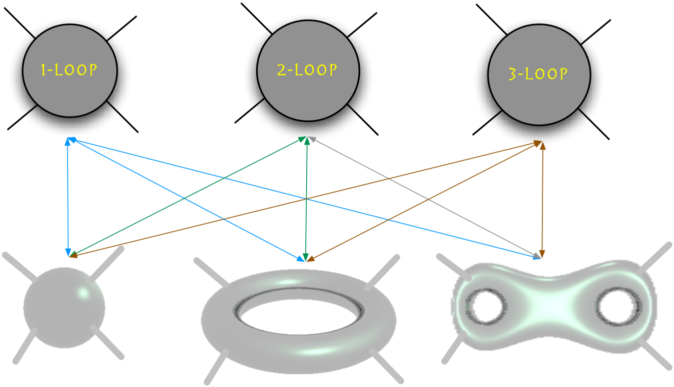
UV behaviour of multiloop amplitudes

Non renormalisation conditions of $D^{2k}R^4$ (or R^{4+k}) operators

- *There are no contributions with string loop genus $h > k$.*
- *The contributions with $h = k$ are determined exactly by finite contributions from the derivative expansion of the one-loop ($L = 1$) diagram in eleven dimensions.*
- *Contributions with $h < k$ are permitted and may arise from any number of supergravity loops greater than one ($L > 1$).*

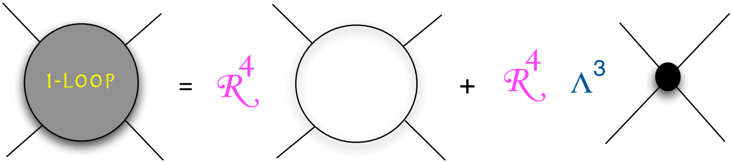
What are the explicit computations in favor of this ?

Relating M-theory to Superstring



One loop in 11D

- $\mathcal{N} = 8$ susy implies that the amplitude factorizes by $\mathcal{R}^4$ Bern et al.



- We have $\beta_1 = 0$ and degree of divergence $\delta_1 = 3$

One loop in 11D

- Compactification on T^2 gives type IIA or type IIB string

$$
\begin{aligned}
S^{IIA} &= \int d^9x\sqrt{g}\mathcal{R}^4\, r_a \left[\frac{2\zeta(3)}{g_s^2} + \frac{4\zeta(2)}{r_a^2} + (\ell_P\Lambda)^3\right] \\
&+ \int d^9x\sqrt{g}\, r_a \sum_{n\geq 2} \frac{\Gamma(n-1/2)\zeta(2n-1)}{n!}\, r_a^{2(n-1)}(\alpha'\mathcal{D}^2)^n\mathcal{R}^4 \\
&+ \int d^9x\sqrt{g}\, r_a \frac{\Gamma(n-1)\zeta(2n-2)}{n!}\, e^{2(n-1)\phi}(\alpha'\mathcal{D}^2)^n\mathcal{R}^4 \\
&+ \text{non-anal.} + \text{non-pert.}
\end{aligned}
$$

Relating M-theory to Superstring

- T-duality invariance at one-loop 4 graviton IIA amplitude = IIB amplitude

$$r_a \leftrightarrow \frac{1}{r_a}; \qquad \frac{r_a}{(g_s^a)^2} = \frac{r_b}{(g_s^b)^2}$$

$$S = \int d^9x\sqrt{g}\mathcal{R}^4 \left[r\frac{2\zeta(3)}{g_s^2} + \frac{4\zeta(2)}{r} + r(\ell_P\Lambda)^3 \right]$$

- Fixes the value of the cut-off

$$4\zeta(2) = (\ell_P\Lambda)^3$$

One loop in 11D

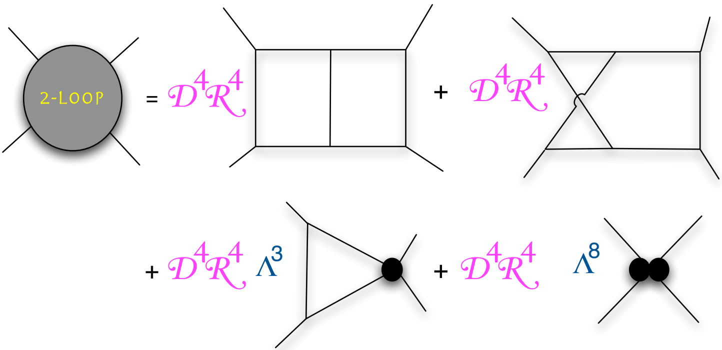
The higher-derivative one string loop terms are not T-duality invariant!!

$$S^{IIA} = \int d^9x \sqrt{g} \sum_{n\geq 2} \frac{\Gamma(n-1/2)\zeta(2n-1)}{n!} \, r_a^{2n-1} (\alpha' \mathcal{D}^2)^n \mathcal{R}^4$$

T-duality will be recovered once the higher-loop amplitudes contributions are included

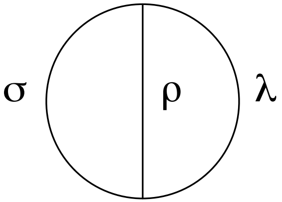
Two loops in 11D

- $\mathcal{N} = 8$ susy implies that the amplitude factorizes $\mathcal{D}^4\,\mathcal{R}^4$ Bern et al.



- This has $\beta_2 = 2$ and the superficial UV behaviour is $\delta_2 = 8$

Two-loop in 11d



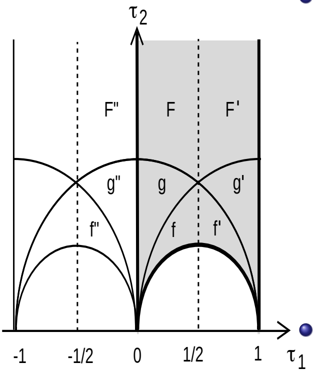
- The planar and non-planar φ^3 double box amplitude sum into a modular invariant amplitude

$$\mathcal{A}^{P+NP} = \mathcal{D}^4\mathcal{R}^4 \int_0^{\Lambda^2} dV V^3 \int_{\Gamma_0(2)} \frac{d^2\tau}{\tau_2^2} f(V,\tau;s,t,u)$$

$$V = \ell_P^2\,(\sigma\rho + \sigma\lambda + \rho\lambda)^{-1/2}$$

- Gives a uniform way of putting the cut-off

Two-loop in 11d



- The planar and non-planar φ^3 double box amplitude sum into a modular invariant amplitude

$$\mathcal{A}^{P+NP} =$$
$$\mathcal{D}^4\mathcal{R}^4 \int_0^{\Lambda^2} dV V^3 \int_{\Gamma_0(2)} \frac{d^2\tau}{\tau_2^2}\, f(V,\tau;s,t,u)$$
$$V = \ell_P^2(\sigma\rho + \sigma\lambda + \rho\lambda)^{-1/2}$$

- Gives a uniform way of putting the cut-off

Two loops in 11D

The 2-loop amplitude on T^2 of volume $\mathcal{V}$ and complex structure Ω

$$\begin{aligned}\mathcal{A}^{P+NP}(\mathcal{V},\Omega) &= \int_0^{\Lambda^2} dV V^3 \int_{\Gamma_0(2)} \frac{d^2\tau}{\tau_2^2}\,\Gamma_{(2,2)}(\mathcal{V},\Omega;V,\tau)\,f(V,\tau;s,t,u)\\ f(V,\tau;s,t,u) &= D^4R^4 + \frac{A_1(\tau,\bar\tau)}{V}D^6R^4 + \frac{A_2(\tau,\bar\tau)}{V^2}D^8R^4 + \cdots\end{aligned}$$

The coupling $A_{(n)}(\tau,\bar\tau) = \sum_i a^i_{(n)}(\tau,\bar\tau)$ satisfies

$$\Delta_\Omega a^i_{(n)} = \lambda_i a^i_{(n)} + P_N(\tau_2)\,\delta(\tau_1)$$

Source term gets contributions from the cusps of $\Gamma_0(2)$

Two loops in 11D

Implies that the 10d coupling satisfies

$$I_{D^4R^4} = \cdots + \frac{(\ell_P \Lambda)^3}{\mathcal{V}^5} E_{5/2} + \cdots$$
$$I_{D^6R^4} = \cdots + \frac{1}{\mathcal{V}^3} E_{(3/2,3/2)} + \cdots$$

where

$$\Delta_\Omega E_{5/2} = \frac{15}{4} E_{5/2}$$
$$\Delta_\Omega E_{(3/2,3/2)} = 12\, E_{(3/2,3/2)} - 6\, E_{3/2}^2$$

Type IIB effective action

- Type IIB theory is obtained by taking M-theory on a 2-torus of *vanishing* volume $\mathcal{V}$ with fixed complex structure $\Omega \equiv \tau$

$$\begin{aligned} \mathcal{S} &= \int d^9x \sqrt{-G^{(9)}}\mathcal{V} \sum_n h^{(n)}(\mathcal{V}, \Omega)\, D^{2n} R^4 \\ h^{(n)}(\mathcal{V}, \Omega) &= \cdots + \frac{1}{\mathcal{V}^{\frac{3+n}{2}}} f^{(n)}(\Omega) + \cdots \end{aligned}$$

- Only this contribution decompactifies to the 10d type limit

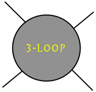
$$\mathcal{S} = \int d^{10}x \sqrt{-g^{(10)}}\, e^{\frac{n-1}{2}\phi} f^{(n)}(\tau)\, D^{2n} R^4$$

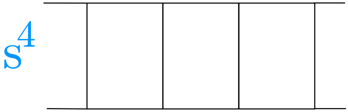
Two loops in 11D

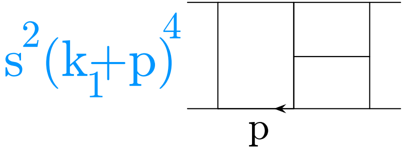
- The D^4R^4 coupling is given by the $E_{5/2}$ series and gets tree-level, no one-loop contributions and a two-loop superstring contributions The perturbative results were checked by Green/Vanhove;D'Hoker/Phong.
- The D^6R^4 is given by the generalized series $E_{(3/2,3/2)}$ and gets tree-level, one-, two- and three-loop superstring contributions.
  The one-loop result was confirmed by Green/Vanhove.
  The 3-loop coefficient matches the one predicts one the type IIA side by the $L=1$ amplitude.
  The F-term computation of Berkovits shows that this contribution is not renormalised after 3-loop.

These ten dimensional couplings should not get corrections from higher-loop amplitudes in 11D

Three loops in 11D

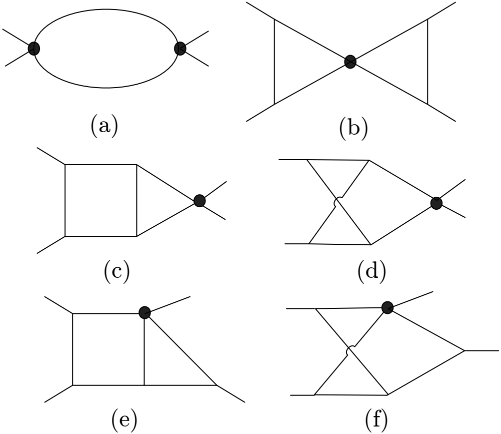






- The ladder diagram has $\beta_3 = 4$ and Mondrian diagram have $\beta_3 = 2$
  Bern et al.

Three loops in 11D (one-loop subdivergences)



Gravitational F-terms

Using the non-minimal pure spinor formalism Berkovits showed that the $D^{2h}R^4$ up to $h=5$ are **F-terms**

At one-loop

$$A_1^{(4)} \sim \int d^{16}\theta_L d^{16}\theta_R \, \theta_L^{11}\theta_R^{11} \, AW^3 \sim R^4$$

At higher-loop up to genus $h=5$

$$\begin{aligned} A_h^{(4)} &\sim \int |d\mu_h|^2 \int d^{16}\theta_L d^{16}\theta_R \, \theta_L^{12-2h}\theta_R^{12-2h} \, W^4 \, |(dx)^{g-2}|^2 \, e^{ik\cdot X} \\ &\sim \int |d\mu_h|^2 \, |(dx)^{h-2}|^2 \, D^{2h}R^4 \, e^{ik\cdot X} \end{aligned}$$

where

$$\begin{aligned} W_{\alpha\beta} &= F_{\alpha\beta} + \cdots + \theta_L^{\gamma}\theta_R^{\delta} \, R_{\alpha\gamma\beta\delta} + \cdots \\ A_{\alpha\beta} &= \theta_L^{\gamma}\theta_R^{\delta} g_{\alpha\gamma\,\beta\delta} + \cdots \end{aligned}$$

In ten-dimensions $\beta_L = L$ for $2 \leq L \leq 5$ and $\beta_L \geq 6$ for $L \geq 6$

UV behaviour of higher-loop amplitudes

When

$$\beta_L = L$$

the amplitude have a reduced UV behaviour since for

$$A_L^{(4)} \sim D^{2L} R^4 \, I(s,t,u;\alpha'/R^2)$$

the residual loop integrals $I(s,t,u;\alpha' R)$ behaves as

$$\delta_L = (D-4)L - 6$$

typical of φ^4 or φ^3 coupled to electromagnetism

Ultraviolet behaviour in $N = 8$ supergravity

- The reduction to $D = 4$ with $\beta_L = L$ leads to

$$A_L^{(4)} \sim D^{2L} R^4 \, I(s, t, u; \alpha'/R^2)$$

- where I has to UV power counting behaviour

$$\delta_L = -6 < 0 \text{ for } L \geq 2$$

- So there is no UV divergences
- But the amplitude can have IR divergences

Ultraviolet behaviour in $N = 8$ supergravity

- The low-energy limit of string theory is obtained by taking

$$\alpha' \to 0; \qquad \kappa_{(4)}^2 = \frac{g_s^2}{V_6} \, \alpha'^4 = \text{fixed}$$

- in this limit NS-branes become massless

$$E_{NS5} = \frac{\alpha'}{\kappa_{(4)}^2}$$

- They could be needed to cure the IR singularities of the amplitude