Towards Absorbing Outer Boundaries in General Relativity

From Geometry to Numerics
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with

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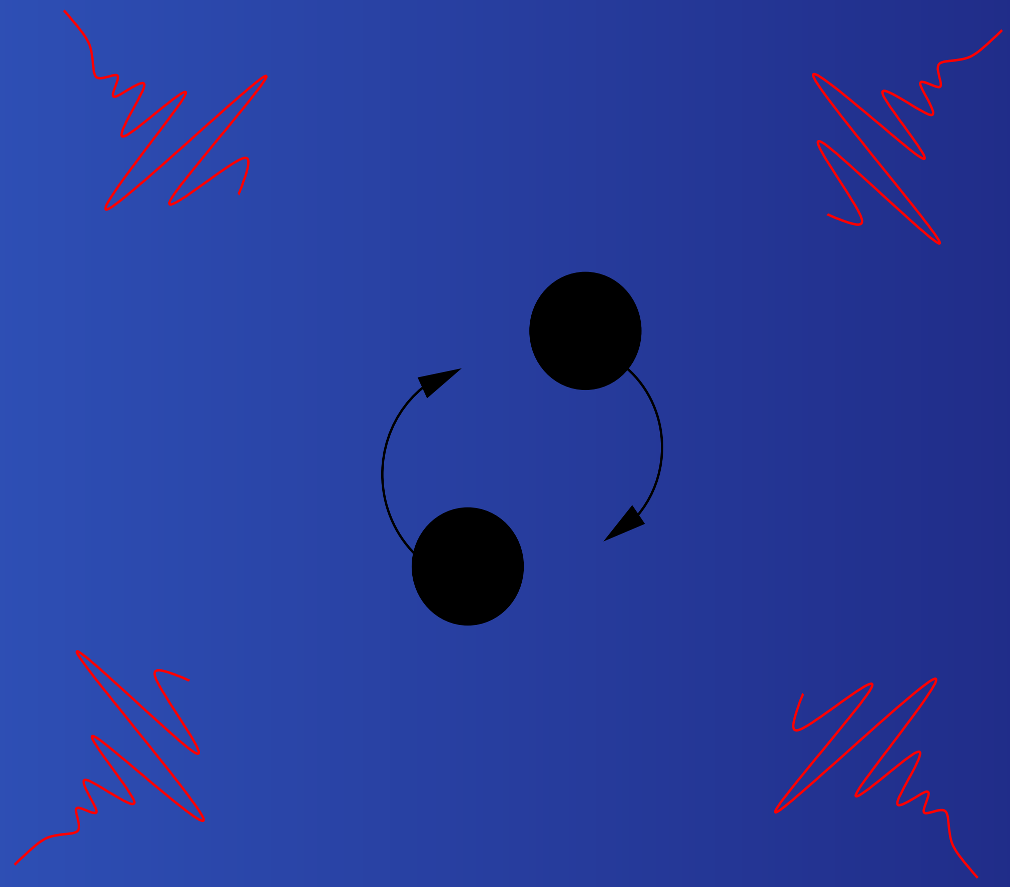
Outline

- Absorbing outer boundaries
- Bianchi equations
- Solutions to IBVP
- Backscatter
- Conclusions

Absorbing outer boundaries

Ideal outer boundary is transparent



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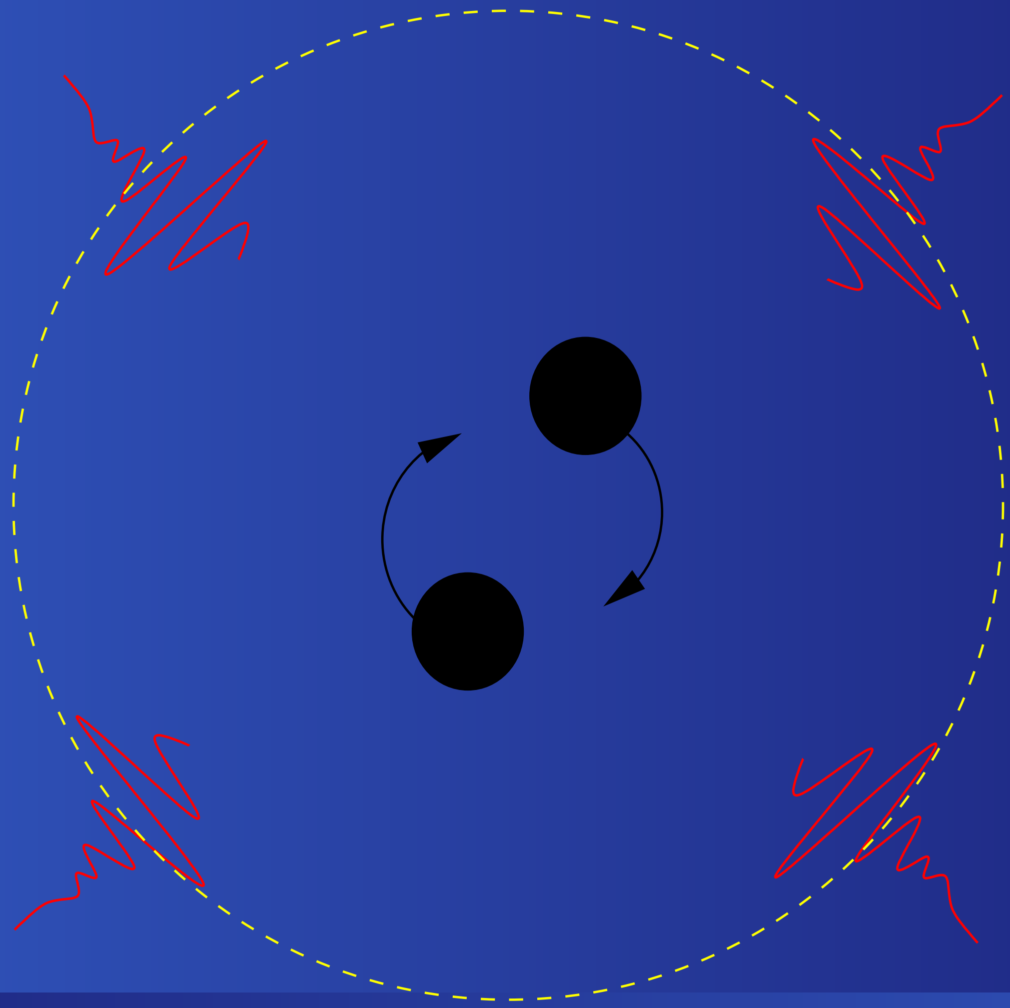

Absorbing outer boundaries

Replace unbounded domain with a compact domain plus an artificial outer boundary.

Ideally, the artificial boundary is completely transparent to the physical problem on the unbounded domain.

Realistically, shoot for boundary conditions (b.c.'s) which:

1. Form a well-posed initial boundary value problem (IBVP).
2. Insure that very little spurious reflection of gravitational radiation occurs from the outer boundary.

Absorbing outer boundaries

Flat Space

- 1D wave equation

$$\left(\partial_t^2 - \partial_x^2\right) u(t,x) = 0, \qquad t > 0,\ x \in [-1,1].$$

General solution is superposition of left- and right-moving solutions

$$u(t,x) = f_{\nwarrow}(x+t) + f_{\nearrow}(x-t),$$

so the b.c.'s

$$(\partial_t - \partial_x)u(t,-1) = 0, \qquad (\partial_t + \partial_x)u(t,+1) = 0,$$

are perfectly absorbing.

Absorbing outer boundaries

Flat Space

- 3D wave equation (much more difficult because of modes propagating tangential to the boundary!)
  Spherical harmonic decomposition

$$u(t,r,\vartheta,\varphi) = \frac{1}{r}\sum_{\ell=0}^{\infty}\sum_{|m|\leq\ell} u_{\ell m}(t,r)Y^{\ell m}(\vartheta,\varphi)$$

  yields

$$\left(\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right)u_{\ell m}(t,r) = 0, \qquad t>0,\ r\in(0,R).$$

  Solutions can be generated from the 1D solutions by applying suitable differential operators to them.

Absorbing outer boundaries

Flat Space

Define the operators $a_\ell \equiv \partial_r + \frac{\ell}{r}, \quad a_\ell^\dagger \equiv -\partial_r + \frac{\ell}{r}$, satisfying the identities

$$a_{\ell+1} a_{\ell+1}^\dagger = a_\ell^\dagger a_\ell = -\partial_r^2 + \frac{\ell(\ell+1)}{r^2} \ .$$

So, for each $\ell = 1, 2, 3...$,

$$\begin{aligned}
\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right] a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger &= \left[\partial_t^2 + a_\ell^\dagger a_\ell\right] a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \\
&= a_\ell^\dagger \left[\partial_t^2 + a_{\ell-1}^\dagger a_{\ell-1}\right] a_{\ell-1}^\dagger ... a_1^\dagger \\
&= a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \left[\partial_t^2 - \partial_r^2\right].
\end{aligned}$$

Absorbing outer boundaries

Flat Space

Explicit in- and outgoing solutions:

$$\phi_{\nwarrow,\ell}(t,r) = a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger V_\ell(r+t),$$
$$\phi_{\nearrow,\ell}(t,r) = a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger U_\ell(r-t).$$

- **Lemma**
  Let $b_- = r^2(\partial_t + \partial_r)$. *Then,* $b_-^{\ell+1}\phi_{\nearrow,\ell}(t,r) = 0$ *for all* $\ell = 0, 1, 2, ...$

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- Therefore, given $L \in \{1, 2, 3, \ldots\}$ the b.c.

$$\mathcal{B}_L: \quad b_-^{L+1}(ru)(t,r,\vartheta,\varphi) = 0\big|_{r=R}$$

leaves the outgoing solutions with $\ell \leq L$ unaltered.

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  leaves the outgoing solutions with $\ell \leq L$ unaltered.
- Can show that each b.c. $\mathcal{B}_L$ yields a well posed problem.

Absorbing outer boundaries

Flat Space

- Therefore, $\mathcal{B}_L$ is perfectly absorbing for waves with $\ell \leq L$.
- Hierarchy of *local* b.c.'s with increasing order of accuracy.
  Bayliss and Turkel, Comm. Pure and Appl. Math., **33**, 707-725 (1980)

Absorbing outer boundaries

General Relativity
A Challenging Problem!

- The future geometry of the outer boundary is not known *a priori*.
- Constraint modes propagate across the boundary.
- "Outgoing" and "ingoing" radiation is difficult to define because of nonlinearities and gauge freedom.

Absorbing outer boundaries

General Relativity

CPBC $\&$ b.c.'s on the gravitational radiation:

- Well-posed IBVP for Einstein's vacuum field equations.
  Friedrich and Nagy 1999
- CPBC $\&$ $\partial_t \Psi_0 \hat{=} 0$ numerically implemented.
  Kidder et al. 2005, Sarbach and Tiglio 2005, Lindblom et al. 2005, Scheel et al. 2006, Rinne 2006
- Hierarchy of local b.c.'s on Ψ_0, which is exact for perturbations on flat spacetime. When 1st order corrections for backscatter are included, the b.c. for quadrupolar radiation gives significantly less reflection than $\partial_t \Psi_0 \hat{=} 0$.
  LTB and O. Sarbach, CQG, **23**, 6709–6744 (2006) (this talk)

Bianchi equations

- Weak field gravity:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ ,$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ is a small ($|h_{\mu\nu}| \ll 1$) perturbation. Neglect quadratic and higher order terms in $h_{\mu\nu}$.

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  OK if boundary is far enough in the wave zone.

- Let the domain be a ball B_R of radius R.
  Not absolutely necessary for our b.c.'s. In any case, modern numerical relativity codes can handle spherical outer boundaries.

Bianchi equations

- Vacuum Bianchi equations:

$$\nabla^a C_{abcd} = 0,$$

  where C_{abcd} is the linearized Weyl tensor.
- Linearized Weyl tensor is invariant w.r.t. infinitesimal coordinate transformations, so there are no gauge modes.
- $3+1$ decomposition yields a symmetric hyperbolic first order system similar to Maxwell's equations.
- Expand the linearized Weyl tensor in spherical tensor harmonics.
- Group the 10 components of the linearized Weyl tensor into 5 complex scalars Ψ_0, Ψ_1, Ψ_2, Ψ_3, Ψ_4, defined w.r.t. the null tetrad: $l = (\partial_t + \partial_r)/\sqrt{2}$, $k = (\partial_t - \partial_r)/\sqrt{2}$, m, $\bar{m}$.

Bianchi equations

Result:

- $\ell = 0$ and $\ell = 1$: solutions are essentially non-dynamical.
- $\ell \geq 2$: dynamics described by two *master equations*.
  From the solutions to these two equations, can reconstruct the linearized Weyl tensor.

Bianchi equations

Master Equations

- Evolution of constraint violations:

$$\left[4\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right]\pi(t,r) = 0.$$

- Evolution of gravitational radiation:

$$\left[\partial_t^2 - \partial_r^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_2(t,r) = S(t,r).$$

- If constraints are satisfied, $S(t,r) = 0$ and the linearized Weyl tensor is entirely determined by the solution ψ_2 of the master equation.

Bianchi equations

Master Equations

- Admit exact analytic solutions, obtained by applying differential operators to solution of 1D flat wave equation (re. 1st sect.).

$$\psi_{2\,\nwarrow,\ell}(t,r) = \frac{1}{r^2} a^\dagger_\ell a^\dagger_{\ell-1} \ldots a^\dagger_1 V_\ell(r+t),$$
$$\psi_{2\,\nearrow,\ell}(t,r) = \frac{1}{r^2} a^\dagger_\ell a^\dagger_{\ell-1} \ldots a^\dagger_1 U_\ell(r-t).$$

- In- and outgoing solutions simply related by $t \mapsto -t$.
- Clear how to quantify amount of spurious reflection and define a reflection coefficient.
- Teukolsky formalism: more complicated!
  Under time reversal, $\Psi_0 \mapsto$ conjugate Ψ_4 and vice versa.

Solutions to IBVP

- Use the exact solutions to construct solutions to the IBVP on B_R corresponding to different boundary conditions on Ψ_0 at ∂B_R (assuming CPBC in place).
- For our exact outgoing solutions, can show that along outgoing null geodesics ($t - r = const.$)

$$\Psi_j = O(r^{j-5}), \quad j = 0, 1, 2, 3, 4.$$

peeling theorem, Penrose, 1965.

- Start with the b.c. $\partial_t \Psi_0 \hat{=} 0$.
- The exact outgoing solutions do not satisfy this b.c. exactly: Ψ_0 falls off as $1/r^5$ along the outgoing null radial geodesics.

Solutions to IBVP

Reflection Coefficients for b.c. $\partial_t \Psi_0 \hat{=} 0$

- A solution to the IBVP corresponding to the b.c. $\partial_t \Psi_0 \hat{=} 0$ consists of a superposition of an out- and an ingoing wave.
- To quantify the amount of reflection, make the monochromatic ansatz

$$\psi_2(t,r) = a_\ell^\dagger a_{\ell-1}^\dagger ... a_1^\dagger \left(e^{ik(r-t)} + \gamma e^{-ik(r+t)} \right),$$

where γ is an *amplitude reflection coefficient*

$$\equiv \frac{ingoing \quad wave \quad amplitude}{outgoing \quad wave \quad amplitude} .$$

Solutions to IBVP

- Reflection coefficients for b.c. $\partial_t \Psi_0 \hat{=} 0$:

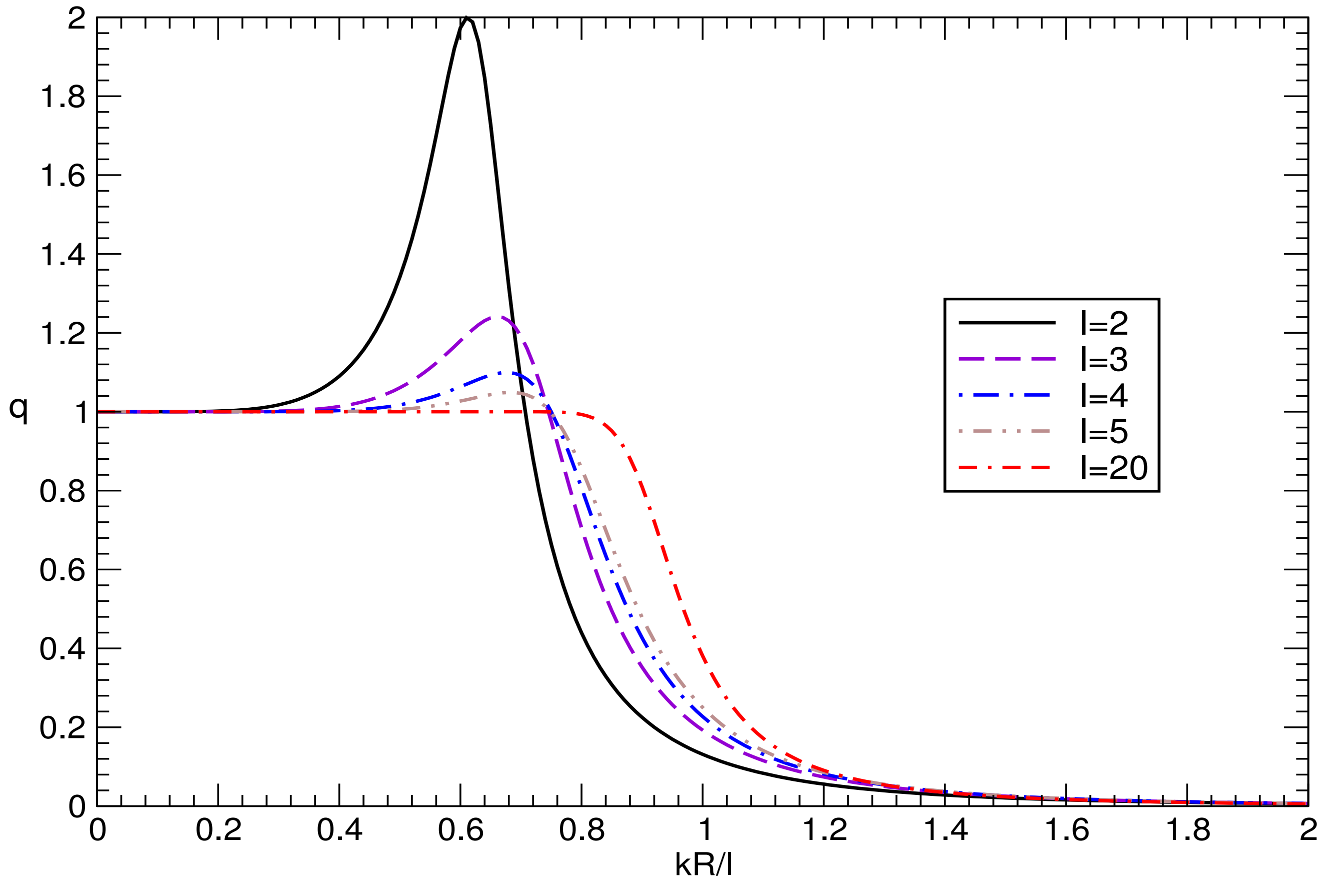
$$q \equiv |\gamma| = \left| \frac{p_{\ell,-2}(-ikR)}{p_{\ell,2}(ikR)} \right|$$

where the polynomials $p_{\ell,m}(z)$, $|m| \leq \ell$, are given by

$$p_{\ell,m}(z) = \sum_{j=0}^{\ell+m} \frac{(\ell+m)!\,(2\ell-j)!}{(\ell+m-j)!\,j!}\,(2z)^j.$$

- $|\gamma|$ is of order unity if $kR < \ell$, and decays as $(kR)^{-4}$ for large kR/ℓ.

q vs. kR/ℓ for b.c. $\partial_t \Psi_0 \hat{=} 0$



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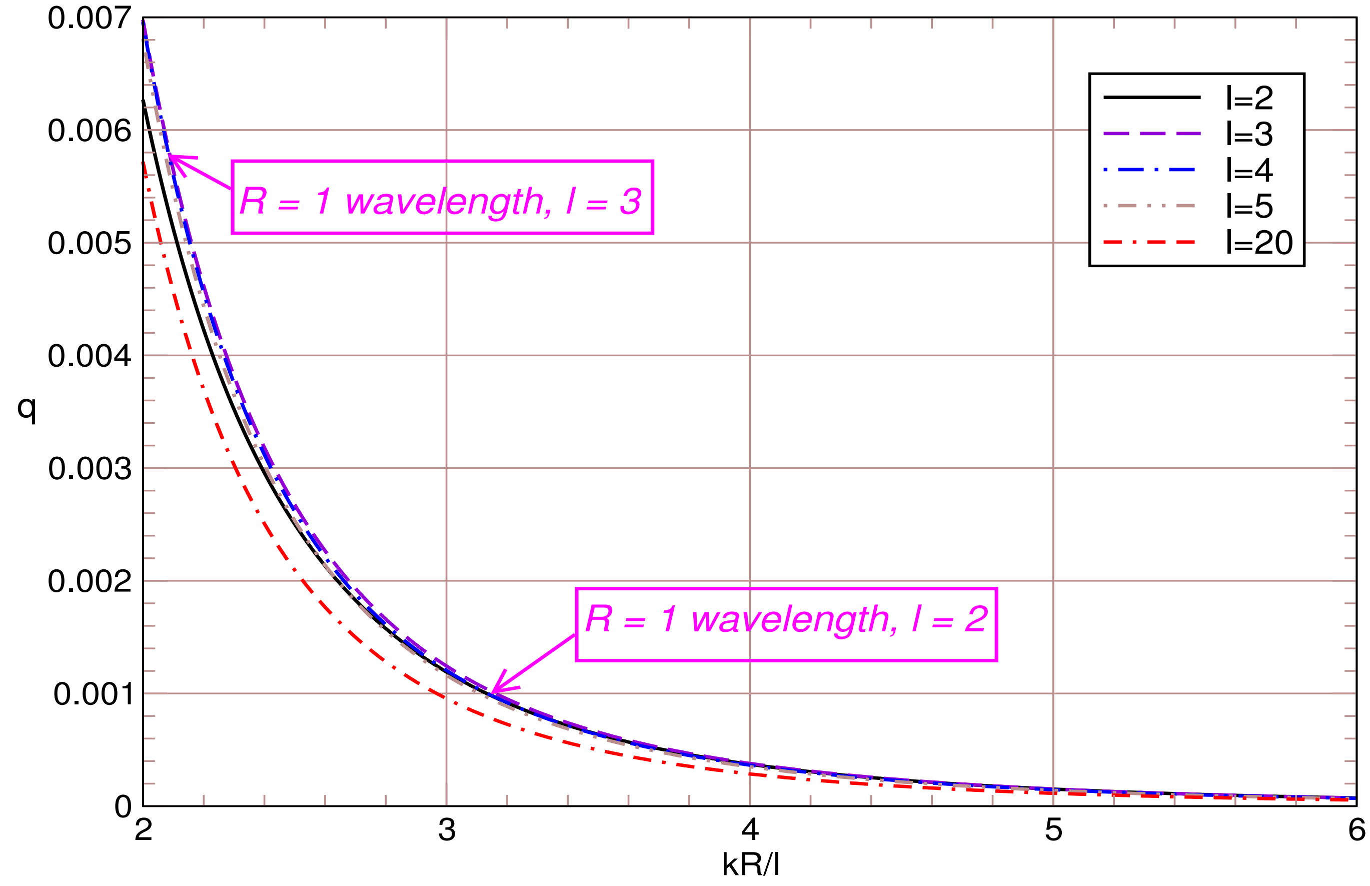

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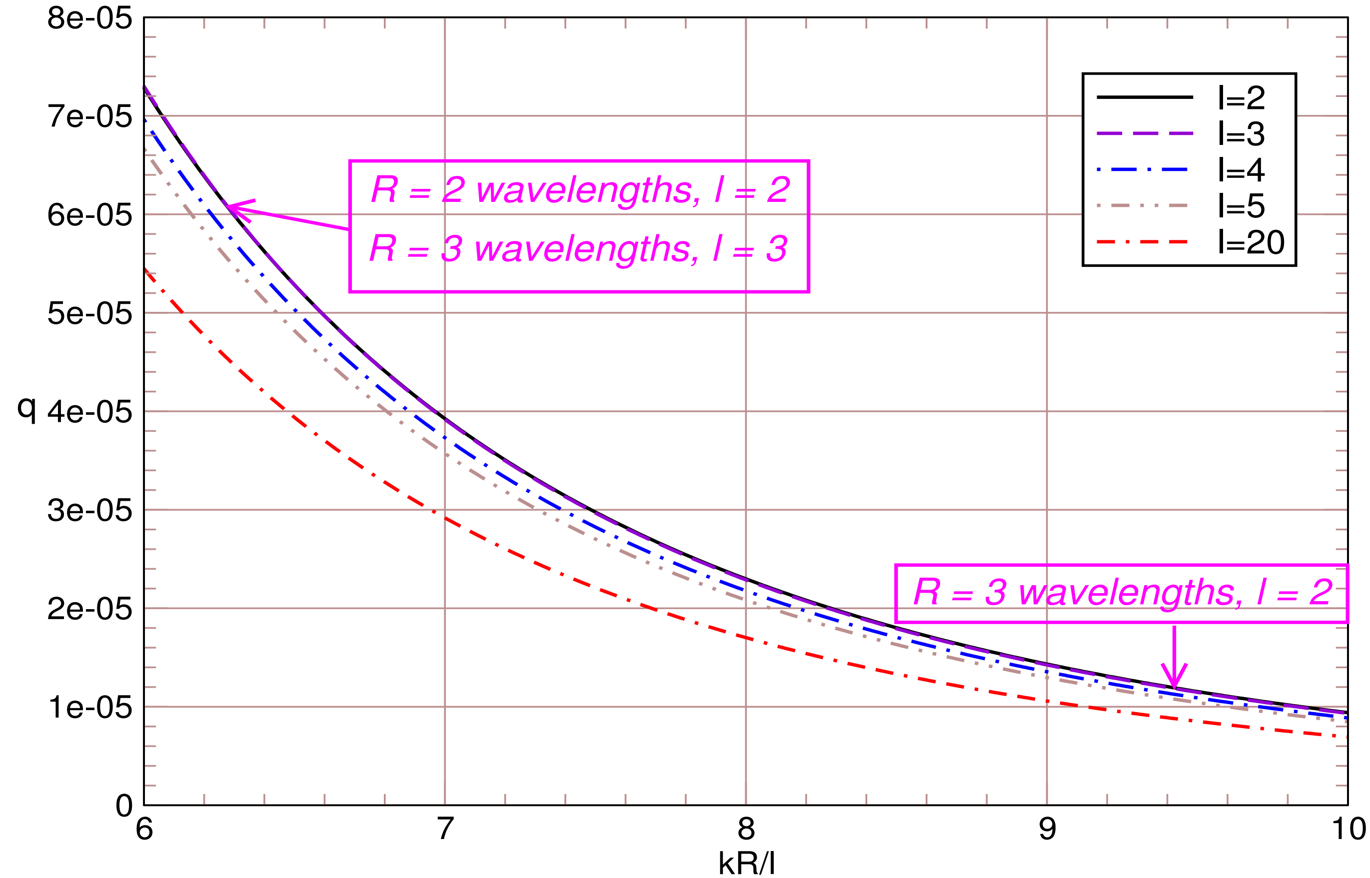

Solutions to IBVP

Hierarchy $\mathcal{B}_L$ of improved b.c.'s

- New b.c. $\mathcal{B}_L$ which, for $L \geq 2$, improve the $\partial_t \Psi_0 \hat{=} 0$ b.c., being *perfectly absorbing* for linearized gravitational radiation in flat space (assumed near the outer boundary) with $\ell \leq L$.

$$\mathcal{B}_L : \quad (b_-)^{L-1}(r^5 \Psi_0) = 0\Big|_{r=R} .$$

- Relation between Ψ_0 and ψ_2:

$$r^5 \Psi_0 \sim (b_-)^2 \psi_2, \qquad b_- = r^2(\partial_t + \partial_r).$$

- Setting $\partial_t \Psi_0 \hat{=} 0$ corresponds to the Bayliss-Turkel b.c. on ψ_2 for $L = 1$.

Solutions to IBVP

- In numerical simulations, expect the few lower multipoles to dominate, so an implementation of this b.c. for $L = 2, 3$ or 4 should suppress much of the spurious reflection.
- For $L = 2$:

$$(\partial_t + \partial_r)\partial_t(r^5\Psi_0) = 0.$$

- Reflection coefficients for $\ell > L$: decay as $(kR)^{-2(L+1)}$ for large kR.

Backscatter

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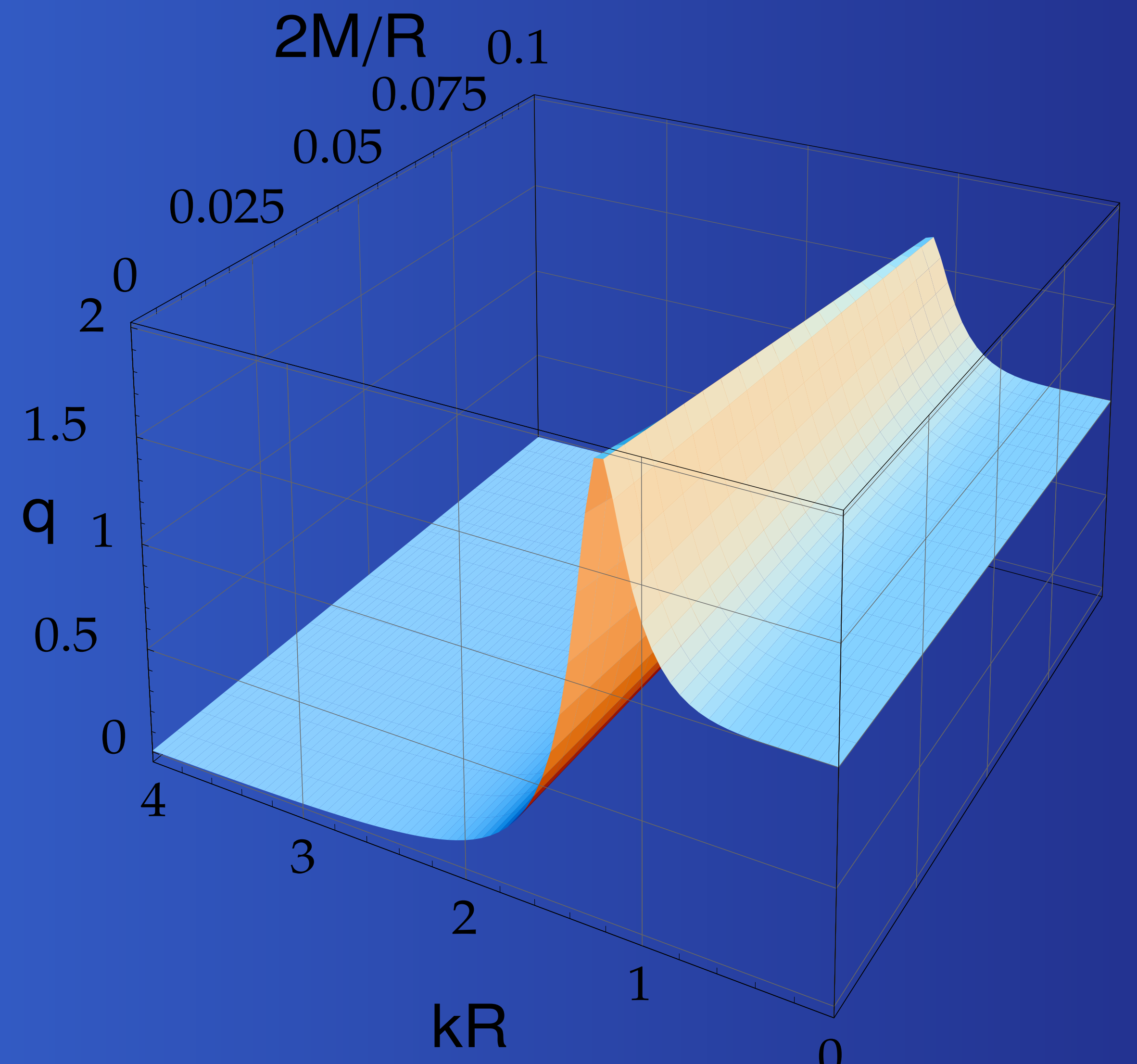
- Result ($\partial_t \Psi_0 \hat{=} 0$) b.c.:
  For $2M/R \ll 1$, the corrected $\ell = 2$ reflection coefficient depends only weakly on $2M/R$.

q vs. kR & $2M/R$ ($\ell = 2$, $\partial_t \Psi_0 \hat{=} 0$)



Backscatter

- Result ($\mathcal{B}_2$ new b.c.):
  Reflection coefficient is smaller than the b.c. $\partial_t \Psi_0 \hat{=} 0$ by a factor of M/R for $kR > 1.05$.

Conclusions

- Estimate amount of spurious reflection off an artificial outer boundary with the b.c. $\partial_t \Psi_0 \hat{=} 0$.
- Propose a hierarchy $\mathcal{B}_L$ $(L = 2, 3, 4, ...)$ of new local b.c.'s which are perfectly absorbing for linearized waves with $\ell \leq L$ on a flat background.
- Including backscatter (to 1st order), these new b.c.'s give a reflection coefficient which is smaller than the one for $\partial_t \Psi_0 \hat{=} 0$ by a factor of M/R for $kR > 1.05$.

Conclusions

For binary black hole simulations:

- New b.c.'s $\mathcal{B}_L$ can be applied to any formulation of the full nonlinear Einstein equations, so long as CPBC are also implemented, and the foliation near the outer boundary resemble the $t =$ const. foliation of Minkowski space.
- Implementation of $\mathcal{B}_L$ may improve accuracy.
- Reflection coefficients provide a way to compute the error in the energy flux due to spurious reflections.
- $\mathcal{B}_L$ may also be useful to minimize reflections of "junk" radiation present in the initial data.

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- More general outer boundary shapes (not just metric spheres).
- Well posedness proof for full nonlinear case.