Gravitational waves in massless scalar-tensor theories

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Outline

Different steps of the presentation :

- define massless scalar-tensor theories
- find the field equations
- solve the field equation using two approximations and the matching equation
- compute waveforms using the matter sources

Full scalar-tensor action : $S_{ST} = S_{\text{gravity}} + S_{\text{matter}}$

Jordan-frame action:

$$S_{\text{gravity}}[g_{ab}, \phi] = \frac{c^3}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right]$$

Scalar field parametrizations : $\phi = \phi_0 \varphi = \phi_0 (1 + \psi)$

Conformal metric : $\tilde{g}_{\mu\nu} := \frac{\phi}{\phi_0} g_{\mu\nu} = \varphi g_{\mu\nu}$

Gothic metric : $\tilde{\mathfrak{g}}^{\mu\nu} = \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$

Einstein-frame action :

$$S_{\text{gravity}}[\tilde{g}_{ab}, \varphi] = \frac{c^3 \phi_0}{16\pi G} \int \mathrm{d}^4x \sqrt{-\tilde{g}} \left[\tilde{R} + \frac{3}{\varphi} \tilde{g}^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi - \frac{9 + 2\omega(\phi)}{2\varphi^2} \tilde{g}^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi \right]$$

Harmonic gauge-fixing

Add gauge-fixing term $-\frac{1}{2}\tilde{g}_{\alpha\beta}\tilde{g}^{\mu\nu}\tilde{g}^{\rho\sigma}\tilde{\Gamma}^{\alpha}_{\mu\nu}\tilde{\Gamma}^{\beta}_{\rho\sigma}$in action which automatically ensures the harmonic gauge condition:

$$\partial_\mu h^{\mu\nu} = 0$$

Find:

$$S_{\text{gravity}}[h^{ab},\psi] = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x \Bigg[-\frac{1}{2}\left(\tilde{\mathfrak{g}}_{\mu\rho}\tilde{\mathfrak{g}}_{\nu\sigma} - \frac{1}{2}\tilde{\mathfrak{g}}_{\mu\nu}\tilde{\mathfrak{g}}_{\rho\sigma}\right)\tilde{\mathfrak{g}}^{\lambda\tau}\partial_\lambda\tilde{\mathfrak{g}}^{\mu\nu}\partial_\tau\tilde{\mathfrak{g}}^{\rho\sigma}$$

$$+\tilde{\mathfrak{g}}_{\mu\nu}\left(\partial_\rho\tilde{\mathfrak{g}}^{\mu\sigma}\partial_\sigma\tilde{\mathfrak{g}}^{\nu\rho} - \partial_\rho\tilde{\mathfrak{g}}^{\mu\rho}\partial_\sigma\tilde{\mathfrak{g}}^{\nu\sigma}\right) - \frac{3+2\omega(\psi)}{\phi_0^2(1+\psi)^2}\tilde{\mathfrak{g}}^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi\Bigg]$$

Field equations

The action principle applied to the full action : $S_{ST} = S_{\text{gravity}} + S_{\text{matter}}$ yields the *exact, coupled and non-linear* field equations:

$$\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$$
$$\Box_\eta \psi = -\frac{8\pi G}{c^4}\tau_s$$

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0}(-g)\,T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h,\psi]$$
$$\tau_s = -\frac{\varphi}{\phi_0(3+2\omega(\phi))}\sqrt{-g}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right) - \frac{c^4}{8\pi G}\Lambda_\phi[h,\psi]$$

$$\Lambda^{\mu\nu}[h,\psi] = \Lambda^{\mu\nu}_{LL}[h] + \Lambda^{\mu\nu}_{H}[h] + \Lambda^{\mu\nu}_{GF}[h] + \Lambda^{\mu\nu}_{S}[h,\psi]$$

N.B. : the harmonic gauge need not be imposed by hand because it is enforced by the field equations via the gauge-fixing term

The strong equivalence problem

Scalar-tensor theories :

- respect the weak equivalence principle (universality of free-fall)
- violate the strong equivalence principle (free-fall independent of the constitution of test-bodies)

Following Eardley 1975, account for this by making the mass ϕ-dependent, and Taylor-expanding around ϕ_0 :

$$m_A(\psi) = \overline{m}_A \left(1 + s_A\psi + \frac{s_A^2 - s_A + s'_A}{2}\psi^2 + \mathcal{O}(\psi^3)\right)$$

where

$$s_A^{(k)} := \left.\frac{\mathrm{d}^{k+1}\ln m_A}{\mathrm{d}(\ln\phi)^{k+1}}\right|_{\phi=\phi_0}$$

- Two-body problem (structureless point particles, but have to account for strong equivalence principle violation)
- Skeletonized matter action for point particles, where $v_A^\mu := (c, v_A^i)$

$$S_{\mathrm{matter}}[g_{ab}, y_1, y_2, v_1, v_2, \phi] = -\sum_A \int \mathrm{d}t\, m_A(\phi) c^2 \sqrt{-\left. g_{\alpha\beta}\right|_{\vec{y}_A(t)} \frac{v_A^\alpha(t) v_B^\beta(t)}{c^2}}$$

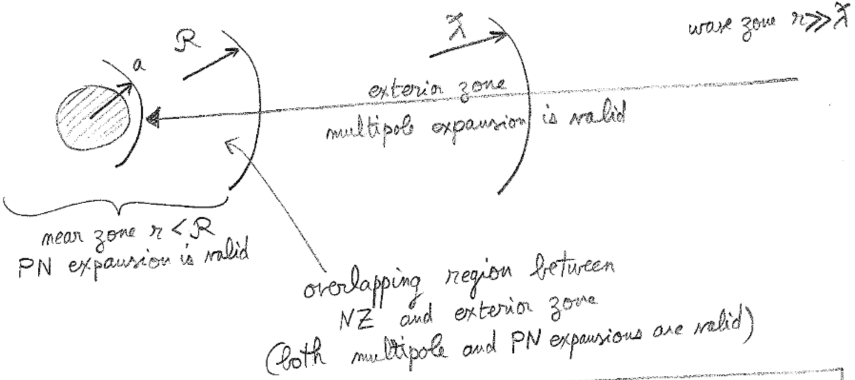
- Energy-momentum tensor:

$$T^{\mu\nu}(t, \vec{x}) := -\frac{2}{\sqrt{-g}} \frac{\delta S_{\mathsf{matter}}}{\delta g_{\mu\nu}}$$

Solving the field equations

The field equations $\Box_\eta h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}$ and $\Box_\eta \psi = -\frac{8\pi G}{c^4}\tau_s$ are valid everywhere, but must be approximated differently :

- in the interior near zone $(r < a \sim R_S \ll \lambda/2\pi,\ T^{\mu\nu} \neq 0)$
- in the exterior near zone $(R_S \sim a < r < \mathcal{R} \sim \lambda/2\pi,\ T^{\mu\nu} = 0)$
- in the wave zone $(r \gg \mathcal{R} \sim \lambda/2\pi,\ T^{\mu\nu} = 0)$



Matching equation

In the exterior near zone and in the wave zone, there is no matter so the most general solution can be parametrized via STF radiative moments U_L :

$$h \sim \sum_{\ell=0}^{\infty} \alpha_\ell \, \partial_L \left[\frac{U_L(t - r/c)}{r} \right] \sim \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} \beta_{\ell,m} \frac{n_L \overset{(m)}{U}_L(t - r/c)}{r^{\ell-m} c^m}$$

Partial proof : $\Box[F(t - r/c)/r] = 0 \Rightarrow \Box h = 0$ (commute derivatives)

In the interior and exterior near zones, matter exists but retardation are small, so we truncate to finite PN order : $\Box \bar{h} \sim \bar{\tau}$ which can be solved by $\bar{h} \sim \mathcal{FP} \Box^{-1} \bar{\tau}$. This can be explicitly expressed as a multipolar expansion :

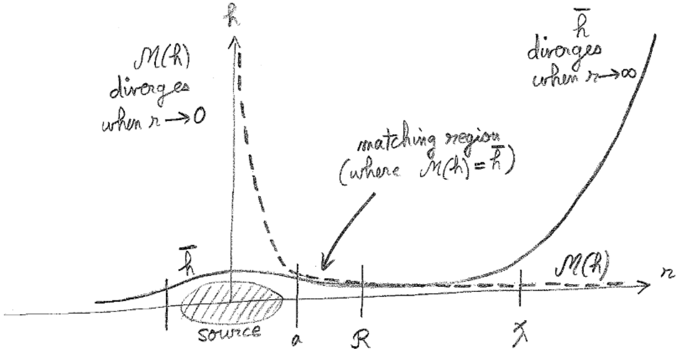
$$\bar{h} \sim \sum_{p=0}^{N} \sum_{q=1}^{M[N]} \gamma_{p,q} \frac{n_{P+Q} F_{P+Q,p,q}[I, J, ...]}{c^p r^q} \sim \frac{1}{r} \sum_{\ell=0}^{\Gamma[N]} \sum_{m=0}^{\ell} \delta_{\ell,m} \frac{n_L G_L^{(m)}[I, J, ...]}{r^{\ell-m} c^m}$$

Matching equation

To link the two approximations, we promote a numerical equality in the overlapping zone to a formal matching of two asymptotic series

$$\overline{\mathcal{M}(h)} \equiv \sum \hat{n}_L r^p (\ln r)^q F(t) \equiv \mathcal{M}(\overline{h})$$

$$\mathsf{NZ}_{r/c\to 0}\Big[\mathsf{Multipolar}_{a/r\to 0}\Big] \equiv \mathsf{FZ}_{r\to\infty}\Big[\mathsf{PN}_{c\to\infty}\Big]$$



Our work

Our work : compute the source multipoles (I_L) for the gravitational and scalar fields (hard at high order !) and deduce *via* the matching equation the asymptotic gravitational and scalar waveforms. These waveforms parametrize reasonable deviations to general relativity: when compared to gravitational wave detector signals, we can either constrain scalar-tensor theories even further, or discover deviations from general relativity.

References

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