Figures of lecture 4

Evolution of black holes and the second law

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS, Observatoire de Paris-PSL, Université Paris Cité, Meudon, France

and

Laboratoire de Mathématiques de Bretagne Atlantique
CNRS, Université de Bretagne Occidentale, Brest, France

https://relativite.obspm.fr/blackholes/aei23/

Albert Einstein Institute

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These lectures

provide an introduction to BH thermodynamics

- focussing on classical (non-quantum) aspects
- keeping the spacetime dimension n general
- not restricting the theory of gravity to general relativity

Home page

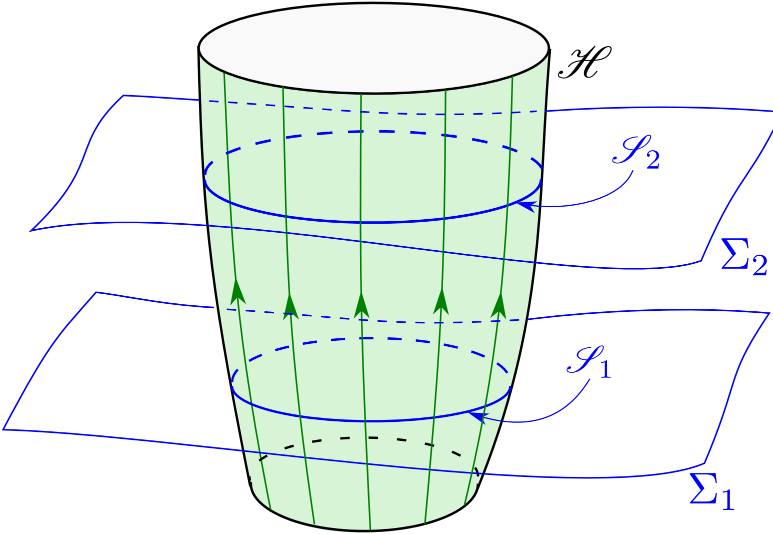
https://relativite.obspm.fr/blackholes/aei23

includes

- the lecture notes (draft)
- some SageMath notebooks
- these slides

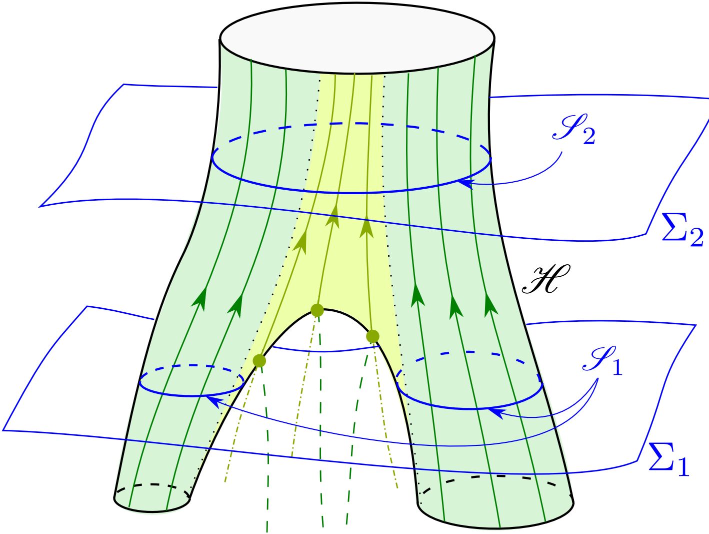
Area theorem

Smooth part of the event horizon

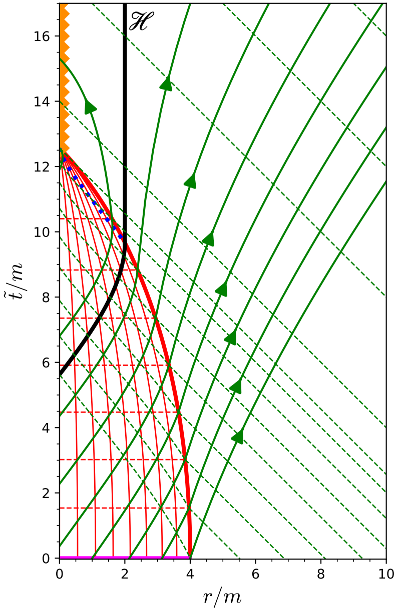


Area theorem

Generic case



Horizon growth in the Oppenheimer-Snyder collapse

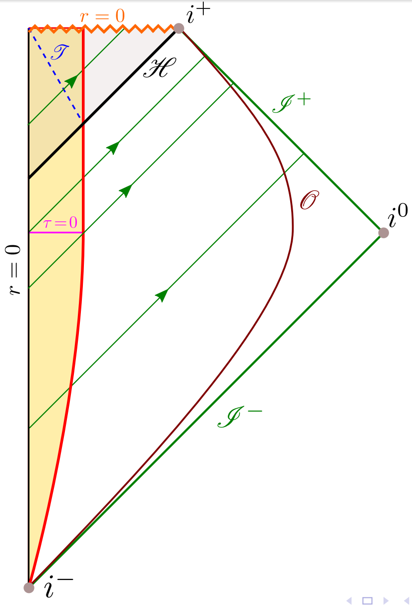


Collapse of a ball of pressureless matter (dust) initially at rest

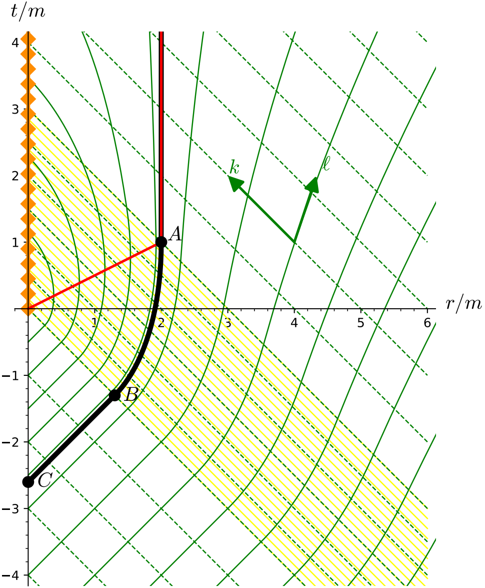
r = areal radius
$\Longrightarrow$ area of a $\tilde{t} = \mathrm{const}$ section of $\mathscr{H}$:
$A = 4\pi r^2$

SageMath notebook: https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Oppenheimer_Snyder.ipynb

Carter-Penrose diagram of the Oppenheimer-Snyder collapse



Horizon growth in the Vaidya collapse

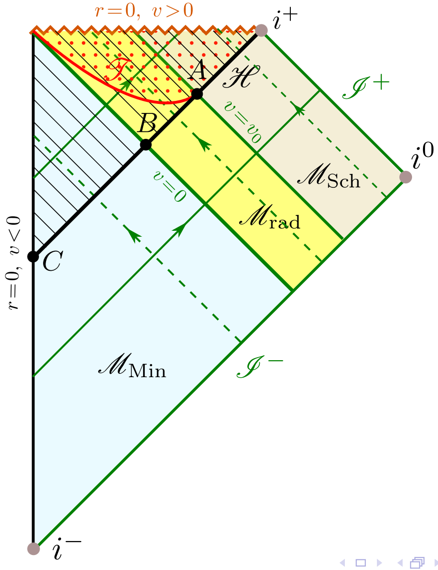


Collapse of shell of electromagnetic radiation

r = areal radius
$\Longrightarrow$ area of a $t = \mathrm{const}$ section of $\mathscr{H}$: $A = 4\pi r^2$

SageMath notebook: https://nbviewer.org/github/egourgoulhon/BHLectures/blob/master/sage/Vaidya.ipynb

Carter-Penrose diagram of the Vaidya collapse



Upper bound on energy extracted via Penrose process

Consider some Penrose process extracting energy from a Kerr black hole of initial mass $m_{\rm i}$ and specific angular momentum $a_{\rm i}$, the extraction taking place until the black hole angular momentum has decayed to zero ($\Longrightarrow$ no longer any ergoregion outside the event horizon).
The final state is then a Schwarzschild black hole of mass $m_{\rm f}$ and the total amount of extracted energy is

$$\Delta E = m_{\rm i} - m_{\rm f}$$

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$$\Delta E = m_{\rm i} - m_{\rm f}$$

Second law $\Longrightarrow$ $A_{\rm f} \geq A_{\rm i}$, i.e. $2m_{\rm f}^2 \geq m_{\rm i}\left(m_{\rm i} + \sqrt{m_{\rm i}^2 - a_{\rm i}^2}\right)$

ΔE is maximal if $m_{\rm f}$ is minimal; given the above inequality, this is achieved for $\boxed{a_{\rm i} = m_{\rm i}} \Longrightarrow 2m_{\rm f}^2 \geq m_{\rm i}^2 \Longrightarrow m_{\rm f} \geq 2^{-1/2} m_{\rm i}$

$$\Longrightarrow \boxed{\Delta E \leq \left(1 - 2^{-1/2}\right) m_{\rm i} \simeq 0.29\, m_{\rm i}}$$

Upper bound on gravitational radiation from a BBH merger (Hawking, 1971)

Consider a binary black hole merger:

- initial stage: two far apart Kerr BHs: (m_1, a_1) and (m_2, a_2)
- final stage: a single Kerr BH: (m_3, a_3)

The total amount of energy radiated via gravitational waves is
$\Delta E = m_1 + m_2 - m_3$

$\Longrightarrow$ **efficiency of gravitational radiation**: $\boxed{\epsilon := \frac{m_1 + m_2 - m_3}{m_1 + m_2}}$

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Second law $\Longrightarrow$ $A_3 \geq A_1 + A_2$, i.e.

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq m_1 \left(m_1 + \sqrt{m_1^2 - a_1^2} \right) + m_2 \left(m_2 + \sqrt{m_2^2 - a_2^2} \right)$$

Upper bound on gravitational radiation from a BBH merger (Hawking, 1971)

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ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_1 = m_1$, $a_2 = m_2$ and $a_3 = 0$

$$\Longrightarrow 2m_3^2 \geq m_1^2 + m_2^2 \Longrightarrow m_3 \geq \sqrt{(m_1^2+m_2^2)/2}$$

$$\Longrightarrow \epsilon \leq 1 - \frac{\sqrt{m_1^2+m_2^2}}{\sqrt{2}(m_1+m_2)}$$

The maximum of the r.h.s. is achieved for $m_1 = m_2$ and is $1/2$, hence the upper bound:

$$\boxed{\epsilon \leq \frac{1}{2}}$$

Upper bound on gravitational radiation from a BBH merger

Case of initially non-spinning equal-mass BHs (Hawking, 1971)

Initially non-spinning equal-mass BHs: $a_1 = a_2 = 0$ and $m_1 = m_2$

The second law yields

$$m_3 \left(m_3 + \sqrt{m_3^2 - a_3^2} \right) \geq 4 m_1^2$$

Again, ϵ is maximal if m_3 is minimal; given the above inequality, this is achieved for $a_3 = 0 \Longrightarrow 2m_3^2 \geq 4m_1^2 \Longrightarrow m_3 \geq \sqrt{2} m_1$

Hence the upper bound:

$$\boxed{\epsilon \leq 1 - 2^{-1/2} \simeq 0.29}$$

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Hence the upper bound:

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The GW efficiency for inspiralling binaries is actually much lower

Inspiralling binary BH merger with $m_1 = m_2$ and $a_1 = a_2 = 0$:
numerical relativity $\implies a_3 = 0.68\, m_3$ and $\epsilon = 0.048$
[Scheel et al., PRD **79**, 024003 (2009)]