Introduction to black hole physics

1. What is a black hole?

Éric Gourgoulhon

Laboratoire Univers et Théories (LUTH)
CNRS / Observatoire de Paris / Université Paris Diderot
Université Paris Sciences et Lettres
92190 Meudon, France

http://luth.obspm.fr/~luthier/gourgoulhon/leshouches18/

École de Physique des Houches
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Home page for the lectures

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Lecture 1: What is a black hole?

Outline

Framework of the lectures

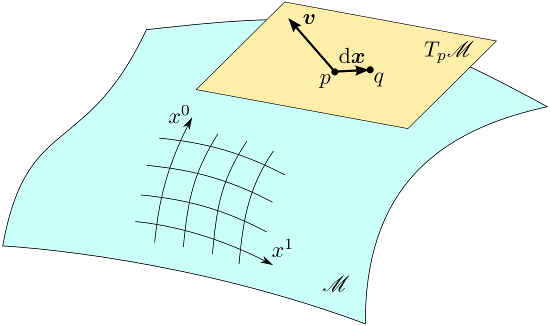
See Thomas Moore lectures for the general relativity background.

Framework regarding the black hole concept:

$$\textbf{spacetime} = (\mathscr{M}, \boldsymbol{g})$$

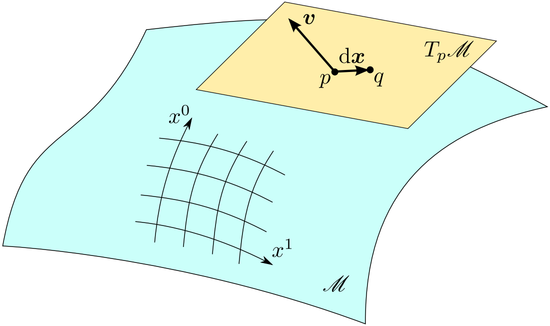
- $\mathscr{M}$: 4-dimensional smooth manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$

Smooth manifold



Smooth manifold: topological space $\mathscr{M}$ that *locally* resembles $\mathbb{R}^n$ (but maybe not globally)
$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

Smooth manifold



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$\Longrightarrow$ **coordinate charts**
$\Longrightarrow$ **tangent vectors**

Remark: vector connecting two points p and q only defined for p and q infinitely close

Lorentzian metric

Metric tensor: (pseudo) scalar product on $\mathscr{M}$, i.e. field $\boldsymbol{g}$ of nondegenerate symmetric bilinear forms on $\mathscr{M}$:

$$\forall p \in \mathscr{M}, \quad \boldsymbol{g}|_p : \begin{array}{ccc} T_p\mathscr{M} \times T_p\mathscr{M} & \longrightarrow & \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) & \longmapsto & \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu \end{array}$$

of signature $(-, +, +, +)$:

$\exists$ basis $(\boldsymbol{e}_\alpha)_{0 \leq \alpha \leq 3}$ of $T_p\mathscr{M}$ such that

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = -u^0 v^0 + u^1 v^1 + u^2 v^2 + u^3 v^3$$

(**Lorentzian** signature)

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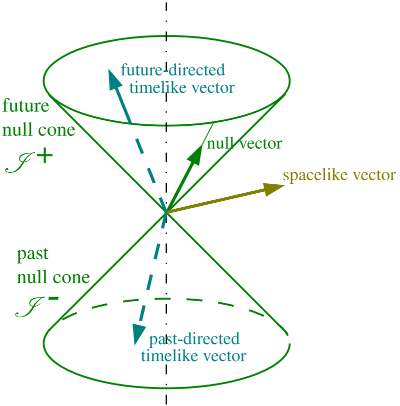
(**Lorentzian** signature)

The "line element":

$$\boxed{\mathrm{d}s^2 := \boldsymbol{g}(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x}) = g_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu}$$

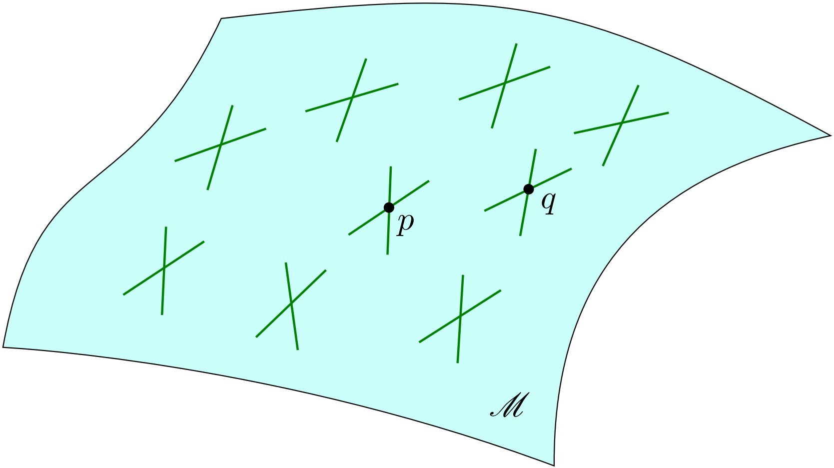
Metric's null cone



Vector $\boldsymbol{v} \in T_p\mathscr{M}$ is

- **spacelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) > 0$
- **null** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) = 0$
- **timelike** $\iff$ $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{v}) < 0$

Lorentzian manifold $(\mathscr{M}, \boldsymbol{g})$



Metric duality

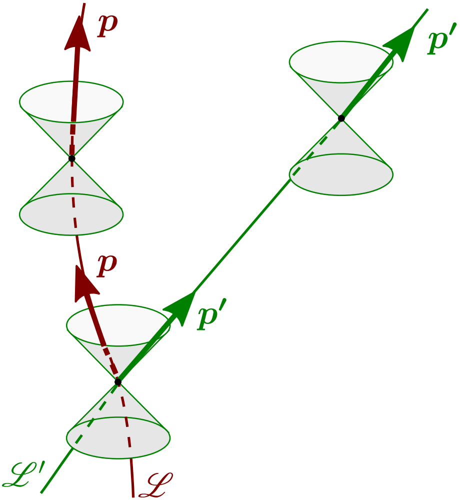
$\boldsymbol{g}$ nondegenerate $\Longrightarrow$ for any vector $\boldsymbol{u} \in T_p\mathscr{M}$, there exists a unique linear form $\underline{\boldsymbol{u}}$ such that
$\forall \boldsymbol{v} \in T_p\mathscr{M}, \quad \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \langle \underline{\boldsymbol{u}}, \boldsymbol{v} \rangle = \underline{u}_\mu v^\mu$

$\underline{\boldsymbol{u}}$ is called the **metric dual** of $\boldsymbol{u}$ and its components are related to those of $\boldsymbol{u}$ by $u_\alpha := \underline{u}_\alpha = g_{\alpha\mu} u^\mu$

Hence the writing

$$\boxed{\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = g_{\mu\nu} u^\mu v^\nu = u_\mu v^\mu}$$

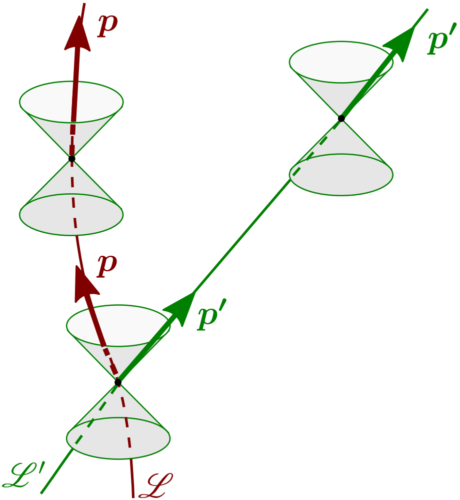
Worldlines



Particle described by its spacetime extent: **worldline** $\mathscr{L}$

massive part. $\iff$ timelike worldline
massless part. $\iff$ null worldline
(tachyon $\iff$ spacelike worldline)

Worldlines



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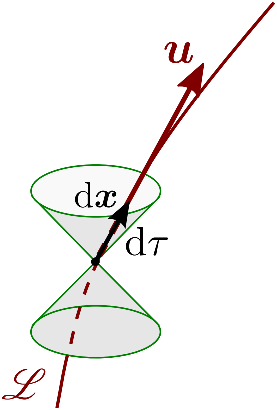
(tachyon $\iff$ spacelike worldline)

Dynamics of a simple particle (no spin, no internal structure) entirely described by a vector field tangent to the worldline: the **4-momentum** $\boldsymbol{p}$

Particle's mass: $\boxed{m = \sqrt{-\boldsymbol{g}(\boldsymbol{p}, \boldsymbol{p})}}$

Physical interpretation of the metric tensor



Elapsed **proper time** corresponding to a small displacement $\mathrm{d}\boldsymbol{x}$ along the worldline $\mathscr{L}$:

$$\boxed{\mathrm{d}\tau = \sqrt{-\boldsymbol{g}(\mathrm{d}\boldsymbol{x}, \mathrm{d}\boldsymbol{x})}}$$

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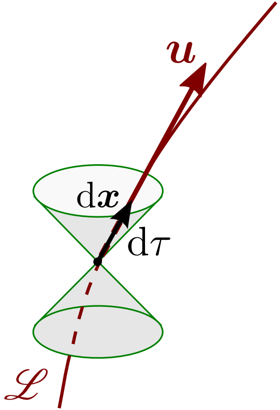

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4-velocity: tangent vector $\boxed{\boldsymbol{u} := \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau}}$

By construction, $\boldsymbol{u}$ is unit timelike:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = -1$$

and is colinear to the 4-momentum $\boldsymbol{p}$:

$$\boldsymbol{p} = m\boldsymbol{u}$$

Einstein equation

General relativity is governed by **Einstein equation:**

$$\boxed{\boldsymbol{R} - \frac{1}{2} R\, \boldsymbol{g} + \Lambda\, \boldsymbol{g} = 8\pi \boldsymbol{T}}$$

where

- $\boldsymbol{R} := \mathrm{Ric}(\boldsymbol{g})$, Ricci tensor: $R_{\alpha\beta} = \mathrm{Riem}(\boldsymbol{g})^{\mu}{}_{\alpha\mu\beta}$
- $R := g^{\mu\nu} R_{\mu\nu}$, Ricci scalar
- Λ cosmological constant
- $\boldsymbol{T}$ energy-momentum tensor of matter

In these lectures, $\Lambda = 0$.

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The definition of a black hole and some of its properties do *not* depend on Einstein equation.
We shall make clear whether some black hole property assumes Einstein equation.

Outline

What is a black hole?

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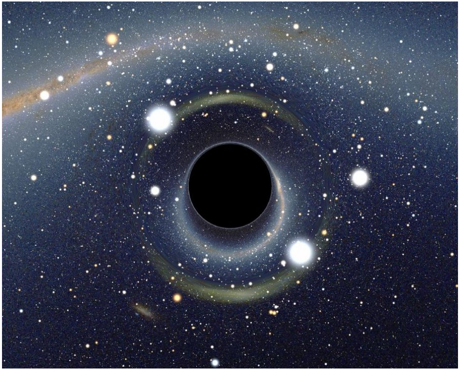
A layperson definition

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.



[A. Riazuelo, arXiv:1511.06025]

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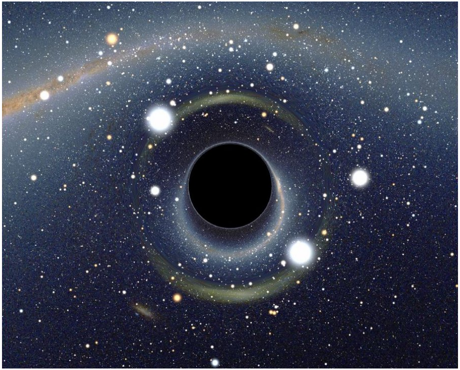

[A. Riazuelo, arXiv:1511.06025]

Two aspects:

- **localization**
- **inescapability**

Boundaries in spacetime

inescapability $\Longrightarrow$ the black hole region is delimited by an impassable boundary: the **event horizon**

Boundary in spacetime: 3-dimensional submanifold, i.e. **hypersurface**

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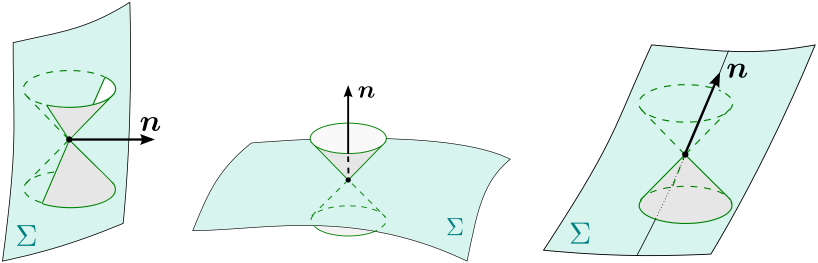
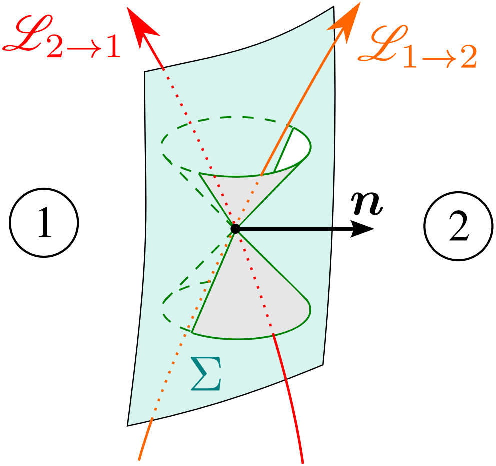
Boundary in spacetime: 3-dimensional submanifold, i.e. **hypersurface**

Locally, 3 types of hypersurfaces Σ in a Lorentzian manifold:



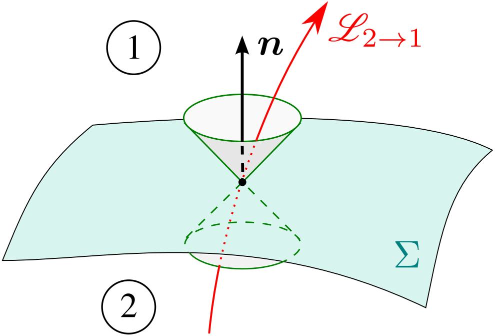
timelike	spacelike	null
$\boldsymbol{n}$ spacelike	$\boldsymbol{n}$ timelike	$\boldsymbol{n}$ null

Timelike hypersurface



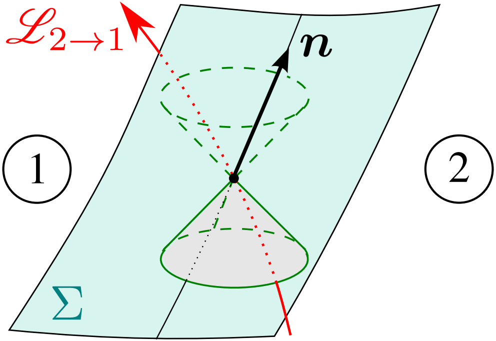
timelike hypersurface = 2-way membrane
$\Longrightarrow$ not eligible for a black hole boundary

Spacelike hypersurface



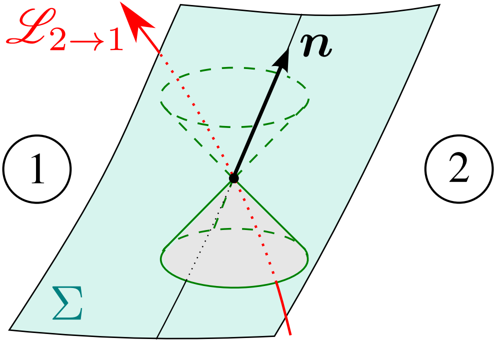
spacelike hypersurface =
1-way membrane
$\Longrightarrow$ eligible for a black hole boundary

Null hypersurface



null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...

Null hypersurface



null hypersurface = 1-way membrane
$\Longrightarrow$ eligible for a black hole boundary...
...and elected! (as a consequence of the formal definition of a black hole, to be given later)

The event horizon of a black horizon is a null hypersurface of spacetime.

Outline

Basic geometry of null hypersurfaces

A hypersurface $\mathscr{H}$ of $(\mathscr{M}, \boldsymbol{g})$ can be (locally) defined as a level set (or "isosurface") of some scalar field $u : \mathscr{M} \to \mathbb{R}$:

$$\forall p \in \mathscr{M}, \quad p \in \mathscr{H} \iff u(p) = 0$$

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$$\forall p \in \mathscr{M}, \quad p \in \mathscr{H} \iff u(p) = 0$$

Any normal $\boldsymbol{\ell}$ to $\mathscr{H}$ must be collinear to the gradient of u:

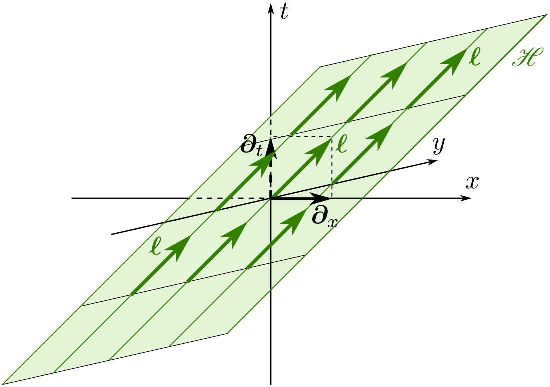
$$\boldsymbol{\ell} = -\mathrm{e}^{\rho} \, \vec{\boldsymbol{\nabla}} u$$

In term of components with respect to a coordinate system (x^α):

$$\ell^\alpha = -\mathrm{e}^\rho \nabla^\alpha u = -\mathrm{e}^\rho g^{\alpha\mu} \nabla_\mu u = -\mathrm{e}^\rho g^{\alpha\mu} \partial_\mu u$$

$\mathscr{H}$ null hypersurface $\iff$ $\boldsymbol{g}(\boldsymbol{\ell}, \boldsymbol{\ell}) = 0$ $\iff$ $g^{\mu\nu} \partial_\mu u \, \partial_\nu u = 0$

Example: null hyperplane in Minkowski spacetime



$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

$$\boxed{u := t - x = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \mathbf{d}x$$

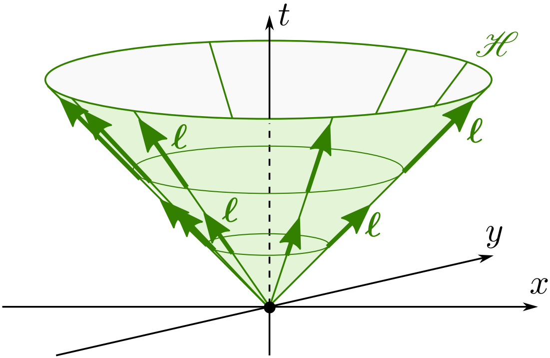
$$\nabla_\alpha u = (1, -1, 0, 0)$$

$$\nabla^\alpha u = (-1, -1, 0, 0)$$

$$\ell^\alpha = (1, 1, 0, 0) \quad (\rho = 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \boldsymbol{\partial}_x}$$

Example: future null cone in Minkowski spacetime



$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\boxed{u := t - \sqrt{x^2 + y^2 + z^2} = 0}$$

$$\boldsymbol{\nabla} u = \mathbf{d}t - \frac{x}{r}\mathbf{d}x - \frac{y}{r}\mathbf{d}y - \frac{z}{r}\mathbf{d}z$$

$$r := \sqrt{x^2 + y^2 + z^2}$$

$$\nabla_\alpha u = \left(1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

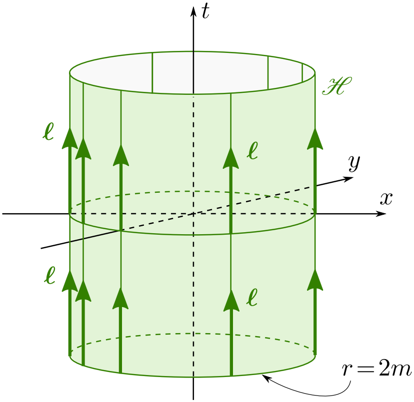
$$\nabla^\alpha u = \left(-1, -\frac{x}{r}, -\frac{y}{r}, -\frac{z}{r}\right)$$

$$\ell^\alpha = \left(1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \qquad (\rho = 0)$$

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{x}{r}\boldsymbol{\partial}_x + \frac{y}{r}\boldsymbol{\partial}_y + \frac{z}{r}\boldsymbol{\partial}_z}$$

Schwarzschild horizon (in Eddington-Finkelstein coordinates)

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \frac{4m}{r}\,\mathrm{d}t\,\mathrm{d}r + \left(1+\frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$



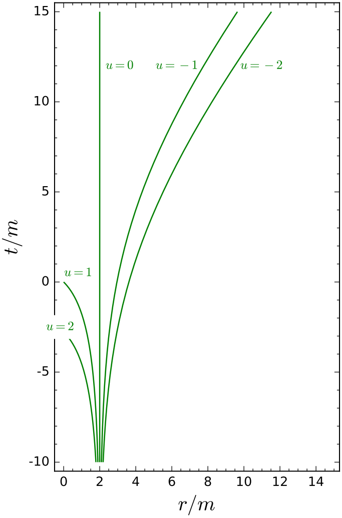
$$\boxed{u := \left(1-\frac{r}{2m}\right)\exp\left(\frac{r-t}{4m}\right) = 0}$$

$$\boldsymbol{\nabla} u = \frac{1}{4m}\mathrm{e}^{(r-t)/(4m)}\left[-\left(1-\frac{r}{2m}\right)\mathbf{d}t - \left(1+\frac{r}{2m}\right)\mathbf{d}r\right]$$

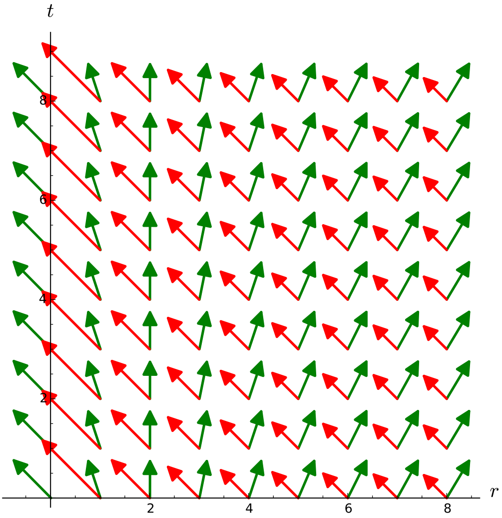
Exercise: compute $\boldsymbol{\ell}$ with ρ chosen so that $\ell^t = 1$ and get

$$\boxed{\boldsymbol{\ell} = \boldsymbol{\partial}_t + \frac{r-2m}{r+2m}\,\boldsymbol{\partial}_r} \Longrightarrow \boxed{\boldsymbol{\ell} \overset{\mathscr{H}}{=} \boldsymbol{\partial}_t}$$

Constant u hypersurfaces around the Schwarzschild horizon



Null normals around the Schwarzschild horizon



Frobenius identity

Starting point: metric dual of the formula relating the null normal ℓ to the gradient of u:

$$\ell_\alpha = -\mathrm{e}^\rho \, \nabla_\alpha u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta = -\mathrm{e}^\rho \nabla_\alpha \rho \nabla_\beta u - \mathrm{e}^\rho \nabla_\alpha \nabla_\beta u$$

$$\Longrightarrow \quad \nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = -\mathrm{e}^\rho \nabla_\alpha \rho \nabla_\beta u + \mathrm{e}^\rho \nabla_\beta \rho \nabla_\alpha u$$

$$\Longrightarrow \quad \boxed{\nabla_\alpha \ell_\beta - \nabla_\beta \ell_\alpha = \nabla_\alpha \rho \, \ell_\beta - \nabla_\beta \rho \, \ell_\alpha}$$

In terms of exterior (Cartan) calculus:

$$\boxed{\mathbf{d}\underline{\boldsymbol{\ell}} = \mathbf{d}\rho \wedge \underline{\boldsymbol{\ell}}}$$

Null geodesic generators

Contract Frobenius identity with $\boldsymbol{\ell}$:

$$\ell^\mu \nabla_\mu \ell_\alpha - \ell^\mu \nabla_\alpha \ell_\mu = \ell^\mu \nabla_\mu \rho \, \ell_\alpha - \underbrace{\ell^\mu \ell_\mu}_{0} \nabla_\alpha \rho$$

Now $\ell^\mu \nabla_\alpha \ell_\mu = \nabla_\alpha (\underbrace{\ell^\mu \ell_\mu}_{0}) - \ell_\mu \nabla_\alpha \ell^\mu \Longrightarrow \ell^\mu \nabla_\alpha \ell_\mu = 0$

Hence

$$\ell^\mu \nabla_\mu \ell_\alpha = \kappa \, \ell_\alpha \quad \text{with } \kappa := \ell^\mu \nabla_\mu \rho = \boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \rho$$

or, by metric duality:

$$\ell^\mu \nabla_\mu \ell^\alpha = \kappa \, \ell^\alpha$$

i.e.

$$\boxed{\boldsymbol{\nabla}_{\boldsymbol{\ell}} \, \boldsymbol{\ell} = \kappa \, \boldsymbol{\ell}}$$

Null geodesic generators

$\boxed{\nabla_{\boldsymbol{\ell}} \boldsymbol{\ell} = \kappa \boldsymbol{\ell}} \implies \boldsymbol{\ell}$ is a pregeodesic vector, i.e. $\exists$ rescaling factor α such that $\boldsymbol{\ell}' = \alpha \boldsymbol{\ell}$ is a geodesic vector field: $\nabla_{\boldsymbol{\ell}'} \boldsymbol{\ell}' = 0$
(*Exercise:* prove it!)
$\implies$ the field lines of $\boldsymbol{\ell}$ are (null) geodesics.

κ is called the **non-affinity coefficient** of the null normal $\boldsymbol{\ell}$, since $\kappa = 0$ iff the parameter λ whose derivative vector is $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda}$$

is an affine parameter of the geodesic.

Null geodesic generators

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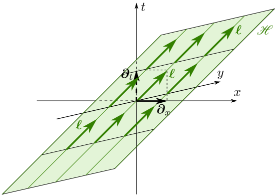
Any null hypersurface $\mathscr{H}$ is ruled by a family of null geodesics, called the **generators of** $\mathscr{H}$, and each vector field $\boldsymbol{\ell}$ normal to $\mathscr{H}$ is tangent to these null geodesics.

Examples of null geodesic generators

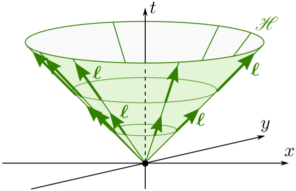
null hyperplane



$$\nabla_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = 0$$

$$\kappa = 0$$

future null cone



$$\nabla_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = 0$$

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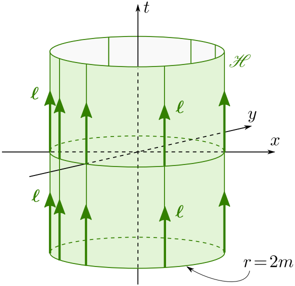
Schwarzschild horizon



$$\nabla_{\boldsymbol{\ell}}\,\boldsymbol{\ell} = \kappa\,\boldsymbol{\ell}$$

$$\kappa = \frac{1}{4m}$$

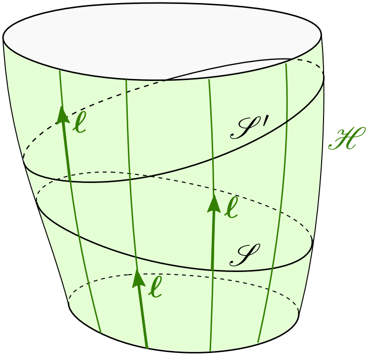
Using SageMath to compute κ for the Schwarzschild horizon

SageMath: free mathematics software system based on Python with tensor calculus capabilities (cf. https://sagemanifolds.obspm.fr)

The computation of κ:
http://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarzschild_horizon.ipynb

See also
https://luth.obspm.fr/~luthier/gourgoulhon/leshouches18/sage.html
for all the notebooks associated with these lectures

Cross-sections of a null hypersurface



cross-section of the null hypersurface $\mathscr{H}$: 2-dimensional submanifold $\mathscr{S} \subset \mathscr{H}$ such that

1. the null normal ℓ is nowhere tangent to $\mathscr{S}$
2. each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{S}$ once, and only once

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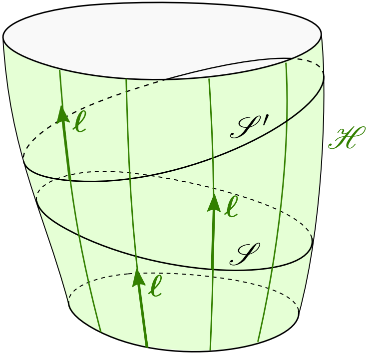

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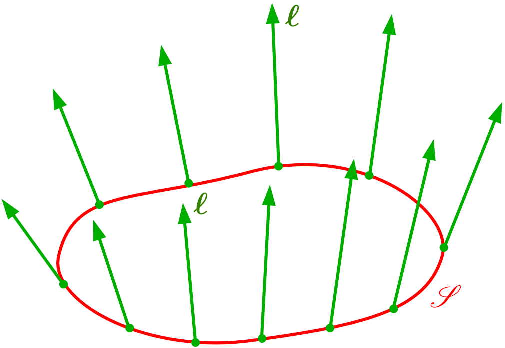
2. each null geodesic generator of $\mathscr{H}$ intersects $\mathscr{S}$ once, and only once

Any cross-section $\mathscr{S}$ is spacelike, i.e. all vectors tangent to $\mathscr{S}$ are spacelike.

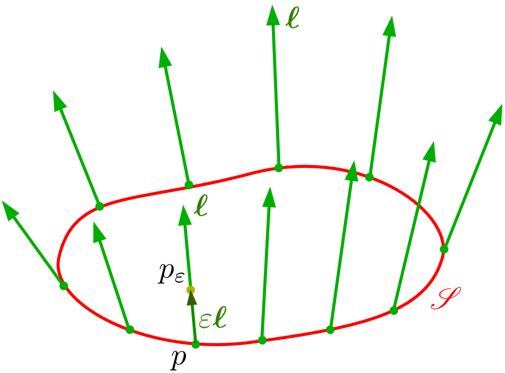
Proof: a vector tangent to $\mathscr{H}$ cannot be timelike, nor null and not normal.

Expansion along a null normal



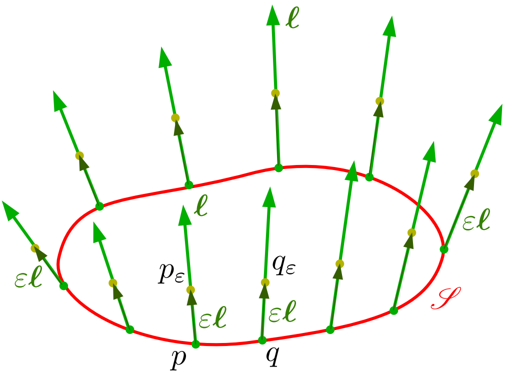
1. Consider a cross-section $\mathscr{S}$ and a null normal ℓ to $\mathscr{H}$

Expansion along a null normal



1. Consider a cross-section $\mathscr{S}$ and a null normal ℓ to $\mathscr{H}$
2. ε being a small parameter, displace the point p by the vector $\varepsilon\boldsymbol{\ell}$ to the point p'

Expansion along a null normal



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3. Do the same for each point in $\mathscr{S}$, keeping the value of ε fixed

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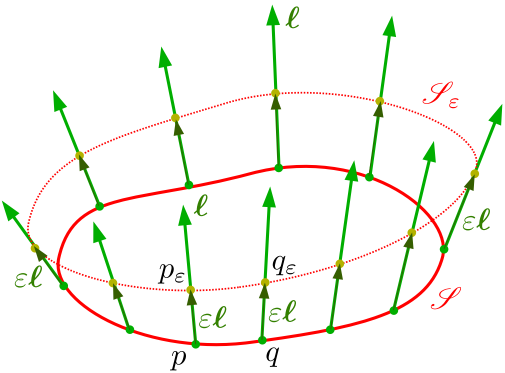

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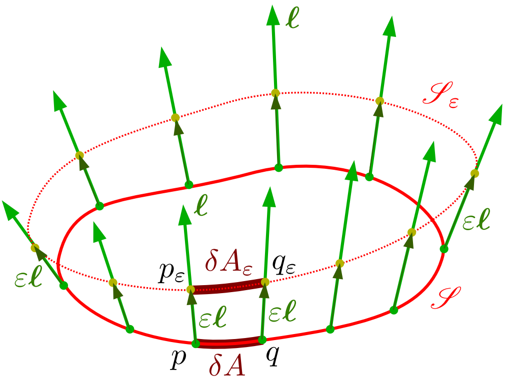

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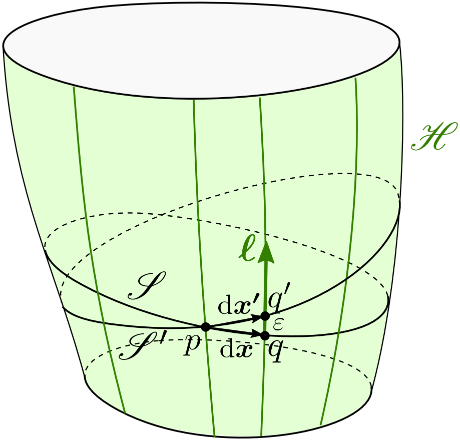
4. This defines a new cross-section $\mathscr{S}_\varepsilon$

At each point, the **expansion along** $\boldsymbol{\ell}$ is defined from the relative change in the area element δA:

$$\boxed{\theta_{(\boldsymbol{\ell})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A_\varepsilon - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{\ell}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu \ell_\nu}$$

Expansion along a null normal



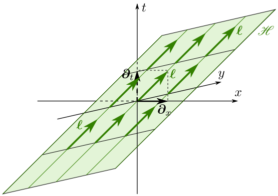
The expansion $\theta_{(\boldsymbol{\ell})}$ depends solely on the null normal $\boldsymbol{\ell}$, not on the choice of the cross-section $\mathscr{S}$.

Dependency of $\theta_{(\boldsymbol{\ell})}$ w.r.t. $\boldsymbol{\ell}$:

$$\boldsymbol{\ell}' = \alpha \boldsymbol{\ell} \Longrightarrow \theta_{(\boldsymbol{\ell}')} = \alpha \theta_{(\boldsymbol{\ell})}$$

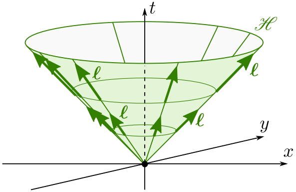
Examples of expansions

null hyperplane



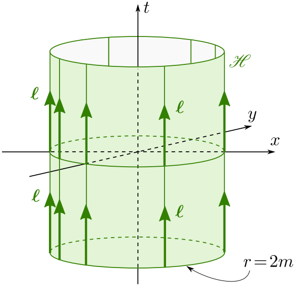
$$\theta_{(\boldsymbol{\ell})} = 0$$

future null cone



$$\theta_{(\boldsymbol{\ell})} = \frac{2}{r}$$

Schwarzschild horizon



$$\theta_{(\boldsymbol{\ell})} = 0$$

Outline

Distinguishing a black hole horizon from a generic null hypersurface

Recall the naive definition stated above:

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.

- no-escape facet $\Longrightarrow$ boundary = null hypersurface

Distinguishing a black hole horizon from a generic null hypersurface

Recall the naive definition stated above:

A **black hole** is a localized region of spacetime from which neither massive particles nor massless ones (photons) can escape.

- no-escape facet $\Longrightarrow$ boundary = null hypersurface
- localized facet: *for equilibrium configurations*, can be enforced by demanding that cross-sections are closed surfaces and have constant area, i.e. vanishing expansion

Non-expanding horizons

Definition

A **non-expanding horizon** is a null hypersurface $\mathscr{H}$ whose cross-sections $\mathscr{S}$ are closed surfaces (i.e. compact without boundary) and such that the expansion along any null normal $\boldsymbol{\ell}$ vanishes identically:

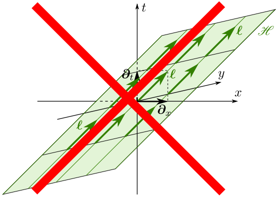
$$\theta_{(\boldsymbol{\ell})} = 0$$

Remark 1: definition independent of $\boldsymbol{\ell}$, due to the scaling $\boldsymbol{\ell}' = \alpha\boldsymbol{\ell} \implies \theta_{(\boldsymbol{\ell}')} = \alpha\theta_{(\boldsymbol{\ell})}$

Remark 2: a non-expanding horizon has the "cylinder" topology $\mathscr{H} \simeq \mathbb{R} \times \mathscr{S}_0$, where $\mathscr{S}_0$ is any cross-section.

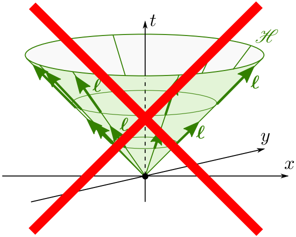
(Counter-)examples of non-expanding horizons

null hyperplane



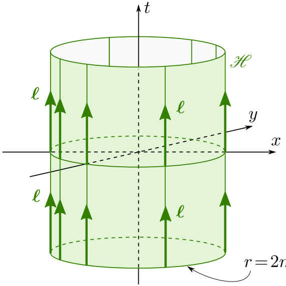
no closed cross-sections

future null cone



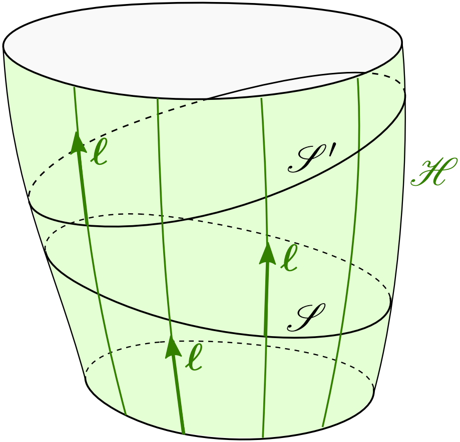
nonzero expansion

Schwarzschild horizon



OK

Area of a non-expanding horizon



Each cross-section $\mathscr{S}$ of $\mathscr{H}$ is a *spacelike* closed surface.

The area of $\mathscr{S}$ is given by the positive definite metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ on $\mathscr{S}$:

$$A = \int_{\mathscr{S}} \sqrt{q}\, \mathrm{d}y^1 \mathrm{d}y^2$$

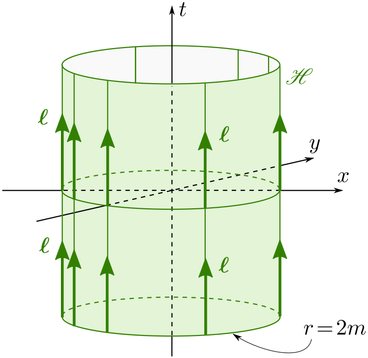
where $y^a = (y^1, y^2)$ are coordinates on $\mathscr{S}$ and $q := \det(q_{ab})$

On a non-expanding horizon, the area A is independent of the choice of the cross-section $\mathscr{S}$ $\Longrightarrow$ **area of** $\mathscr{H}$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{4m}{r}\,\mathrm{d}t\,\mathrm{d}r + \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$



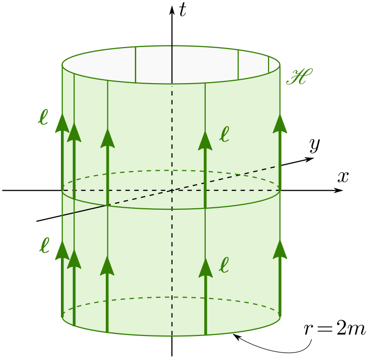
$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

Example: area of the Schwarzschild horizon

Spacetime metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r} dt\, dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$



$\mathscr{H}$: $r = 2m$; coord: (t, θ, φ)

$\mathscr{S}$: $r = 2m$ and $t = t_0$; coord: $y^a = (\theta, \varphi)$

$\Longrightarrow$ induced metric on $\mathscr{S}$:

$$q_{ab}\, dy^a dy^b = (2m)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\Longrightarrow q := \det(q_{ab}) = (2m)^4 \sin^2\theta$$

$$\Longrightarrow A = \int_{\mathscr{S}} (2m)^2 \sin\theta\, d\theta d\varphi$$

$$\Longrightarrow \boxed{A = 16\pi m^2}$$

Killing horizon

Killing vector: generator $\boldsymbol{\xi}$ of a 1-parameter group of symmetry of the spacetime $(\mathscr{M}, \boldsymbol{g})$ (isometries)
$(\mathscr{M}, \boldsymbol{g})$ is invariant along the field lines of $\boldsymbol{\xi}$:

$$\mathcal{L}_{\boldsymbol{\xi}}\, \boldsymbol{g} = 0 \iff \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$$

Definition

A **Killing horizon** is a null hypersurface $\mathscr{H}$ in a spacetime $(\mathscr{M}, \boldsymbol{g})$ admitting a Killing vector field $\boldsymbol{\xi}$ such that, on $\mathscr{H}$, $\boldsymbol{\xi}$ is normal to $\mathscr{H}$.

$\Longrightarrow$ $\boldsymbol{\xi}$ is null on $\mathscr{H}$

$\Longrightarrow$ the null geodesic generators of $\mathscr{H}$ are orbits of the 1-parameter group of isometries of $(\mathscr{M}, \boldsymbol{g})$ generated by $\boldsymbol{\xi}$.

Killing horizons as non-expanding horizons

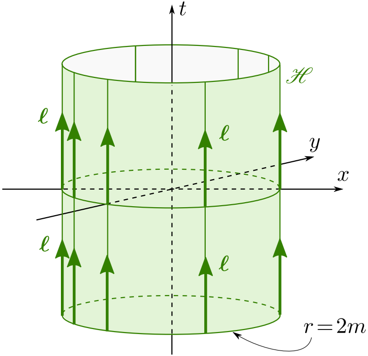
A Killing horizon with closed cross-sections is a non-expanding horizon.

Proof: since $\boldsymbol{\xi}$ is a symmetry generator and $\boldsymbol{\xi} = \boldsymbol{\ell}$ on $\mathscr{H}$, the area δA of an element of cross-section does not vary when Lie-dragged along $\boldsymbol{\ell}$, hence $\theta_{(\boldsymbol{\ell})} = 0$.

Example of Killing horizon: the Schwarzschild horizon

Spacetime metric:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \frac{4m}{r}\,\mathrm{d}t\,\mathrm{d}r + \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$



Killing vector field of Schwarzschild spacetime associated with stationarity: $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

On $\mathscr{H}$: $\boldsymbol{\xi} = \boldsymbol{\ell}$

Outline

Limitation of the concept of non-expanding horizon

Non-expanding horizons capture well the "localized-in-space" feature of the no-escape region. However they do so only for *steady-state configurations*: the area of any cross section must remain constant. In particular, Killing horizons assume that the spacetime is endowed with some symmetry, usually the spacetime is (asymptotically) *stationary*.

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To define black holes in *dynamical* spacetimes, one shall first define properly some "infinitely far" region (concept of *asymptotic flatness*) and distinguish between the timelike or null worldlines, those that can reach this far region from those that cannot.
The definition of the "infinitely far" region is best performed via Penrose's concept of conformal completion.

Conformal completion of Minkowski spacetime

1. Introducing "compactified" coordinates

Spacetime metric:

$$\boxed{ds^2 = -dt^2 + dx^2 + dy^2 + dz^2}$$

Move to spherical coordinates (t, r, θ, φ) via $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$\Longrightarrow \boxed{ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2}$$

Move to coordinates $(\tau, \chi, \theta, \varphi)$ with $0 \leq \chi < \pi$ and $\chi - \pi < \tau < \pi - \chi$

via $$\left\{ \begin{array}{l} \tau = \arctan(t+r) + \arctan(t-r) \\ \chi = \arctan(t+r) - \arctan(t-r) \end{array} \right. \iff \left\{ \begin{array}{l} t = \dfrac{\sin\tau}{\cos\tau + \cos\chi} \\ r = \dfrac{\sin\chi}{\cos\tau + \cos\chi} \end{array} \right.$$

$$\Longrightarrow \boxed{ds^2 = (\cos\tau + \cos\chi)^{-2}\left[-d\tau^2 + d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]}$$

Conformal completion of Minkowski spacetime

2. The conformal metric

Thus we may write $\boldsymbol{g} = \Omega^{-2}\tilde{\boldsymbol{g}}$, or equivalently

$$\boxed{\tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}}$$

with

- $\Omega := \cos\tau + \cos\chi = \dfrac{2}{\sqrt{(t-r)^2+1}\sqrt{(t+r)^2+1}}$
- $\tilde{\boldsymbol{g}}$ is the metric defined by

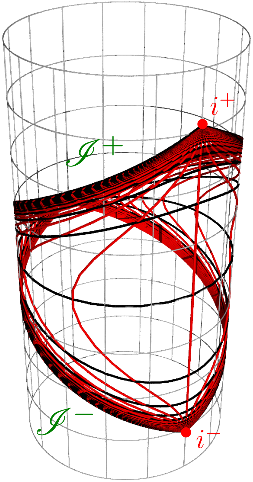
$$d\tilde{s}^2 = -d\tau^2 + \underbrace{d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}_{\text{standard metric on } \mathbb{S}^3}$$

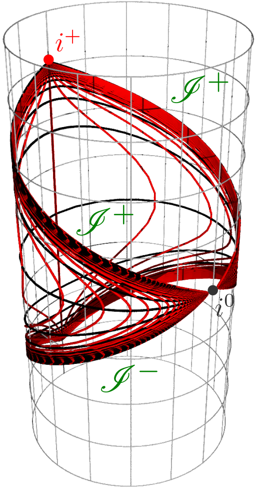
$\tilde{\boldsymbol{g}}$ is a Lorentzian metric on the **Einstein cylinder** $\mathscr{E} = \mathbb{R}\times\mathbb{S}^3$

$(\mathscr{E}, \tilde{\boldsymbol{g}})$ is a solution of Einstein equation with a cosmological constant $\Lambda > 0$ and some pressureless matter of uniform density $\rho = \Lambda/(4\pi)$.

Conformal completion of Minkowski spacetime

3. Embedding into the Einstein cylinder

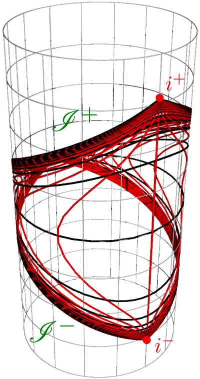


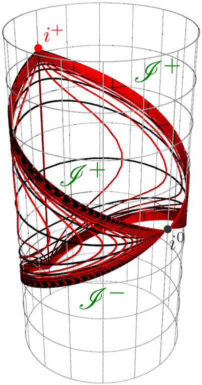


- on $\mathscr{E}$:
  $-\infty < \tau < +\infty$
  $0 \leq \chi \leq \pi$
- on $\mathscr{M}$:
  $\chi - \pi < \tau < \pi - \chi$
  $0 \leq \chi < \pi$

Conformal completion of Minkowski spacetime

3. Embedding into the Einstein cylinder





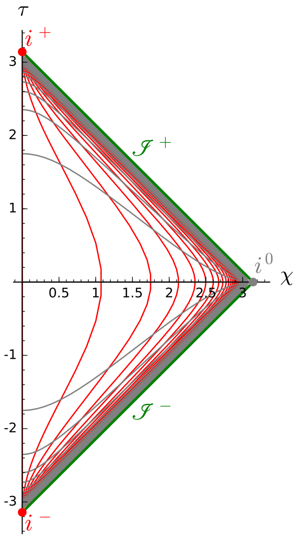
Boundaries of the embedding of $\mathscr{M}$ into $\mathscr{E}$:

- $\mathscr{I}^+$ = hypersurface $\{\tau = \pi - \chi,\ 0 < \tau < \pi\}$
- $\mathscr{I}^-$ = hypersurface $\{\tau = \chi - \pi,\ -\pi < \tau < 0\}$
- i^0 = point $(\tau, \chi) = (0, \pi)$
- i^+ = point $(\tau, \chi) = (\pi, 0)$
- i^- = point $(\tau, \chi) = (-\pi, 0)$

Closure of $\mathscr{M}$ in $\mathscr{E}$: $\overline{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}^+ \cup \mathscr{I}^- \cup \{i^0, i^+, i^-\}$

NB: $\mathscr{I}^+$ and $\mathscr{I}^-$ are *not* parts of $\mathscr{M}$ and i^0, i^+ and i^- are *not* points of $\mathscr{M}$

Conformal completion of Minkowski spacetime
4. Conformal diagram

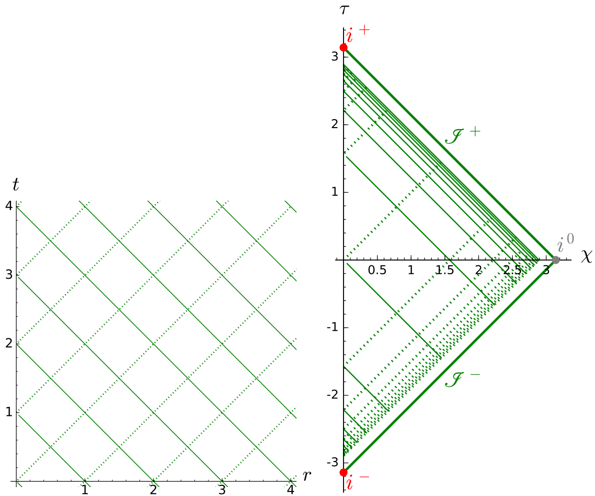


red: $r = \text{const}$

grey: $t = \text{const}$

Conformal completion of Minkowski spacetime

4. Conformal diagram

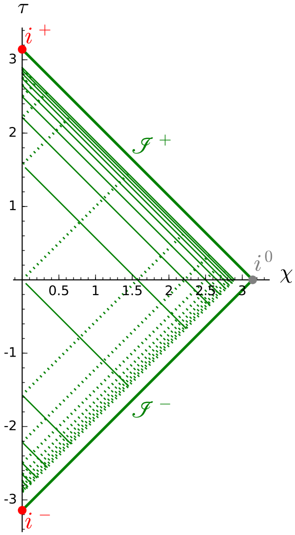


solid:
$v := t + r = \mathrm{const}$
dotted:
$u := t - r = \mathrm{const}$

Radial null geodesics appear as straight lines with $\pm 45°$ slope (conformal diagram)

Conformal completion of Minkowski spacetime

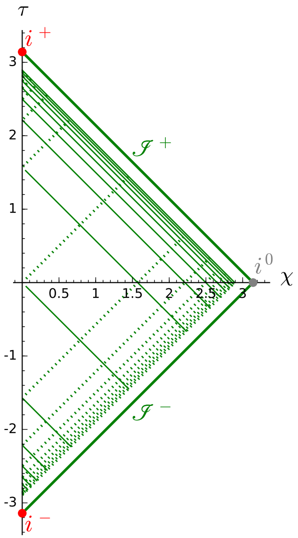
4. Conformal diagram



- $\mathscr{I}^+$: where all radial future-directed null geodesics terminate $\Longrightarrow$ **future null infinity**
- $\mathscr{I}^-$: where all radial future-directed null geodesics originate $\Longrightarrow$ **past null infinity**

Conformal completion of Minkowski spacetime

4. Conformal diagram



- $\mathscr{I}^+$: where all radial future-directed null geodesics terminate $\Longrightarrow$ **future null infinity**
- $\mathscr{I}^-$: where all radial future-directed null geodesics originate $\Longrightarrow$ **past null infinity**

Let $\mathscr{I} := \mathscr{I}^+ \cup \mathscr{I}^-$ and $\tilde{\mathscr{M}} := \mathscr{M} \cup \mathscr{I}$
$\tilde{\mathscr{M}}$ is a manifold with boundary, and its boundary is $\mathscr{I}$. Moreover the conformal factor Ω relating $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ vanishes at the boundary:

$$\boxed{\Omega \stackrel{\mathscr{I}}{=} 0}$$

Conformal completion

Definition 1

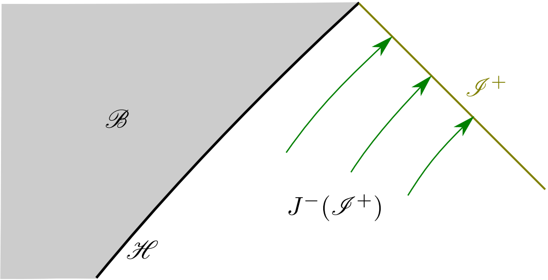
A spacetime $(\mathscr{M}, \boldsymbol{g})$ admits a **conformal completion** iff there exists a Lorentzian manifold with boundary $(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ equipped with a smooth non-negative scalar field $\Omega : \tilde{\mathscr{M}} \to \mathbb{R}^+$ such that

- $\tilde{\mathscr{M}} = \mathscr{M} \cup \mathscr{I}$, with $\mathscr{I} := \partial\tilde{\mathscr{M}}$ (the boundary of $\tilde{\mathscr{M}}$);
- on $\mathscr{M}$, $\tilde{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$;
- on $\mathscr{I}$, $\Omega = 0$;
- on $\mathscr{I}$, $\mathbf{d}\Omega \neq 0$.

Definition 2

$(\tilde{\mathscr{M}}, \tilde{\boldsymbol{g}})$ is a **conformal completion at null infinity** of $(\mathscr{M}, \boldsymbol{g})$ iff the boundary $\mathscr{I} := \partial\tilde{\mathscr{M}}$ obeys $\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-$, with $\mathscr{I}^+$ (resp. $\mathscr{I}^-$) being never intersected by any past-directed (resp. future-directed) causal curve originating in $\mathscr{M}$. $\mathscr{I}^+$ is called the **future null infinity** and $\mathscr{I}^-$ the **past null infinity** of $(\mathscr{M}, \boldsymbol{g})$.

General definition of a black hole



Causal past $J^-(\mathscr{I}^+)$: set of points of $\tilde{\mathscr{M}}$ that can be reached from a point of $\mathscr{I}^+$ by a past-directed causal (i.e. null or timelike) curve.

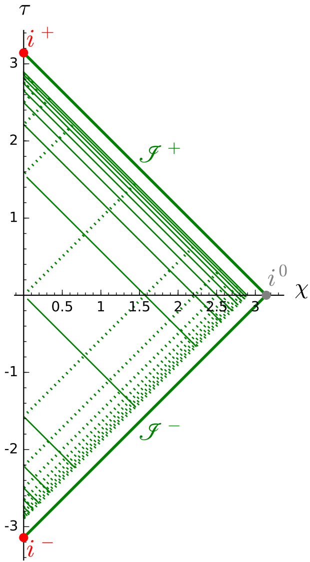
Definition

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime with a conformal completion at null infinity such that $\mathscr{I}^+$ is complete; the **black hole region**, or simply **black hole**, is the set of points of $\mathscr{M}$ that are not in the causal past of the future null infinity:

$$\boxed{\mathscr{B} := \mathscr{M} \setminus \left(J^-(\mathscr{I}^+) \cap \mathscr{M}\right)}$$

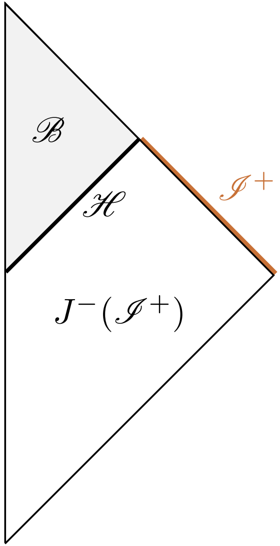
The boundary of $\mathscr{B}$ is called the **(future) event horizon**: $\mathscr{H} = \partial\mathscr{B}$

No black hole in Minkowski spacetime



$$J^-(\mathscr{I}^+) \cap \mathscr{M} = \mathscr{M}$$

Completeness of $\mathscr{I}^+$ to avoid spurious BH



If $\mathscr{I}^+$ is a null hypersurface, $\mathscr{I}^+$ complete $\iff$ $\mathscr{I}^+$ generated by complete null geodesics.

$\leftarrow$ Spurious black hole region $\mathscr{B}$ in Minkowski spacetime resulting from a conformal completion with a non-complete $\mathscr{I}^+$.

Properties of the event horizon

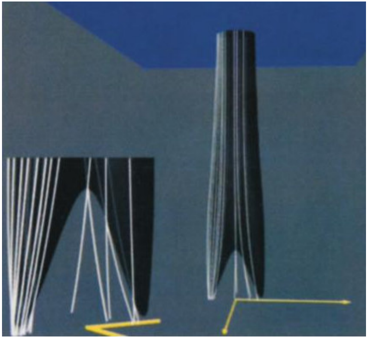
Property 1

The event horizon $\mathscr{H}$ is an **achronal set**, i.e. no pair of points of $\mathscr{H}$ can be connected by a timelike curve of $\mathscr{M}$.

Property 2

$\mathscr{H}$ is a topological manifold of dimension 3.

Properties of the event horizon



[R.A. Matzner et al., Science **270**, 941 (1995)]

Property 3 (Penrose 1968)

$\mathscr{H}$ is ruled by a family of *null geodesics* that

- either lie entirely in $\mathscr{H}$ or never leave $\mathscr{H}$ when followed into the future from the point where they arrived in $\mathscr{H}$
- have no endpoint in the future.

Moreover, there is exactly one null geodesic through each point of $\mathscr{H}$, except at special points where null geodesics enter in contact with $\mathscr{H}$.

Property 4

Wherever it is smooth, $\mathscr{H}$ is a null hypersurface.