Introduction to black hole physics

2. The Schwarzschild black hole

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École de Physique des Houches
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Lecture 2: The Schwarzschild black hole

Outline

The Schwarzschild solution (1915)

Spacetime manifold

$$\mathscr{M}_{\rm SD} := \mathscr{M}_{\rm I} \cup \mathscr{M}_{\rm II}$$
$$\mathscr{M}_{\rm I} := \mathbb{R} \times (2m, +\infty) \times \mathbb{S}^2, \quad \mathscr{M}_{\rm II} := \mathbb{R} \times (0, 2m) \times \mathbb{S}^2$$

Schwarzschild-Droste (SD) coordinates

$$(t, r, \theta, \varphi)$$

$t \in \mathbb{R}$, $r \in (2m, +\infty)$ on $\mathscr{M}_{\rm I}$, $r \in (0, 2m)$ on $\mathscr{M}_{\rm II}$
$\theta \in (0, \pi)$, $\varphi \in (0, 2\pi)$

Spacetime metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right)$$

Schwarzschild original work (1915)

Karl Schwarzschild (letter to Einstein 22 Dec. 1915; publication submitted 13 Jan 1916)
Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitz. Preuss. Akad. Wiss., Phys. Math. Kl. 1916, 189 (1916)

$\Longrightarrow$ First exact non-trivial solution of Einstein equation, with

- coordinates[1] $(t, \bar{r}, \theta, \varphi)$
- "auxiliary quantity": $r := (\bar{r}^3 + 8m^3)^{1/3}$
- parameter m = gravitational mass of the "mass point"

[1] Schwarzschild's notations: $r = \bar{r}$, $R = r$, $\alpha = 2m$

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The "center"

Origin of coordinates: $\bar{r} = 0 \iff r = 2m$

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Droste's contribution (1916)

Johannes Droste (communication 27 May 1916)
The Field of a Single Centre in Einstein's Theory of Gravitation, and the Motion of a Particle in that Field, Kon. Neder. Akad. Weten. Proc. **19**, 197 (1917)

$\Longrightarrow$ derives the Schwarzschild solution (independently of Schwarzschild) via some coordinates (t, r', θ, φ) such that $g_{r'r'} = 1$; presents the result in the standard SD form via a change of coordinates leading to the areal radius r
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Apparent barrier at $r = 2m$

A particle falling from infinity never reaches $r = 2m$ within a finite amount of "time" t.

Basic properties

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right)$$

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- $(\mathscr{M}_{\mathrm{SD}}, \boldsymbol{g})$ is asymptotically flat:

$$\mathrm{d}s^2 \sim -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2\right) \quad \text{when } r \to +\infty$$

Radial null geodesics

First thing to do to explore a given spacetime: investigate the null geodesics.

Radial null geodesics: $\theta = \text{const}$ and $\varphi = \text{const} \implies \mathrm{d}\theta = 0$ and $\mathrm{d}\varphi = 0$ along them.
A null geodesic is a null curve (NB: the converse is not true):

$$\mathrm{d}s^2 = 0 \iff \mathrm{d}t^2 = \frac{\mathrm{d}r^2}{\left(1 - \frac{2m}{r}\right)^2} \iff \mathrm{d}t = \pm\frac{\mathrm{d}r}{1 - \frac{2m}{r}}$$

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Hence two families of radial null geodesics:

- the **outgoing radial null geodesics**: $t = r + 2m \ln\left|\frac{r}{2m} - 1\right| + u$, $u = \mathrm{const}$
- the **ingoing radial null geodesics**: $t = -r - 2m \ln\left|\frac{r}{2m} - 1\right| + v$, $v = \mathrm{const}$

Radial null geodesics

Exercice: check that the two families of radial null curves do satisfy the geodesic equation

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} + \Gamma^\alpha_{\ \mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$$

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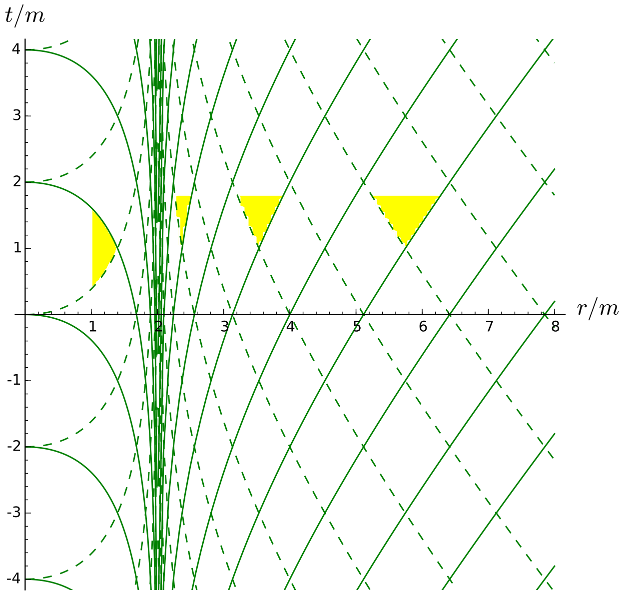
Hint: write $x^\alpha(r) = \left(r + 2m \ln\left| \frac{r}{2m} - 1 \right| + u,\ r,\ \theta,\ \varphi \right)$, so that

$\frac{\mathrm{d}x^\alpha}{\mathrm{d}r} = \left(\frac{r}{r-2m}, 1, 0, 0 \right)$ and $\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}r^2} = \left(-\frac{2m}{(r-2m)^2}, 0, 0, 0 \right)$,

then use the Christoffel symbols given by the SageMath notebook:
http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.2/SM_basic_Schwarzschild.ipynb

Radial null geodesics



Radial null geodesics in Schwarzschild-Droste coordinates:

- *solid:* outgoing family
- *dashed:* ingoing family
- *yellow:* interior of some future null cones

Outline

Ingoing Eddington-Finkelstein (IEF) coordinates

Use the ingoing family, parametrized by v, to introduce a new coordinate system $(\tilde{t}, r, \theta, \varphi)$ with

$$\boxed{\tilde{t} := v - r} \iff \boxed{\tilde{t} := t + 2m \ln\left|\frac{r}{2m} - 1\right|}$$

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Spacetime metric in IEF coordinates

$$ds^2 = -\left(1 - \frac{2m}{r}\right) d\tilde{t}^2 + \frac{4m}{r} d\tilde{t}\, dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

NB: $\tilde{t}$ was denoted t in lecture 1.

Coordinate singularity vs. curvature singularity

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}\tilde{t}^2 + \frac{4m}{r}\mathrm{d}\tilde{t}\,\mathrm{d}r + \left(1 + \frac{2m}{r}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)$$

All the metric components w.r.t. IEF coordinates are regular at $r = 2m$!
$\Longrightarrow$ the divergence of g_{rr} as $r \to 2m$ in Schwarzschild-Droste (SD) coordinates is a mere coordinate singularity.

Coordinate singularity vs. curvature singularity

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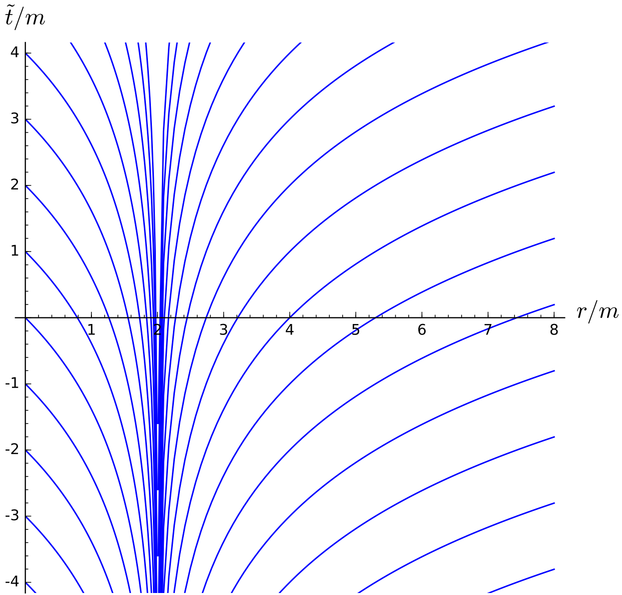
The metric components in both SD and IEF coordinates do exhibit divergences as $r \to 0$. The **Kretschmann scalar** $K := R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is

$$K = \frac{48m^2}{r^6} \xrightarrow[r\to 0]{} +\infty$$

Since K is a scalar field representing some "square" of the Riemann tensor, this corresponds to a curvature singularity.

Pathology of Schwarzschild-Droste coordinates at $r = 2m$



Hypersurfaces of constant t in terms of IEF coordinates

$\Longrightarrow$ singular slicing on $\mathscr{H}$

Extending the spacetime manifold

Metric components in IEF coordinates regular for all $r \in (0, +\infty)$
$\Longrightarrow$ consider

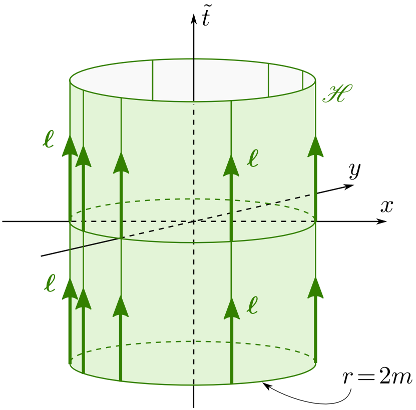
$$\boxed{\mathscr{M}_{\rm IEF} := \mathbb{R} \times (0, +\infty) \times \mathbb{S}^2}$$

for the spacetime manifold.
$\mathscr{M}_{\rm IEF}$ extends the Schwarzschild-Droste domain $\mathscr{M}_{\rm SD}$ according to

$$\mathscr{M}_{\rm IEF} = \mathscr{M}_{\rm SD} \cup \mathscr{H} = \mathscr{M}_{\rm I} \cup \mathscr{M}_{\rm II} \cup \mathscr{H}$$

where $\mathscr{H}$ is the hypersurface of $\mathscr{M}_{\rm IEF}$ defined by $r = 2m$.

The Schwarzschild horizon



$\mathscr{H}$: hypersurface $r = 2m$

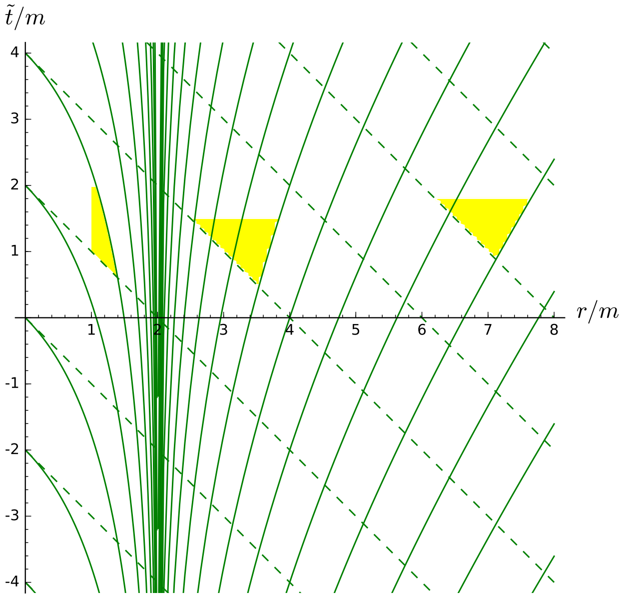
Recall from lecture 1 that

$\mathscr{H}$ is a Killing horizon, the null normal of which is $\boldsymbol{\ell} = \boldsymbol{\partial}_{\tilde{t}}$.

Topology: $\mathscr{H} \simeq \mathbb{R} \times \mathbb{S}^2$

$\mathscr{H}$ is a non-expanding horizon, whose area is $A = 16\pi m^2$

Black hole character



Radial null geodesics in IEF coordinates:

- *solid:* "outgoing" family
- *dashed:* ingoing family ($\tilde{t} = v - r$)
- *yellow:* interior of some future null cones

The region $r < 2m$ ($\mathscr{M}_{\rm II}$) is a black hole, the event horizon of which is $\mathscr{H}$.

Outline

Kruskal-Szekeres coordinates

Coordinates (T, X, θ, φ) such that

$$\begin{cases} T = \mathrm{e}^{r/4m} \left[\cosh\left(\frac{\tilde{t}}{4m}\right) - \frac{r}{4m} \mathrm{e}^{-\tilde{t}/4m} \right] \\ X = \mathrm{e}^{r/4m} \left[\sinh\left(\frac{\tilde{t}}{4m}\right) + \frac{r}{4m} \mathrm{e}^{-\tilde{t}/4m} \right] \end{cases}$$

and $-X < T < \sqrt{X^2 + 1}$ on $\mathscr{M}_{\rm IEF}$.

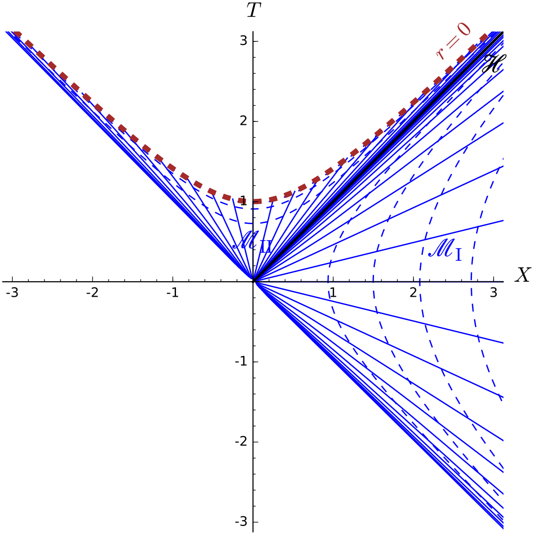
Spacetime metric

$$\mathrm{d}s^2 = \frac{32m^3}{r}\, \mathrm{e}^{-r/2m} \left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2\right)$$

with $r = r(T, X)$ implicitly defined by $\mathrm{e}^{r/2m}\left(\frac{r}{2m} - 1\right) = X^2 - T^2$

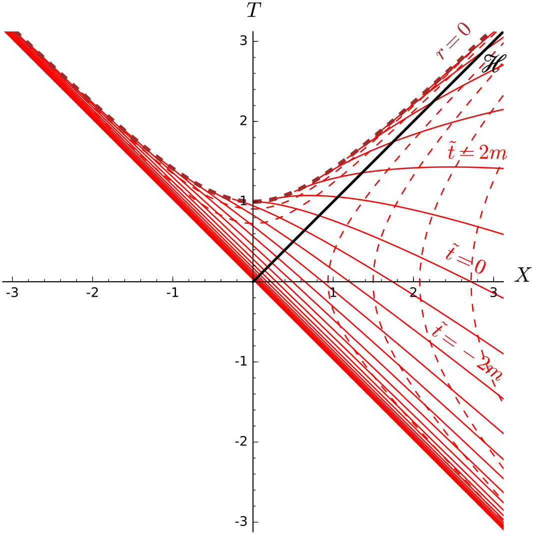
$\Longrightarrow$ radial null geodesics: $\mathrm{d}s^2 = 0 \iff \mathrm{d}T = \pm \mathrm{d}X$

SD coordinates in terms of KS coordinates



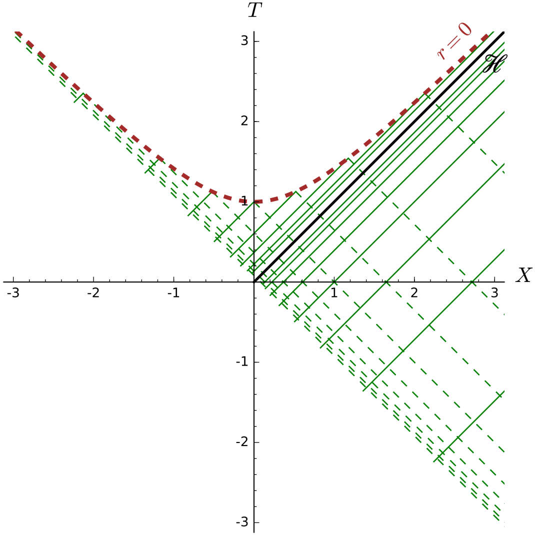
- *solid:* $t = \mathrm{const}$
- *dashed:* $r = \mathrm{const}$

IEF coordinates in terms of KS coordinates



- *solid:* $\tilde{t} = \text{const}$
- *dashed:* $r = \text{const}$

Radial null geodesics



Radial null geodesics

$\mathrm{d}T = \pm \mathrm{d}X$

$\iff T = \pm X + T_0$

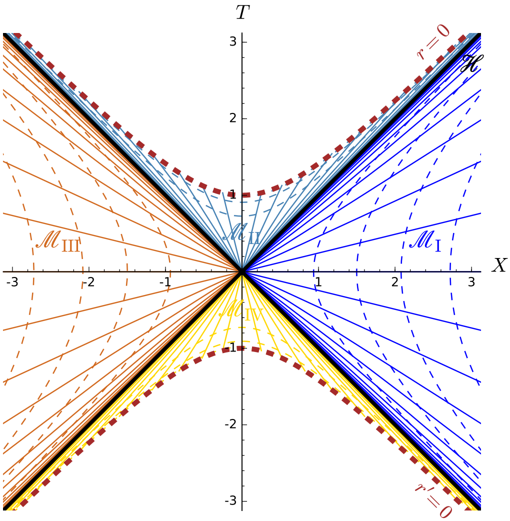
($\pm 45°$ straight lines!)

- *solid:* outgoing family
- *dashed:* ingoing family

$\Longrightarrow$ outgoing null geodesics are incomplete (to the past)

$\Longrightarrow$ spacetime can be extended...

Maximally extended Schwarzschild spacetime
Kruskal diagram



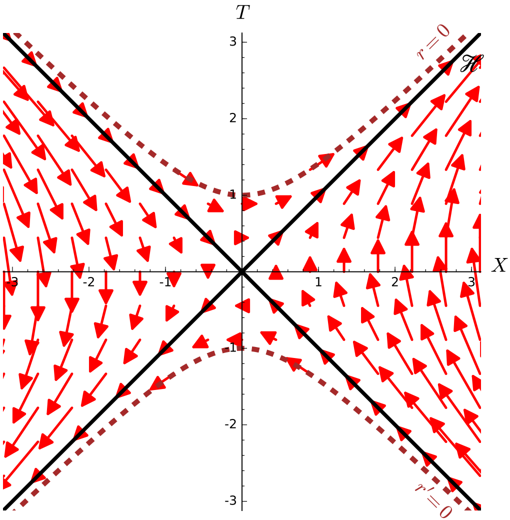
- *solid:* $t = \mathrm{const}$
- *dashed:* $r = \mathrm{const}$

Null geodesics are either complete or terminating at a curvature singularity
$\Longrightarrow$ maximal extension $\mathscr{M}$

Each point of the diagram is a sphere: topology $\mathscr{M} \simeq \mathbb{R}^2 \times \mathbb{S}^2$

Maximally extended Schwarzschild spacetime

"Stationary" Killing vector field

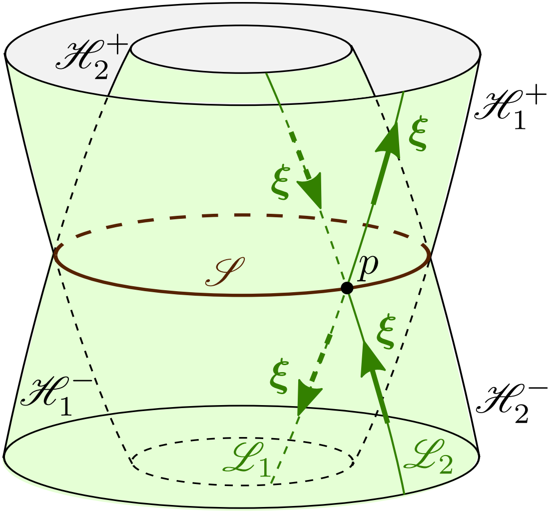


Killing vector field

$\boldsymbol{\xi} = \boldsymbol{\partial}_t = \boldsymbol{\partial}_{\tilde{t}}$

- $\boldsymbol{\xi}$ timelike in $\mathscr{M}_{\rm I}$ and $\mathscr{M}_{\rm III}$
- $\boldsymbol{\xi}$ spacelike in $\mathscr{M}_{\rm II}$ and $\mathscr{M}_{\rm IV}$
- $\boldsymbol{\xi}$ null on the null hypersurfaces $T = X$ (includes $\mathscr{H}$) and $T = -X$
- $\boldsymbol{\xi}$ vanishes on the central 2-sphere $T = X = 0$ (the **bifurcation sphere**)

Bifurcate Killing horizon

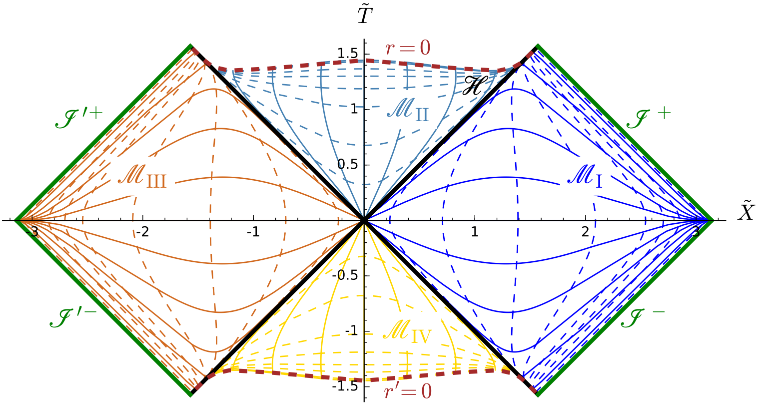


$\hat{\mathscr{H}} = \mathscr{H}_1 \cup \mathscr{H}_2$

- $\mathscr{H}_1$: null hypersurface $T = X$
- $\mathscr{H}_2$: null hypersurface $T = -X$
- $\mathscr{H}_1^+ = \mathscr{H}$ Schwarzschild horizon $T = X$ and $X > 0$
- $\mathscr{S}$: bifurcation sphere $T = X = 0$

Carter-Penrose diagram

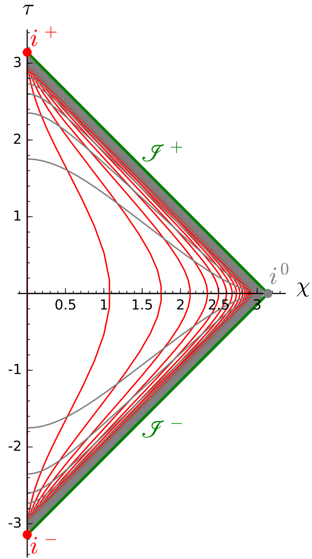
based on Frolov-Novikov coordinates



solid: $t = \text{const}$, *dashed:* $r = \text{const}$

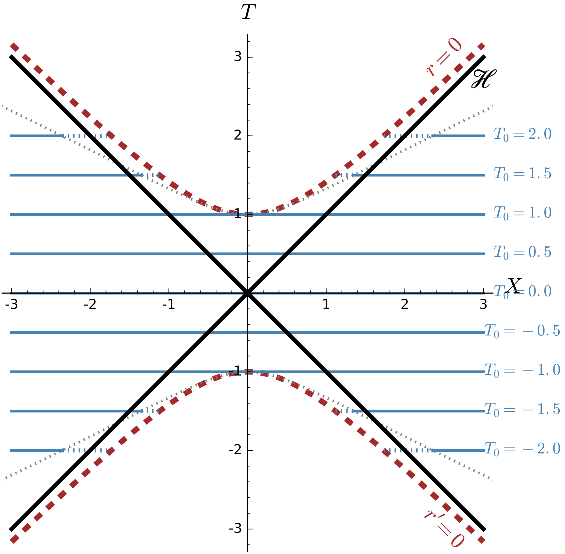
http://nbviewer.jupyter.org/github/egourgoulhon/BHLectures/blob/master/sage/Schwarz_conformal.ipynb

Comparison with the conf. diagram of Minkowski spacetime

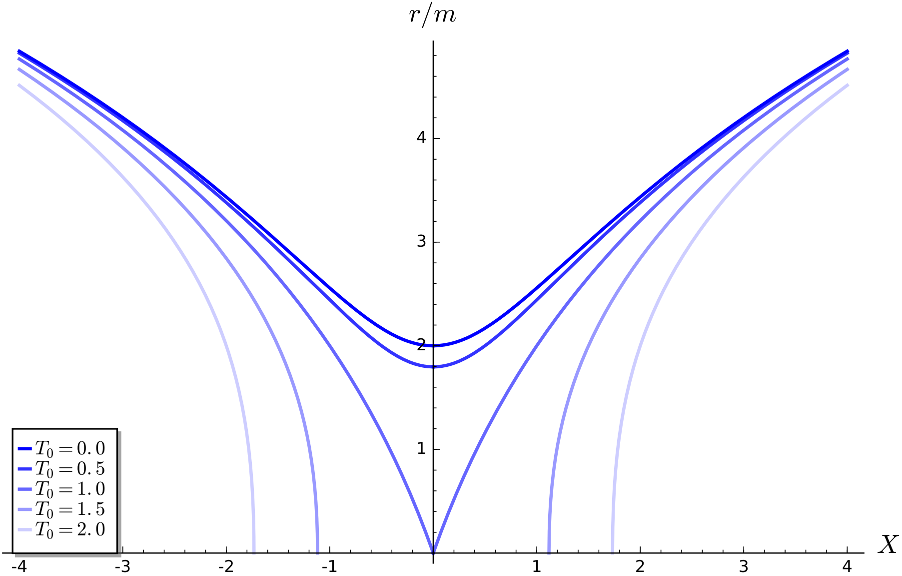


Outline

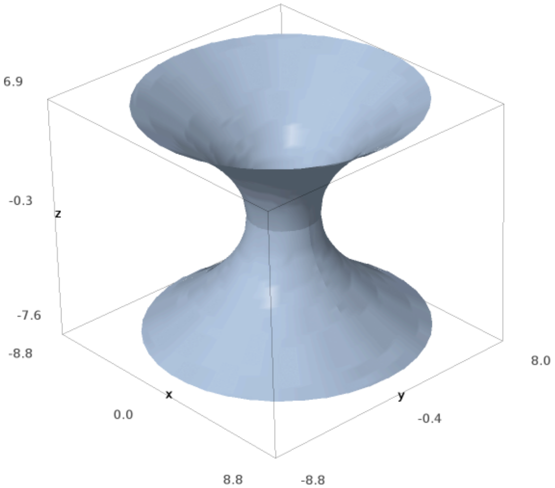
Constant KS-time hypersurfaces Σ_{T_0}



Areal radius as a function of X on Σ_{T_0}



Flamm paraboloid

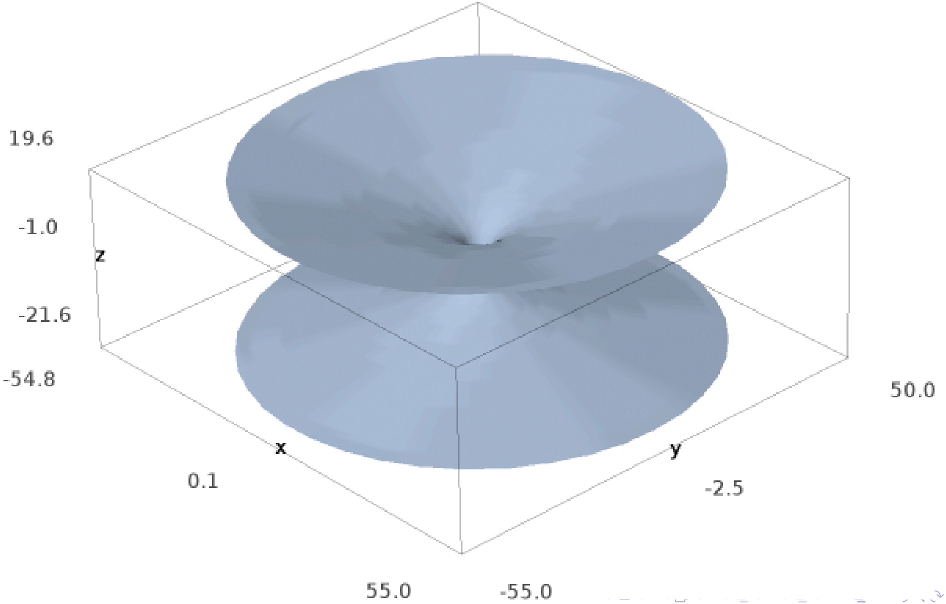


Flamm paraboloid

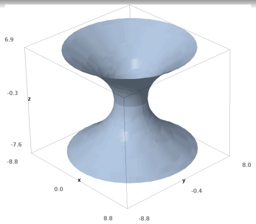
Isometric embedding of the equatorial slice $\theta = \pi/2$ of the (spacelike) hypersurface $T = 0$ of the extended Schwarzschild spacetime into the Euclidean 3-space $\mathbb{E}^3$

Topology: $\Sigma^{\rm eq}_{T=0} \simeq \mathbb{R} \times \mathbb{S}^1$

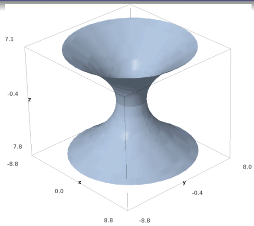
Flamm paraboloid (zoom out)



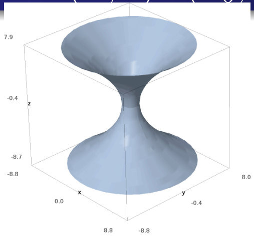
Sequence of isometric embeddings of slices $(T, \theta) = (T_0, \frac{\pi}{2})$



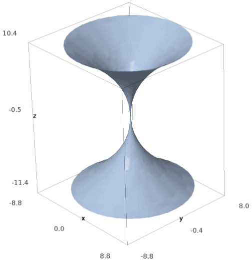
$T_0 = 0$



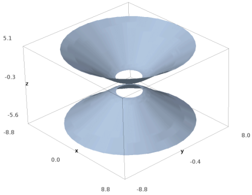
$T_0 = 0.5$



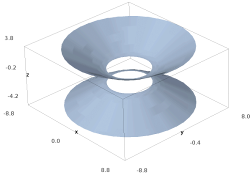
$T_0 = 0.9$



$T_0 = 1$



$T_0 = 1.5$



$T_0 = 2$