# Trapping horizons and black holes

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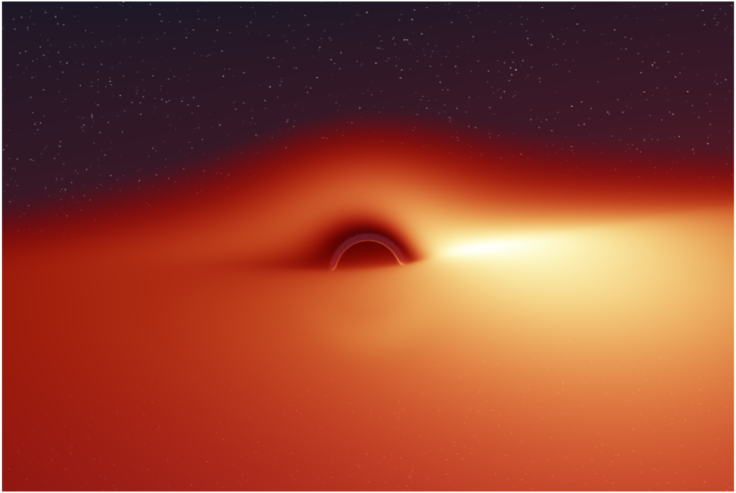
Université de Bretagne Occidentale
5 June 2009

# Plan

# Outline

# What is a black hole ?

... for the astrophysicist: a very deep gravitational potential well



[J.A. Marck, CQG **13**, 393 (1996)]

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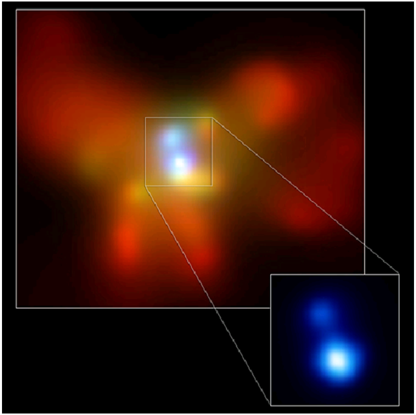
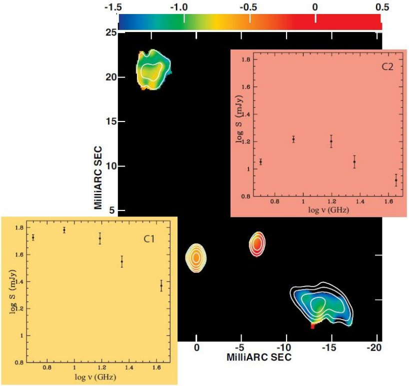


Binary BH in galaxy NGC 6240
$d = 1.4$ kpc
[Komossa et al., ApJ **582**, L15 (2003)]




Binary BH in radio galaxy 0402+379
$d = 7.3$ pc
[Rodriguez et al., ApJ **646**, 49 (2006)]

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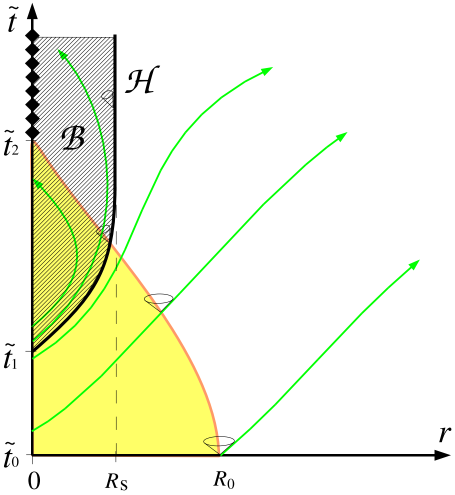


... for the mathematical physicist:

$$\mathcal{B} := \mathscr{M} - J^-(\mathscr{I}^+)$$

i.e. the region of spacetime where light rays cannot escape to infinity

- $(\mathscr{M}, \boldsymbol{g})$ = asymptotically flat manifold
- $\mathscr{I}^+$ = future null infinity
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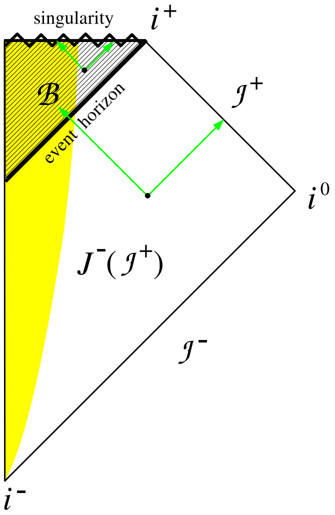


Compactified (Carter-Penrose) diagram

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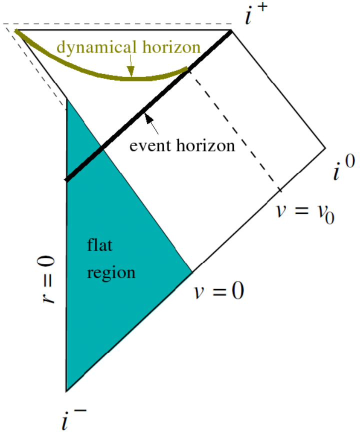
Determination of $\dot{J}^-(\mathscr{I}^+)$ requires the knowledge of the entire future null infinity. Moreover this is *not locally linked with the notion of strong gravitational field:*




Example of event horizon in a **flat** region of spacetime:

Vaidya metric, describing incoming radiation from infinity:

$$ds^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2dv\, dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

with $m(v) = 0$ for $v < 0$
$dm/dv > 0$ for $0 \leq v \leq v_0$
$m(v) = M_0$ for $v > v_0$

[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

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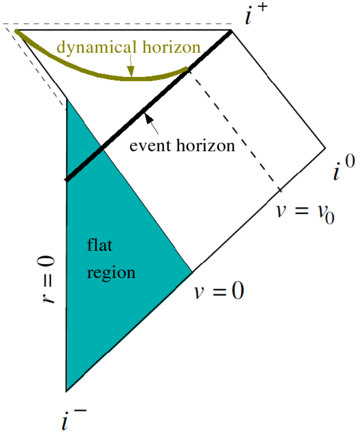


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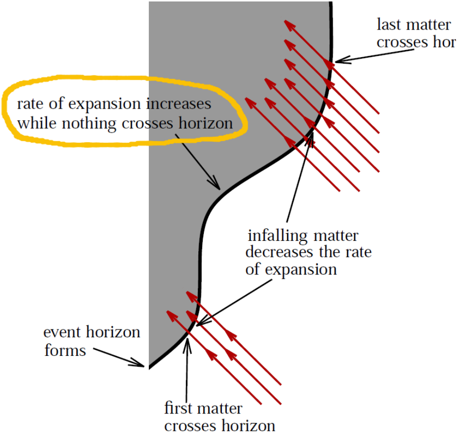
$\Rightarrow$ no local physical experiment whatsoever can locate the event horizon

# Another non-local feature: teleological nature of event horizons

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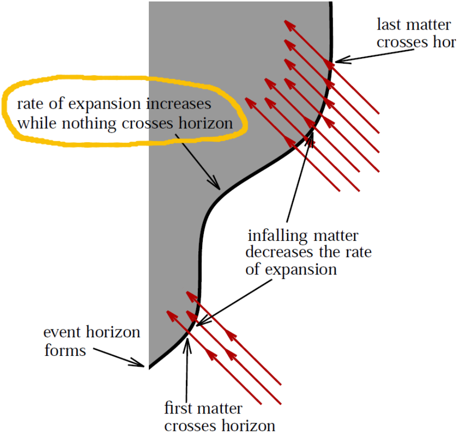


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To deal with black holes as ordinary physical objects, a **local** definition would be desirable
→ quantum gravity, numerical relativity

# Outline

# Local characterizations of black holes

Recently a **new paradigm** appeared in the theoretical approach of black holes: instead of *event horizons*, black holes are described by

- trapping horizons (Hayward 1994)
- isolated horizons (Ashtekar et al. 1999)
- dynamical horizons (Ashtekar and Krishnan 2002)
- slowly evolving horizons (Booth and Fairhurst 2004)

All these concepts are **local** and are based on the notion of trapped surfaces

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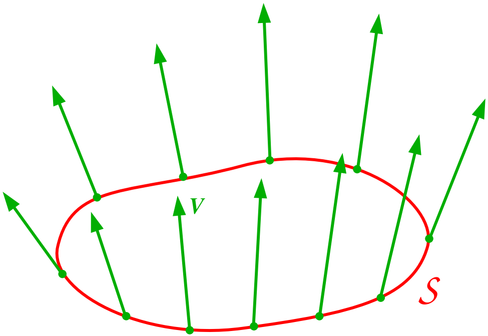


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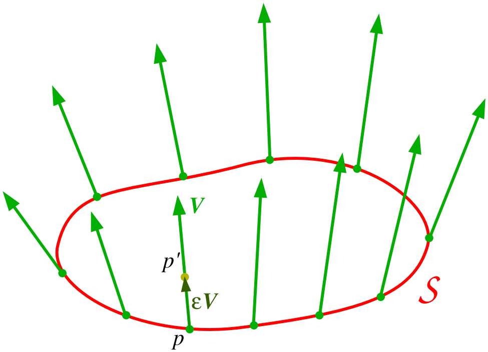


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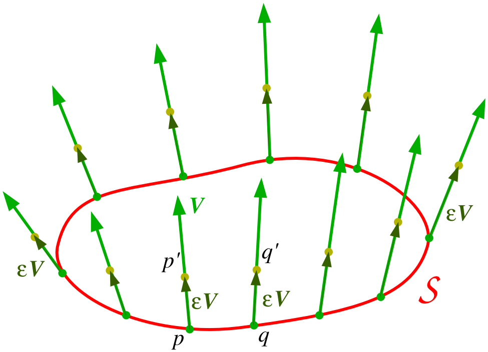


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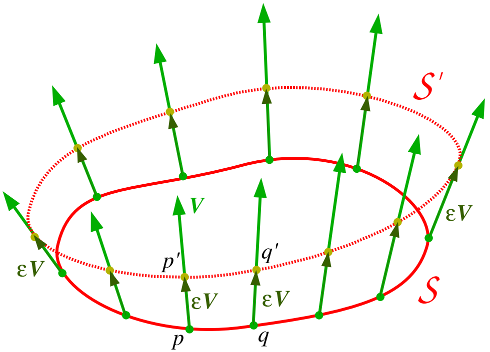


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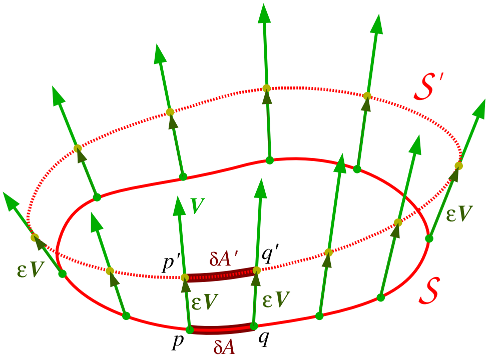


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At each point, the **expansion of $\mathcal{S}$ along $\boldsymbol{v}$** is defined from the relative change in the area element $\delta A$:

$$\theta^{(\boldsymbol{v})} := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\delta A' - \delta A}{\delta A} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln \sqrt{q} = q^{\mu\nu} \nabla_\mu v_\nu$$

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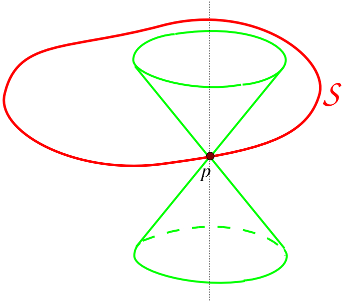


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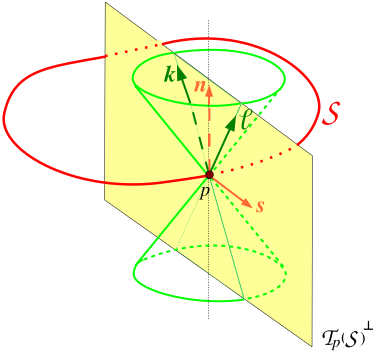


Being spacelike, $\mathcal{S}$ lies outside the light cone

$\exists$ two future-directed null directions orthogonal to $\mathcal{S}$:

$\boldsymbol{\ell}$ = outgoing, expansion $\theta^{(\boldsymbol{\ell})}$

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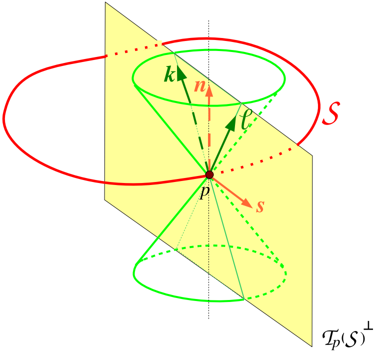


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$\mathcal{S}$ is trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} < 0$ [Penrose 1965]

$\mathcal{S}$ is marginally trapped $\iff$ $\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0$

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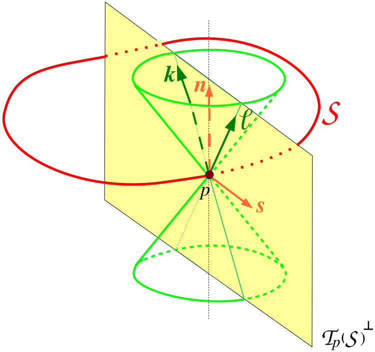


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*trapped surface* = **local** concept characterizing very strong gravitational fields

# Link with apparent horizons

A closed spacelike 2-surface $\mathcal{S}$ is said to be outer trapped (resp. marginally outer trapped (MOTS)) iff [Hawking & Ellis 1973]

- the notions of *interior* and *exterior* of $\mathcal{S}$ can be defined (for instance spacetime asymptotically flat) $\Rightarrow$ $\boldsymbol{\ell}$ is chosen to be the *outgoing* null normal and $\boldsymbol{k}$ to be the *ingoing* one
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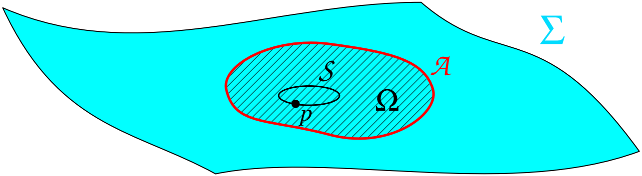



$\Sigma$: spacelike hypersurface extending to spatial infinity (Cauchy surface)

outer trapped region of $\Sigma$ : $\Omega$ = set of points $p \in \Sigma$ through which there is a outer trapped surface $\mathcal{S}$ lying in $\Sigma$

apparent horizon in $\Sigma$: $\mathcal{A}$ = connected component of the boundary of $\Omega$

*Proposition* [Hawking & Ellis 1973]: $\mathcal{A}$ smooth $\Longrightarrow$ $\mathcal{A}$ is a MOTS

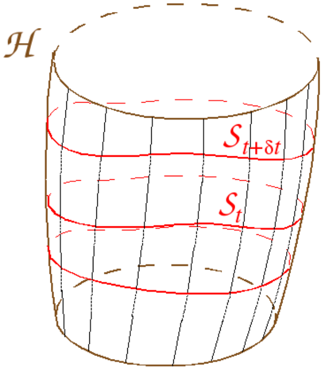
# Connection with singularities and black holes

*Proposition* [Penrose (1965)]:
provided that the weak energy condition holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a singularity in $(\mathscr{M}, \boldsymbol{g})$ (in the form of a future inextendible null geodesic)

*Proposition* [Hawking & Ellis (1973)]:
provided that the cosmic censorship conjecture holds,
$\exists$ a trapped surface $\mathcal{S} \Longrightarrow \exists$ a black hole $\mathcal{B}$ and $\mathcal{S} \subset \mathcal{B}$

# Local definitions of "black holes"

A hypersurface $\mathcal{H}$ of $(\mathscr{M}, \boldsymbol{g})$ is said to be




- a **future outer trapping horizon (FOTH)** iff
  (i) $\mathcal{H}$ foliated by marginally trapped 2-surfaces $(\theta^{(\boldsymbol{k})} < 0$ and $\theta^{(\boldsymbol{\ell})} = 0)$
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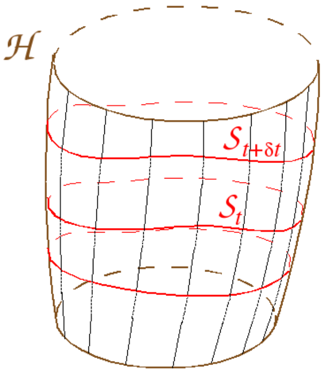


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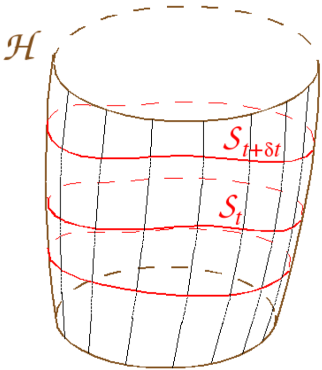


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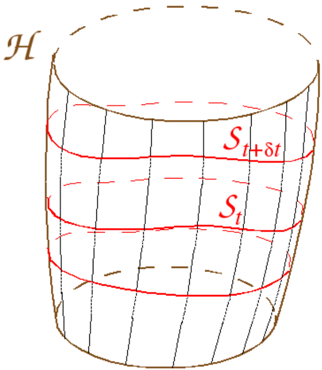


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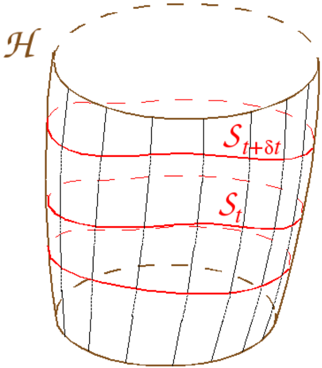


BH in equilibrium = IH
(e.g. Kerr)
BH out of equilibrium = DH
generic BH = FOTH

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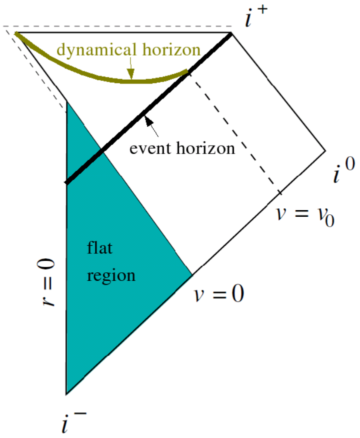
# Example: Vaidya spacetime




[Ashtekar & Krishnan, LRR **7**, 10 (2004)]

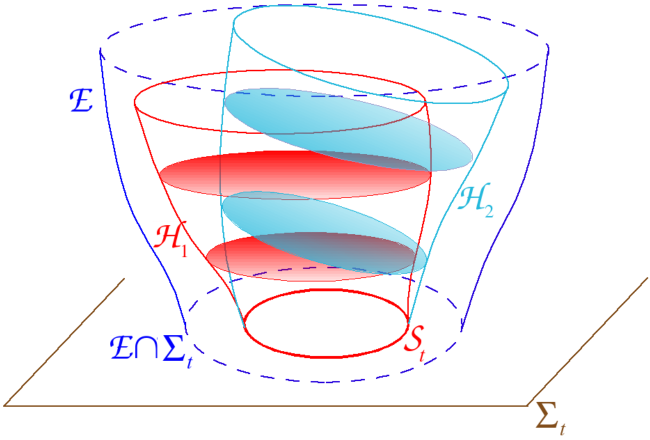
- The event horizon crosses the flat region
- The dynamical horizon lies entirely outside the flat region

# Dynamics of these new horizons

The *trapping horizons* and *dynamical horizons* have their **own dynamics**, ruled by Einstein equations.
In particular, one can establish for them

- existence and (partial) uniqueness theorems
  [Andersson, Mars & Simon, PRL **95**, 111102 (2005)],
  [Ashtekar & Galloway, Adv. Theor. Math. Phys. **9**, 1 (2005)]
- first and second laws of black hole mechanics
  [Ashtekar & Krishnan, PRD **68**, 104030 (2003)], [Hayward, PRD **70**, 104027 (2004)]
- a viscous fluid bubble analogy ("membrane paradigm", as for the event horizon)
  [Gourgoulhon, PRD **72**, 104007 (2005)], [Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

# Non-uniqueness of trapping horizons




*NB:* uniqueness in spherical symmetry

# Outline

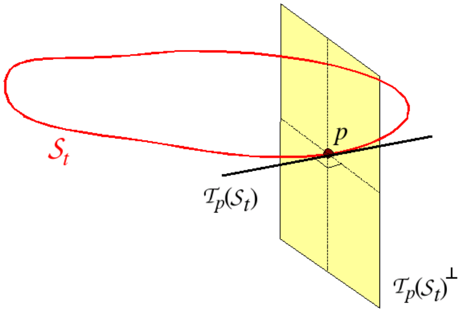
# Closed spacelike surfaces

$\mathcal{S}$ : closed (i.e. compact without boundary) spacelike 2-dimensional surface embedded in spacetime $(\mathscr{M}, \boldsymbol{g})$

$\mathcal{S}$ spacelike $\iff$ metric $\boldsymbol{q}$ induced by $\boldsymbol{g}$ is positive definite

$\boldsymbol{q}$ not degenerate $\Longrightarrow$ orthogonal decomposition of the tangent space at any $p \in \mathscr{M}$:

$$\mathcal{T}_p(\mathscr{M}) = \mathcal{T}_p(\mathcal{S}) \oplus \mathcal{T}_p(\mathcal{S})^{\perp}$$




$\boldsymbol{q}$: induced metric on $\mathcal{S}$, components: $q_{\alpha\beta}$

$\vec{\boldsymbol{q}}$: orthogonal projector onto $\mathcal{S}$, components: $q^{\alpha}{}_{\beta}$

# Expansion and shear along normal vectors

Let $\boldsymbol{v}$ be a vector field on $\mathscr{M}$, defined at least at $\mathcal{S}$ and everywhere normal to $\mathcal{S}$.
*NB: $\boldsymbol{v}$ is not assumed to be null*

**Deformation tensor of $\mathcal{S}$ along $v$:** $\boxed{\boldsymbol{\Theta}^{(\boldsymbol{v})} := \vec{\boldsymbol{q}}^{*}\boldsymbol{\nabla}\underline{\boldsymbol{v}}}$ or $\Theta^{(\boldsymbol{v})}_{\alpha\beta} := \nabla_\nu v_\mu \, q^\mu_{\ \alpha} q^\nu_{\ \beta}$

$\boldsymbol{v}$ normal to a 2-surface ($\mathcal{S}$) $\Longrightarrow$ $\boldsymbol{\Theta}^{(\boldsymbol{v})}$ is a symmetric bilinear form

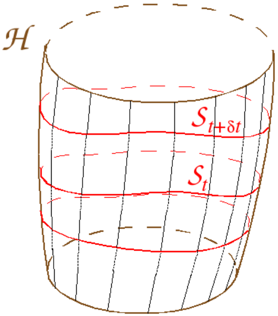
*Prop:* $\boldsymbol{\Theta}^{(\boldsymbol{v})} = \frac{1}{2}\, \vec{\boldsymbol{q}}^{*} \boldsymbol{\mathcal{L}}_{\boldsymbol{v}}\, \boldsymbol{q}$

Decomposition into traceless part (**shear** $\boldsymbol{\sigma}^{(\boldsymbol{v})}$) and trace part (**expansion** $\theta^{(\boldsymbol{v})}$ ):

$\boxed{\boldsymbol{\Theta}^{(\boldsymbol{v})} = \boldsymbol{\sigma}^{(\boldsymbol{v})} + \frac{1}{2}\theta^{(\boldsymbol{v})}\, \boldsymbol{q}}$ with $\theta^{(\boldsymbol{v})} := q^{\mu\nu}\Theta^{(\boldsymbol{v})}_{\mu\nu} = \boldsymbol{\mathcal{L}}_{\boldsymbol{v}} \ln\sqrt{q}$, $q := \det q_{ab}$

*Prop:* $\boxed{\boldsymbol{\mathcal{L}}_{\boldsymbol{v}}\, {}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \theta^{(\boldsymbol{v})}\, {}^{\mathcal{S}}\!\boldsymbol{\epsilon}}$ with ${}^{\mathcal{S}}\!\boldsymbol{\epsilon}$ surface element of $(\mathcal{S}, \boldsymbol{q})$ : ${}^{\mathcal{S}}\!\boldsymbol{\epsilon} = \sqrt{q}\, \mathbf{d}x^2 \wedge \mathbf{d}x^3$
$\Longrightarrow$ hence the name *expansion*

# Foliation of a hypersurface by spacelike 2-surfaces



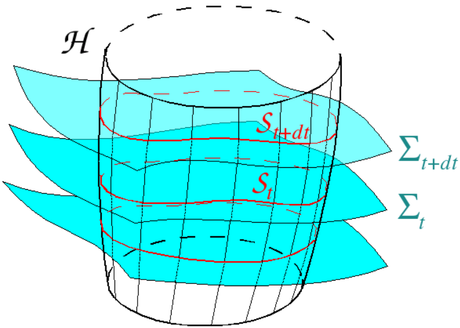

hypersurface $\mathcal{H}$ = submanifold of spacetime $(\mathscr{M}, \boldsymbol{g})$ of codimension 1

$\mathcal{H}$ can be $\begin{cases} \text{spacelike} \\ \text{null} \\ \text{timelike} \end{cases}$

$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

$\mathcal{S}_t$ = spacelike 2-surface

# Foliation of a hypersurface by spacelike 2-surfaces

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$\Longleftarrow$ 3+1 perspective

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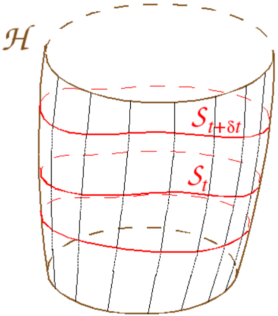


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$$\mathcal{H} = \bigcup_{t\in\mathbb{R}} \mathcal{S}_t$$

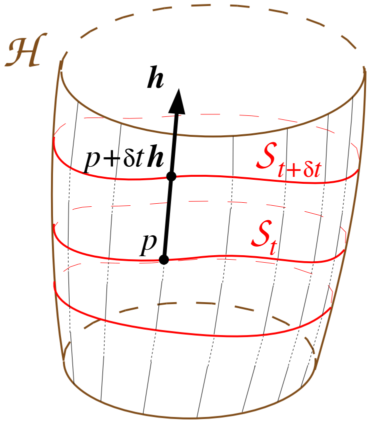
$\mathcal{S}_t$ = spacelike 2-surface

intrinsic viewpoint adopted here (i.e. not relying on extra-structure such as a 3+1 foliation)

$\boldsymbol{q}$ : induced metric on $\mathcal{S}_t$ (positive definite)

$\boldsymbol{\mathcal{D}}$ : connection associated with $\boldsymbol{q}$

# Evolution vector on the horizon

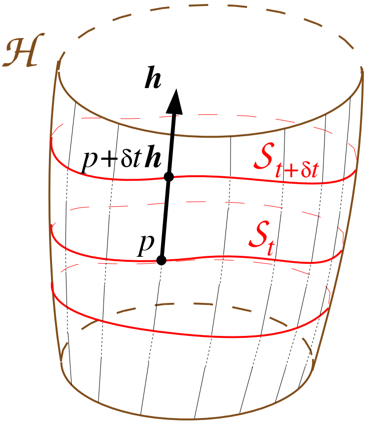



Vector field $\boldsymbol{h}$ on $\mathcal{H}$ defined by

- (i) $\boldsymbol{h}$ is tangent to $\mathcal{H}$
- (ii) $\boldsymbol{h}$ is orthogonal to $\mathcal{S}_t$
- (iii) $\mathcal{L}_{\boldsymbol{h}}\, t = h^\mu \partial_\mu t = \langle \mathbf{d}t, \boldsymbol{h} \rangle = 1$

NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

# Evolution vector on the horizon

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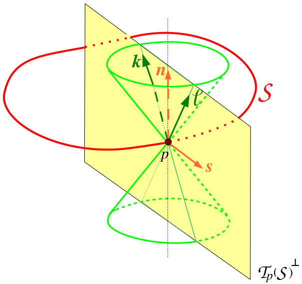
NB: (iii) $\Longrightarrow$ the 2-surfaces $\mathcal{S}_t$ are Lie-dragged by $\boldsymbol{h}$

Define $C := \frac{1}{2}\boldsymbol{h}\cdot\boldsymbol{h}$

| | | | | |
|---|---|---|---|---|
| $\mathcal{H}$ is spacelike | $\iff$ | $C > 0$ | $\iff$ | $\boldsymbol{h}$ is spacelike |
| $\mathcal{H}$ is null | $\iff$ | $C = 0$ | $\iff$ | $\boldsymbol{h}$ is null |
| $\mathcal{H}$ is timelike | $\iff$ | $C < 0$ | $\iff$ | $\boldsymbol{h}$ is timelike. |

# Frames normal to $\mathcal{S}_t$




Two natural types of choice for a vector basis of $\mathcal{T}_p(\mathcal{S}_t)^\perp$ :

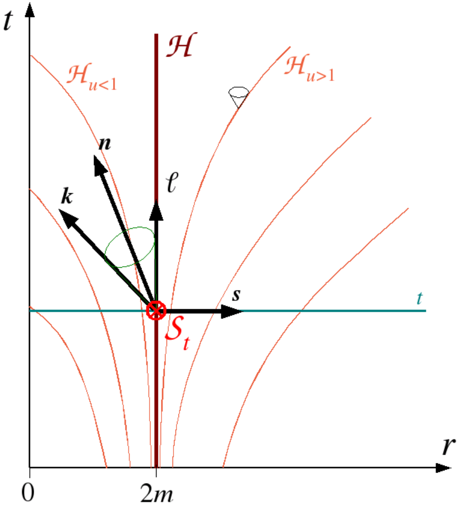
1. an orthonormal basis $(\boldsymbol{n}, \boldsymbol{s})$ ($\boldsymbol{n}$ = timelike, $\boldsymbol{s}$ = spacelike):
   $\boldsymbol{n}\cdot\boldsymbol{n} = -1, \quad \boldsymbol{s}\cdot\boldsymbol{s} = 1, \quad \boldsymbol{n}\cdot\boldsymbol{s} = 0$
2. a pair linearly independent future-directed null vectors $(\boldsymbol{\ell}, \boldsymbol{k})$:
   $\boldsymbol{\ell}\cdot\boldsymbol{\ell} = 0, \quad \boldsymbol{k}\cdot\boldsymbol{k} = 0, \quad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$

Degrees of freedom:

1. boost : $\left\{ \begin{array}{l} \boldsymbol{n}' = \cosh\eta\, \boldsymbol{n} + \sinh\eta\, \boldsymbol{s} \\ \boldsymbol{s}' = \sinh\eta\, \boldsymbol{n} + \cosh\eta\, \boldsymbol{s} \end{array} \right. , \quad \eta\in\mathbb{R}$
2. rescaling : $\left\{ \begin{array}{l} \boldsymbol{\ell}' = \lambda\boldsymbol{\ell}, \quad \lambda > 0 \\ \boldsymbol{k}' = \lambda^{-1}\,\boldsymbol{k} \end{array} \right.$

Orthogonal projector: $\vec{\boldsymbol{q}} = \mathbf{1} + \langle \underline{\boldsymbol{n}}, .\rangle\, \boldsymbol{n} - \langle \underline{\boldsymbol{s}}, .\rangle\, \boldsymbol{s} = \mathbf{1} + \langle \underline{\boldsymbol{k}}, .\rangle\, \boldsymbol{\ell} + \langle \underline{\boldsymbol{\ell}}, .\rangle\, \boldsymbol{k}$

# Example of normal frames




$\mathcal{H}$ = event horizon of Schwarzschild black hole

$\mathcal{S}_t$ = slice of constant Eddington-Finkelstein time

# Second fundamental tensor of $\mathcal{S}_t$

Tensor $\boldsymbol{\mathcal{K}}$ of type $(1,2)$ relating the covariant derivative of a vector tangent to $\mathcal{S}_t$ taken by the spacetime connection $\boldsymbol{\nabla}$ to that taken by the connection $\boldsymbol{\mathcal{D}}$ in $\mathcal{S}_t$ compatible with the induced metric $\boldsymbol{q}$:

$$\forall(\boldsymbol{u},\boldsymbol{v}) \in \mathcal{T}(\mathcal{S}_t)^2, \quad \boldsymbol{\nabla}_{\boldsymbol{u}}\boldsymbol{v} = \boldsymbol{\mathcal{D}}_{\boldsymbol{u}}\boldsymbol{v} + \boldsymbol{\mathcal{K}}(\boldsymbol{u},\boldsymbol{v})$$

*Prop:*

$\mathcal{K}^\alpha_{\ \ \beta\gamma} = \nabla_\mu q^\alpha_{\ \ \nu}\, q^\mu_{\ \ \beta} q^\nu_{\ \ \gamma}$

$\mathcal{K}^\alpha_{\ \ \beta\gamma} = n^\alpha\, \Theta^{(\boldsymbol{n})}_{\beta\gamma} - s^\alpha\, \Theta^{(\boldsymbol{s})}_{\beta\gamma} = k^\alpha\, \Theta^{(\boldsymbol{\ell})}_{\beta\gamma} + \ell^\alpha\, \Theta^{(\boldsymbol{k})}_{\beta\gamma}$

*Remark:* for a hypersurface of normal $\boldsymbol{n}$ and extrinsic curvature $\boldsymbol{K}$,
$\mathcal{K}^\alpha_{\ \ \beta\gamma} = -n^\alpha K_{\beta\gamma}$

# Normal fundamental forms

**Extrinsic geometry of $\mathcal{S}_t$ not entirely specified by $\mathcal{K}$ (contrary to the hypersurface case)**

$\mathcal{K}$ involves only the deformation tensors $\boldsymbol{\Theta}^{(.)}$ of the normals to $\mathcal{S}_t \Longrightarrow \mathcal{K}$ encodes only the part of the variation of $\mathcal{S}_t$'s normals which is parallel to $\mathcal{S}_t$

Variation of the two normals with respect to each other: encoded by the **normal fundamental forms** (also called *external rotation coefficients* or *connection on the normal bundle*, or if $\mathcal{H}$ is null, *Hájíček 1-form*):

1. $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{n})} := \boldsymbol{s} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{n}}$ or $\Omega^{(\boldsymbol{n})}_{\alpha} := s_\mu \nabla_\nu n^\mu \, q^\nu{}_\alpha$

   $\boldsymbol{\Omega}^{(\boldsymbol{s})} := \boldsymbol{n} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{s}$
2. $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := -\boldsymbol{k} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{\ell}}$ or $\Omega^{(\boldsymbol{\ell})}_{\alpha} := -k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$

   $\boldsymbol{\Omega}^{(\boldsymbol{k})} := -\boldsymbol{\ell} \cdot \boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\, \boldsymbol{k}$

# Basic properties of the normal fundamental forms

From the definition: $\boldsymbol{\Omega}^{(\boldsymbol{s})} = -\boldsymbol{\Omega}^{(\boldsymbol{n})}$ and $\boldsymbol{\Omega}^{(\boldsymbol{k})} = -\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$

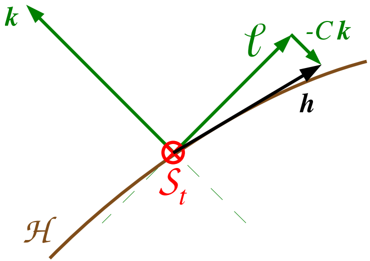
Relation between the $(\boldsymbol{n}, \boldsymbol{s})$-type and the $(\boldsymbol{\ell}, \boldsymbol{k})$-type:
$\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\Omega}^{(\boldsymbol{n})}$ $[\boldsymbol{\ell} = \boldsymbol{n} + \boldsymbol{s}]$ and $\boldsymbol{\Omega}^{(\boldsymbol{k})} = -\boldsymbol{\Omega}^{(\boldsymbol{n})}$ $[\boldsymbol{k} = \boldsymbol{n} - \boldsymbol{s}]$

**The normal fundamental forms are not unique**

(contrary to the second fundamental tensor $\mathcal{K}$)
Dependence of the normal frame

1. $(\boldsymbol{n}, \boldsymbol{s}) \mapsto (\boldsymbol{n}', \boldsymbol{s}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{n}')} = \boldsymbol{\Omega}^{(\boldsymbol{n})} + \boldsymbol{\mathcal{D}}\eta}$
2. $(\boldsymbol{\ell}, \boldsymbol{k}) \mapsto (\boldsymbol{\ell}', \boldsymbol{k}') \Longrightarrow$ $\boxed{\boldsymbol{\Omega}^{(\boldsymbol{\ell}')} = \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \boldsymbol{\mathcal{D}} \ln \lambda}$

# Normal null frame associated with the evolution vector




The foliation $(\mathcal{S}_t)_{t\in\mathbb{R}}$ entirely fixes the ambiguities in the choice of the null normal frame $(\boldsymbol{\ell}, \boldsymbol{k})$, via the evolution vector $\boldsymbol{h}$: there exists a unique normal null frame $(\boldsymbol{\ell}, \boldsymbol{k})$ such that

$$\boldsymbol{h} = \boldsymbol{\ell} - C\boldsymbol{k} \qquad \text{and} \qquad \boldsymbol{\ell}\cdot\boldsymbol{k} = -1$$

Normal fundamental form: $\boldsymbol{\Omega}^{(\boldsymbol{\ell})} := -\boldsymbol{k}\cdot\boldsymbol{\nabla}_{\vec{\boldsymbol{q}}}\,\boldsymbol{\ell}$ or $\Omega^{(\boldsymbol{\ell})}_{\alpha} := -k_\mu \nabla_\nu \ell^\mu \, q^\nu{}_\alpha$

Evolution of $\boldsymbol{h}$ along itself: $\boldsymbol{\nabla}_{\boldsymbol{h}}\boldsymbol{h} = \kappa\,\boldsymbol{\ell} + (C\kappa - \boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,C)\boldsymbol{k} - \boldsymbol{\mathcal{D}}C$

NB: null limit : $C = 0$, $\boldsymbol{h} = \boldsymbol{\ell} \Longrightarrow \boldsymbol{\nabla}_{\boldsymbol{\ell}}\boldsymbol{\ell} = \kappa\,\boldsymbol{\ell} \Longrightarrow \kappa$ = surface gravity

# Outline

# Concept of black hole viscosity

- Hartle and Hawking (1972, 1973): introduced the concept of **black hole viscosity** when studying the response of the *event horizon* to external perturbations
- Damour (1979): 2-dimensional **Navier-Stokes** like equation for the event horizon $\Longrightarrow$ *shear viscosity* and *bulk viscosity*
- Thorne and Price (1986): **membrane paradigm** for black holes

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- Thorne and Price (1986): **membrane paradigm** for black holes

Shall we restrict the analysis to the event horizon ?

Can we extend the concept of viscosity to the local characterizations of black hole recently introduced, i.e. future outer trapping horizons and dynamical horizons ?

NB: *event horizon* = null hypersurface
*future outer trapping horizon* = null or spacelike hypersurface
*dynamical horizon* = spacelike hypersurface

# Navier-Stokes equation in Newtonian fluid dynamics

$$\rho\left(\frac{\partial v^i}{\partial t} + v^j \nabla_j v^i\right) = -\nabla^i P + \mu \Delta v^i + \left(\zeta + \frac{\mu}{3}\right) \nabla^i (\nabla_j v^j) + f^i$$

or, in terms of fluid momentum density $\pi_i := \rho v_i$,

$$\boxed{\frac{\partial \pi_i}{\partial t} + v^j \nabla_j \pi_i + \theta \pi_i = -\nabla_i P + 2\mu \nabla^j \sigma_{ij} + \zeta \nabla_i \theta + f_i}$$

where $\theta$ is the fluid expansion:

$$\theta := \nabla_j v^j$$

and $\sigma_{ij}$ the velocity shear tensor:

$$\sigma_{ij} := \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right) - \frac{1}{3} \theta \, \delta_{ij}$$

$P$ is the pressure, $\mu$ the shear viscosity, $\zeta$ the bulk viscosity and $f_i$ the density of external forces

# Original Damour-Navier-Stokes equation

*Hyp:* $\mathcal{H}$ = null hypersurface (particular case: black hole **event horizon**)
Then $\boldsymbol{h} = \boldsymbol{\ell}$ ($C = 0$)

Damour (1979) has derived from Einstein equation the relation

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{\ell})} + 8\pi \vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{\ell}$$

or equivalently

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\pi} + \theta^{(\boldsymbol{\ell})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + 2\mu \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \zeta \boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{\ell})} + \boldsymbol{f} \quad (*)$$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\mu := \frac{1}{16\pi}$ shear viscosity

$\zeta := -\frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{\boldsymbol{q}}^{*}\boldsymbol{T} \cdot \boldsymbol{\ell}$ external force surface density ($\boldsymbol{T}$ = stress-energy tensor)

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($*$) is identical to a 2-dimensional Navier-Stokes equation

# Original Damour-Navier-Stokes equation (con't)

Introducing a coordinate system $(t, x^1, x^2, x^3)$ such that

- $t$ is compatible with $\boldsymbol{\ell}$: $\mathcal{L}_{\boldsymbol{\ell}}\, t = 1$
- $\mathcal{H}$ is defined by $x^1 = \mathrm{const}$, so that $x^a = (x^2, x^3)$ are coordinates spanning $\mathcal{S}_t$

then

$$\boldsymbol{\ell} = \frac{\partial}{\partial t} + \boldsymbol{V}$$

with $\boldsymbol{V}$ tangent to $\mathcal{S}_t$: velocity of $\mathcal{H}$'s null generators with respect to the coordinates $x^a$ [Damour, PRD **18**, 3598 (1978)].

Then

$$\theta^{(\boldsymbol{\ell})} = \mathcal{D}_a V^a + \frac{\partial}{\partial t} \ln \sqrt{q} \qquad q := \det q_{ab}$$

$$\sigma^{(\boldsymbol{\ell})}_{ab} = \frac{1}{2}\left(\mathcal{D}_a V_b + \mathcal{D}_b V_a\right) - \frac{1}{2}\theta^{(\boldsymbol{\ell})}\, q_{ab} + \frac{1}{2}\frac{\partial q_{ab}}{\partial t}$$

compare

# Negative bulk viscosity of event horizons

From the Damour-Navier-Stokes equation, $\zeta = -\frac{1}{16\pi} < 0$

This negative value would yield to a *dilation or contraction instability* in an ordinary fluid

It is in agreement with the tendency of a null hypersurface to continually contract or expand

The event horizon is stabilized by the teleological condition imposing its expansion to vanish in the far future (equilibrium state reached)

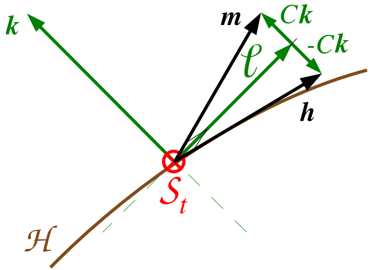
# Generalization to the non-null case

**Starting remark: in the null case (event horizon), $\ell$ plays two different roles:**

- evolution vector along $\mathcal{H}$ (e.g. term ${}^{\mathcal{S}}\!\mathcal{L}_{\ell}$ )
- normal to $\mathcal{H}$ (e.g. term $\vec{q}^{*}\boldsymbol{T}\cdot\boldsymbol{\ell}$)

When $\mathcal{H}$ is no longer null, these two roles have to be taken by two different vectors:

- evolution vector: obviously $\boldsymbol{h}$
- vector normal to $\mathcal{H}$: a natural choice is $\boxed{\boldsymbol{m} := \boldsymbol{\ell} + C\boldsymbol{k}}$



# Generalized Damour-Navier-Stokes equation

From the contracted Ricci identity applied to the vector $\boldsymbol{m}$ and projected onto $\mathcal{S}_t$: $(\nabla_\mu \nabla_\nu m^\mu - \nabla_\nu \nabla_\mu m^\mu)\, q^\nu{}_\alpha = R_{\mu\nu} m^\mu q^\nu{}_\alpha$ and using Einstein equation to express $R_{\mu\nu}$, one gets an evolution equation for $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ along $\mathcal{H}$:

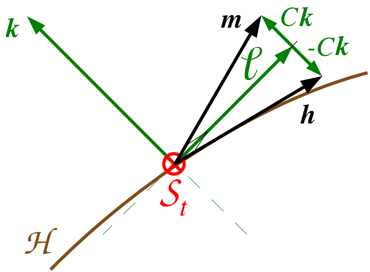
$$ {}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\boldsymbol{m})} - \theta^{(\boldsymbol{k})} \boldsymbol{\mathcal{D}} C + 8\pi \vec{\boldsymbol{q}}^* \boldsymbol{T} \cdot \boldsymbol{m} $$

- $\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ : normal fundamental form of $\mathcal{S}_t$ associated with null normal $\boldsymbol{\ell}$
- $\theta^{(\boldsymbol{h})}$, $\theta^{(\boldsymbol{m})}$ and $\theta^{(\boldsymbol{k})}$: expansion scalars of $\mathcal{S}_t$ along the vectors $\boldsymbol{h}$, $\boldsymbol{m}$ and $\boldsymbol{k}$ respectively
- $\boldsymbol{\mathcal{D}}$ : covariant derivative within $(\mathcal{S}_t, \boldsymbol{q})$
- $\kappa$ : component of $\boldsymbol{\nabla}_{\boldsymbol{h}} \boldsymbol{h}$ along $\boldsymbol{\ell}$
- $\boldsymbol{\sigma}^{(\boldsymbol{m})}$ : shear tensor of $\mathcal{S}_t$ along the vector $\boldsymbol{m}$
- $C$ : half the scalar square of $\boldsymbol{h}$

# Null limit (event horizon)

If $\mathcal{H}$ is a null hypersurface,

$$\boldsymbol{h} = \boldsymbol{m} = \boldsymbol{\ell} \qquad \text{and} \qquad C = 0$$




and we recover the original Damour-Navier-Stokes equation:

$$^{\mathcal{S}}\mathcal{L}_{\boldsymbol{\ell}}\, \boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{\ell})} \boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}} \cdot \boldsymbol{\sigma}^{(\boldsymbol{\ell})} + \frac{1}{2} \boldsymbol{\mathcal{D}} \theta^{(\boldsymbol{\ell})} + 8\pi \vec{\boldsymbol{q}}^{*} \boldsymbol{T} \cdot \boldsymbol{\ell}$$

# Case of future trapping horizons

Definition [Hayward, PRD **49**, 6467 (1994)] :
$\mathcal{H}$ is a **future trapping horizon** iff $\theta^{(\boldsymbol{\ell})} = 0$ and $\theta^{(\boldsymbol{k})} < 0$.
The generalized Damour-Navier-Stokes equation reduces then to

$$ {}^{\mathcal{S}}\!\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\Omega}^{(\boldsymbol{\ell})} + \theta^{(\boldsymbol{h})}\,\boldsymbol{\Omega}^{(\boldsymbol{\ell})} = \boldsymbol{\mathcal{D}}\kappa - \boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{m})} - \frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} - \theta^{(\boldsymbol{k})}\boldsymbol{\mathcal{D}}C + 8\pi\vec{\boldsymbol{q}}^{*}\boldsymbol{T}\cdot\boldsymbol{m} $$

[Gourgoulhon, PRD **72**, 104007 (2005)]

*NB:* Notice the change of sign in the $-\frac{1}{2}\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})}$ term with respect to the original Damour-Navier-Stokes equation

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[Gourgoulhon, PRD **72**, 104007 (2005)]

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The explanation: it is $\theta^{(\boldsymbol{m})}$ which appears in the general equation and

$$\theta^{(\boldsymbol{m})} + \theta^{(\boldsymbol{h})} = 2\theta^{(\boldsymbol{\ell})} \Longrightarrow \begin{cases} \text{event horizon } (\boldsymbol{m} = \boldsymbol{h}) & : \quad \theta^{(\boldsymbol{m})} = \theta^{(\boldsymbol{\ell})} \\ \text{trapping horizon } (\theta^{(\boldsymbol{\ell})} = 0) & : \quad \theta^{(\boldsymbol{m})} = -\theta^{(\boldsymbol{h})} \end{cases}$$

# Viscous fluid form

$$ {}^{\mathcal{S}}\mathcal{L}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f} $$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

$\zeta := \frac{1}{16\pi}$ bulk viscosity

$\boldsymbol{f} := -\vec{q}^{\,*}\boldsymbol{T}\cdot\boldsymbol{m} + \frac{\theta^{(\boldsymbol{k})}}{8\pi}\boldsymbol{\mathcal{D}}C$ external force surface density

Similar to the Damour-Navier-Stokes equation for an event horizon, except

- the **Newtonian-fluid** relation between *stress* and *strain* does not hold: $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi \neq 2\mu\boldsymbol{\sigma}^{(\boldsymbol{h})}$, rather $\boldsymbol{\sigma}^{(\boldsymbol{m})}/8\pi = [\boldsymbol{\sigma}^{(\boldsymbol{h})} + 2C\boldsymbol{\sigma}^{(\boldsymbol{k})}]/8\pi$

# Viscous fluid form

$$ {}^{\mathcal{S}}\!\boldsymbol{\mathcal{L}}_{\boldsymbol{h}}\,\boldsymbol{\pi} + \theta^{(\boldsymbol{h})}\boldsymbol{\pi} = -\boldsymbol{\mathcal{D}}P + \frac{1}{8\pi}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\sigma}^{(\boldsymbol{m})} + \zeta\boldsymbol{\mathcal{D}}\theta^{(\boldsymbol{h})} + \boldsymbol{f} $$

with $\boldsymbol{\pi} := -\frac{1}{8\pi}\boldsymbol{\Omega}^{(\boldsymbol{\ell})}$ momentum surface density

$P := \frac{\kappa}{8\pi}$ pressure

$\frac{1}{8\pi}\boldsymbol{\sigma}^{(\boldsymbol{m})}$ shear stress tensor

$\zeta := \frac{1}{16\pi}$ bulk viscosity

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- positive bulk viscosity

This positive value of bulk viscosity shows that FOTHs and DHs behave as "ordinary" physical objects, in perfect agreement with their local nature

# Outline

# Angular momentum of trapping horizons

Definition [Booth & Fairhurst, CQG **22**, 4545 (2005)]: Let $\varphi$ be a vector field on $\mathcal{H}$ which

- is tangent to $\mathcal{S}_t$
- has closed orbits
- has vanishing divergence with respect to the induced metric: $\boldsymbol{\mathcal{D}} \cdot \boldsymbol{\varphi} = 0$ (weaker than being a Killing vector of $(\mathcal{S}_t, \boldsymbol{q})$ !)

For dynamical horizons, $\theta^{(\boldsymbol{h})} \neq 0$ and there is a unique choice of $\boldsymbol{\varphi}$ as the generator (conveniently normalized) of the curves of constant $\theta^{(\boldsymbol{h})}$

[Hayward, PRD **74**, 104013 (2006)]

The *generalized angular momentum associated with* $\boldsymbol{\varphi}$ is then defined by

$$J(\boldsymbol{\varphi}) := -\frac{1}{8\pi} \oint_{\mathcal{S}_t} \langle \boldsymbol{\Omega}^{(\boldsymbol{\ell})}, \boldsymbol{\varphi} \rangle \, {}^{\mathcal{S}}\boldsymbol{\epsilon},$$

*Remark 1:* does not depend upon the choice of null vector $\boldsymbol{\ell}$, thanks to the divergence-free property of $\boldsymbol{\varphi}$

*Remark 2:*

- coincides with Ashtekar & Krishnan's definition for a dynamical horizon
- coincides with Brown-York angular momentum if $\mathcal{H}$ is timelike and $\boldsymbol{\varphi}$ a Killing vector

# Angular momentum flux law

Under the supplementary hypothesis that $\varphi$ is transported along the evolution vector $\boldsymbol{h}$ : $\mathcal{L}_{\boldsymbol{h}}\,\varphi = 0$, the generalized Damour-Navier-Stokes equation leads to

$$\frac{d}{dt}J(\varphi) = -\oint_{\mathcal{S}_t} \boldsymbol{T}(\boldsymbol{m},\varphi)\,{}^{\mathcal{S}}\boldsymbol{\epsilon} - \frac{1}{16\pi}\oint_{\mathcal{S}_t}\left[\boldsymbol{\sigma}^{(\boldsymbol{m})} : \mathcal{L}_{\varphi}\,\boldsymbol{q} - 2\theta^{(\boldsymbol{k})}\varphi\cdot\boldsymbol{\mathcal{D}}C\right]{}^{\mathcal{S}}\boldsymbol{\epsilon}$$

[Gourgoulhon, PRD **72**, 104007 (2005)]

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Two interesting limiting cases:

- $\mathcal{H} =$ null hypersurface : $C = 0$ and $\boldsymbol{m} = \boldsymbol{\ell}$ :

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- $\mathcal{H}$ = future trapping horizon :

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# Area evolution law for an event horizon

$A(t)$ : area of the 2-surface $\mathcal{S}_t$; ${}^{\mathcal{S}}\!\boldsymbol{\epsilon}$ : volume element of $\mathcal{S}_t$; $\bar{\kappa}(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa \, {}^{\mathcal{S}}\!\boldsymbol{\epsilon}$

Integrating the null Raychaudhuri equation on $\mathcal{S}_t$, one gets

$$\frac{d^2 A}{dt^2} - \bar{\kappa} \frac{dA}{dt} = - \int_{\mathcal{S}_t} \left[ 8\pi \boldsymbol{T}(\boldsymbol{\ell}, \boldsymbol{\ell}) + \boldsymbol{\sigma}^{(\boldsymbol{\ell})} : \boldsymbol{\sigma}^{(\boldsymbol{\ell})} - \frac{(\theta^{(\boldsymbol{\ell})})^2}{2} + (\bar{\kappa} - \kappa) \theta^{(\boldsymbol{\ell})} \right] {}^{\mathcal{S}}\!\boldsymbol{\epsilon} \quad (1)$$

[Damour, 1979]

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Simplified analysis : assume $\bar{\kappa} = \mathrm{const} > 0$ :

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Simplified analysis : assume $\bar{\kappa} = \mathrm{const} > 0$ :

- Cauchy problem $\Longrightarrow$ diverging solution of the homogeneous equation: $\frac{dA}{dt} = \alpha \exp(\bar{\kappa} t)$ !
- correct treatment: impose $\frac{dA}{dt} = 0$ at $t = +\infty$ (teleological !)

$$\frac{dA}{dt} = \int_t^{+\infty} D(u) e^{\bar{\kappa}(t-u)} \, du \qquad D(t) : \text{r.h.s. of Eq. (1)}$$

**Non causal evolution**

# Area evolution law for a dynamical horizon

Dynamical horizon : $C > 0$; $\kappa' := \kappa - \mathcal{L}_{\boldsymbol{h}} \ln C$; $\bar{\kappa}'(t) := \frac{1}{A(t)} \int_{\mathcal{S}_t} \kappa' \, {}^{\mathcal{S}}\boldsymbol{\epsilon}$

From the $(\boldsymbol{m}, \boldsymbol{h})$ component of Einstein equation, one gets

$$\frac{d^2 A}{dt^2} + \bar{\kappa}' \frac{dA}{dt} = \int_{\mathcal{S}_t} \left[ 8\pi \boldsymbol{T}(\boldsymbol{m}, \boldsymbol{h}) + \boldsymbol{\sigma}^{(\boldsymbol{h})} : \boldsymbol{\sigma}^{(\boldsymbol{m})} + \frac{(\theta^{(\boldsymbol{h})})^2}{2} + (\bar{\kappa}' - \kappa')\theta^{(\boldsymbol{h})} \right] {}^{\mathcal{S}}\boldsymbol{\epsilon} \quad (2)$$

[Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

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[Gourgoulhon & Jaramillo, PRD **74**, 087502 (2006)]

Simplified analysis : assume $\bar{\kappa}' = \mathrm{const} > 0$
(OK for small departure from equilibrium [Booth & Fairhurst, PRL **92**, 011102 (2004)]):
Standard Cauchy problem :

$$\frac{dA}{dt} = \left. \frac{dA}{dt} \right|_{t=0} + \int_0^t D(u) e^{\bar{\kappa}'(u-t)} \, du \qquad D(t) \text{ : r.h.s. of Eq. (2)}$$

Causal evolution, in agreement with local nature of dynamical horizons

# Outline

# Applications to numerical relativity

- **Initial data**: isolated horizons (helical symmetry)
  [Gourgoulhon, Grandclément & Bonazzola, PRD **65**, 044020 (2002)]
  [Grandclément, Gourgoulhon & Bonazzola, PRD **65**, 044021 (2002)]
  [Cook & Pfeiffer, PRD **70**, 104016 (2004)]

- **A posteriori analysis**: estimating mass, linear and angular momentum of formed black holes
  [Schnetter, Krishnan & Beyer, PRD **74**, 024028 (2006)]
  [Cook & Whiting, PRD **76**, 041501 (2007)]
  [Krishnan, Lousto & Zlochower, PRD **76**, 081501(R) (2007)]

- **Numerical construction of spacetime**: inner boundary conditions for a constrained scheme with "black hole excision"
  [Jaramillo, Gourgoulhon, Cordero-Carrión, & J.M. Ibáñez, PRD **77**, 047501 (2008)]
  [Vasset, Novak & Jaramillo, arXiv:0901??]

# Outline

# Review articles


- A. Ashtekar and B. Krishnan : *Isolated and dynamical horizons and their applications*, Living Rev. Relativity **7**, 10 (2004) ; http://www.livingreviews.org/lrr-2004-10
- I. Booth : *Black hole boundaries*, Canadian J. Phys. **83**, 1073 (2005) ; http://arxiv.org/abs/gr-qc/0508107
- E. Gourgoulhon and J.L. Jaramillo : *A 3+1 perspective on null hypersurfaces and isolated horizons*, Phys. Rep. **423**, 159 (2006) ; http://arxiv.org/abs/gr-qc/0503113
- E. Gourgoulhon and J. L. Jaramillo : *New theoretical approaches to black holes*, New Astron. Rev. **51**, 791 (2008) ; http://arxiv.org/abs/0803.2944
- S.A. Hayward : *Dynamics of black holes*, Adv. Sci. Lett. in press; http://arxiv.org/abs/0810.0923
- B. Krishnan : *Fundamental properties and applications of quasi-local black hole horizons*, Class. Quantum Grav. **25**, 114005 (2008); http://arxiv.org/abs/0712.1575