Symmetries and peeling in the extreme Reissner-Nordström spacetime

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Black holes and their symmetries
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Outline

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The extreme Reissner-Nordström (ERN) black hole

ERN = static and spherically symmetric solution $(\mathscr{M}, \boldsymbol{g}, \boldsymbol{F})$ to the electrovacuum Einstein-Maxwell equations defined in a patch $\mathbb{R} \times (0, +\infty) \times \mathbb{S}^2$ spanned by Schwarzschild-like coordinates (t, r, θ, φ) by

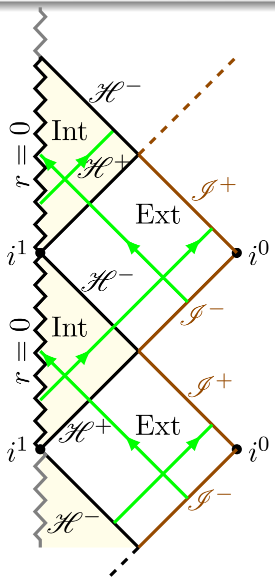
$$\boldsymbol{g} = -\left(1 - \frac{M}{r}\right)^2 \mathbf{d}t^2 + \left(1 - \frac{M}{r}\right)^{-2} \mathbf{d}r^2 + r^2 \left(\mathbf{d}\theta^2 + \sin^2\theta \mathbf{d}\varphi^2\right)$$

$$\boldsymbol{F} = -\frac{Q}{r^2} \mathbf{d}t \wedge \mathbf{d}r + P \sin\theta \, \mathbf{d}\theta \wedge \mathbf{d}\varphi$$

where the electric charge Q and the magnetic charge P obey $\sqrt{Q^2 + P^2} = M$.

$\Longrightarrow$ describes a black hole with the event horizon located at $r = M$.

Carter-Penrose diagram of ERN maximal analytic extension



Compactified diagram of the maximal analytic extension of the ERN spacetime [Carter, Phys. Lett. **21**, 423 (1966)]

Eddington-Finkelstein-type (EF) coordinates:

- outgoing: $u := t - r_*$, $\quad r_* := \frac{r(r-2M)}{r-M} + 2M \ln\left|\frac{r}{M} - 1\right|$
- ingoing: $v := t + r_*$

Compactified coordinates: $U := \arctan\left(\frac{u}{2M}\right)$, $V = \arctan\left(\frac{v}{2M}\right)$

i^0: spatial infinity
i^1: internal infinity (infinitely long throat along any $t = \mathrm{const}$ hypersuface)

The degenerate event horizon

The black hole event horizon $\mathscr{H}^+$ is the hypersurface $r = M$ in a ingoing patch (v, r, θ, φ).
$\mathscr{H}^+$ is a degenerate Killing horizon with respect to the Killing vector $\boldsymbol{\xi} = \boldsymbol{\partial}_t$

degenerate Killing horizon $\iff$ surface gravity κ, defined by $\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi} \stackrel{\mathscr{H}^+}{=} \kappa\,\boldsymbol{\xi}$, is vanishing:

$$\kappa = 0$$

$\Longrightarrow$ $\boldsymbol{\xi}$ is a geodesic vector field on $\mathscr{H}^+$: $\boldsymbol{\nabla}_{\boldsymbol{\xi}}\boldsymbol{\xi} \stackrel{\mathscr{H}^+}{=} 0$

$\Longrightarrow$ t is an affine parameter along the null geodesic generators of $\mathscr{H}^+$

$\Longrightarrow$ the null geodesic generators of $\mathscr{H}^+$ are complete geodesics (no bifurcation surface); internal infinity i^1 = limit $t \to -\infty$ along the null geodesic generators

$\boldsymbol{\xi}$ is null on $\mathscr{H}^+$ and is timelike both in the black hole exterior and in the black hole interior (contrary to Schwarzschild)

Near-horizon geometry

"Near-horizon magnifying" coordinates ($\varepsilon \neq 0$): $\begin{cases} T := \varepsilon \dfrac{t}{M} \\ R := \dfrac{r-M}{\varepsilon M} \end{cases} \iff \begin{cases} t =: M \dfrac{T}{\varepsilon} \\ r =: M(1+\varepsilon R) \end{cases}$

At fixed (T, R), $\lim_{\varepsilon \to 0} t = +\infty$ and $\lim_{\varepsilon \to 0} r = M$

$$\Longrightarrow \quad \boldsymbol{g} = M^2 \left[-\frac{R^2}{(1+\varepsilon R)^2} \mathbf{d}T^2 + (1+\varepsilon R)^2 \frac{\mathbf{d}R^2}{R^2} + (1+\varepsilon R)^2 \left(\mathbf{d}\theta^2 + \sin^2\theta \, \mathbf{d}\varphi^2 \right) \right]$$

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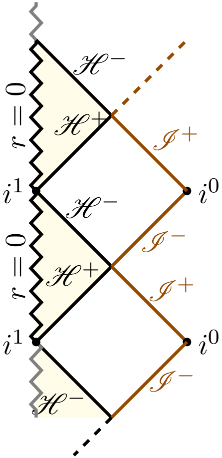
Near-horizon (NHERN) metric: $\boldsymbol{h} = \lim_{\varepsilon \to 0} \boldsymbol{g} \Longrightarrow$ product metric of $\mathrm{AdS}_2 \times \mathbb{S}^2$ [Carter 1973]:

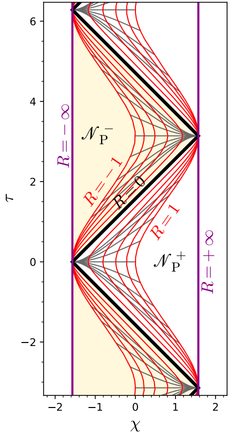
$$\boldsymbol{h} = M^2 \Big(\underbrace{-R^2 \mathbf{d}T^2 + \frac{\mathbf{d}R^2}{R^2}}_{\mathrm{AdS}_2} + \underbrace{\mathbf{d}\theta^2 + \sin^2\theta \, \mathbf{d}\varphi^2}_{\mathbb{S}^2} \Big)$$

also known as Bertotti-Robinson metric — another solution (1959) of the electrovacuum Einstein-Maxwell equations.

Near-horizon region mapped to AdS_2



ERN



AdS_2

(T, R) are Poincaré coordinates in a Poincaré patch $\mathscr{N}_\mathrm{P}^\pm$ of AdS_2, bounded by the Poincaré horizon $\mathscr{H}_\mathrm{P}$ at $R=0$.

Global AdS_2 coordinates: $(\tau, \chi) \in \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

such that $\begin{cases} T := \dfrac{\sin\tau}{\cos\tau + \sin\chi} \\ R := \dfrac{\cos\tau + \sin\chi}{\cos\chi} \end{cases}$

Both the ERN horizon and the Poincaré horizon are degenerate Killing horizons.

Near-horizon enhanced symmetries

NHERN metric: $\boldsymbol{h} = M^2 \left(-R^2 \mathbf{d}T^2 + \frac{\mathbf{d}R^2}{R^2} + \mathbf{d}\theta^2 + \sin^2\theta \mathbf{d}\varphi^2 \right)$

Killing vectors of the AdS_2 sector:

- $\boldsymbol{\xi}_1 = \boldsymbol{\partial}_T$ $\leftarrow$ inherited from ENR stationarity
- $\boldsymbol{\xi}_2 = T\boldsymbol{\partial}_T - R\boldsymbol{\partial}_R$ $\leftarrow$ generates the isometries $(T, R) \mapsto \left(\alpha T, \frac{R}{\alpha}\right)$, $\alpha > 0$
- $\boldsymbol{\xi}_3 = \frac{1}{2}\left(T^2 + \frac{1}{R^2}\right)\boldsymbol{\partial}_T - RT\boldsymbol{\partial}_R$ $\leftarrow$ from the global stationarity of AdS_2: $\boldsymbol{\xi}_3 = \boldsymbol{\partial}_\tau - \frac{1}{2}\boldsymbol{\xi}_1$

One has

$[\boldsymbol{\xi}_2, \boldsymbol{\xi}_1] = -\boldsymbol{\xi}_1, \quad [\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = \boldsymbol{\xi}_3, \quad [\boldsymbol{\xi}_1, \boldsymbol{\xi}_3] = \boldsymbol{\xi}_2 \quad \Longrightarrow \quad \mathfrak{sl}(2, \mathbb{R})$ algebra

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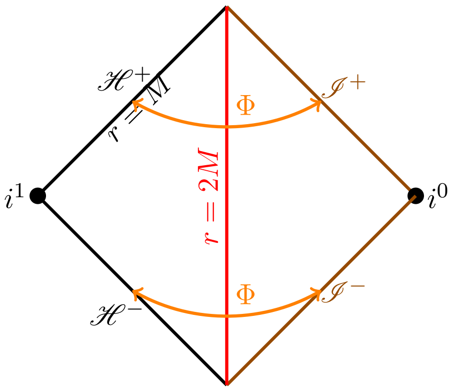
Isometry groups

$G_{\mathrm{ERN}} = \mathbb{R} \times \mathrm{SO}(3)$	$G_{\mathrm{NHERN}} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(3)$
$\dim G_{\mathrm{ERN}} = 4$	$\dim G_{\mathrm{NHERN}} = 6$

A peculiar feature of ERN: the Couch-Torrence inversion



The map $\Phi : \mathrm{Ext} \to \mathrm{Ext}$ defined by

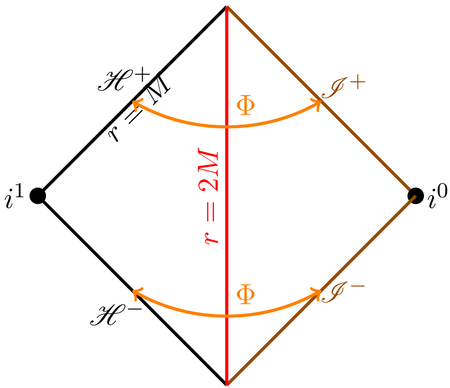
$$\Phi(t, r, \theta, \varphi) = \left(t, \frac{rM}{r - M}, \theta, \varphi\right)$$

or equivalently by

$$\Phi(t, r_*, \theta, \varphi) = (t, -r_*, \theta, \varphi)$$

is an involution that fixes the photon sphere $\{r = 2M\}$ and interchanges $\mathscr{H}^+$ and $\mathscr{I}^+$, as well as $\mathscr{H}^-$ and $\mathscr{I}^-$.

A peculiar feature of ERN: the Couch-Torrence inversion



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is an involution that fixes the photon sphere $\{r = 2M\}$ and interchanges $\mathscr{H}^+$ and $\mathscr{I}^+$, as well as $\mathscr{H}^-$ and $\mathscr{I}^-$.

Φ is a conformal isometry of the exterior region:

$$\Phi^* \boldsymbol{g} = \frac{M^2}{(r-M)^2}\, \boldsymbol{g}$$

[Couch & Torrence, Gen. Relat. Gravit. **16**, 789 (1984)]

Outline

Peeling in Minkowski spacetime

Peeling of massless fields (Sachs, 1961)

An outgoing massless field $\mathbf{\Psi}$ of spin s, along a null geodesic $\mathscr{L}$ to $\mathscr{I}^+$, can be expanded as

$$\mathbf{\Psi} = \sum_{k=0}^{2s-1} \frac{\mathbf{\Psi}_k}{r^{k+1}} + O\left(\frac{1}{r^{2s+1}}\right),$$

where r is an affine parameter along $\mathscr{L}$ and $\mathbf{\Psi}_k$ has $2s-k$ principal null directions (PND) that coincide with the null tangent to $\mathscr{L}$.

- $s=1$: $\mathbf{\Psi} = \boldsymbol{F}$, $\boldsymbol{\ell}$ PND $\iff \ell^a F_{a[b}\ell_{c]} = 0$, $\quad \boldsymbol{F} = \frac{\boldsymbol{N}}{r} + \frac{\boldsymbol{I}}{r^2} + O\left(\frac{1}{r^3}\right)$
- $s=2$: $\mathbf{\Psi} = \boldsymbol{C}$, $\boldsymbol{\ell}$ PND $\iff \ell^b\ell^c\ell_{[e}C_{a]bc[d}\ell_{f]} = 0$, $\quad \boldsymbol{C} = \frac{\boldsymbol{N}}{r} + \frac{\boldsymbol{III}}{r^2} + \frac{\boldsymbol{II}}{r^3} + \frac{\boldsymbol{I}}{r^4} + O\left(\frac{1}{r^5}\right)$

Extending the peeling to asymptotically flat spacetimes

Penrose conformal completion

Physical spacetime manifold $(\mathscr{M}, \boldsymbol{g})$ admits a *conformal completion at infinity* iff $\exists$ a Lorentzian manifold $(\hat{\mathscr{M}}, \hat{\boldsymbol{g}})$ with boundary $\mathscr{I}$ and a smooth function $\Omega : \hat{\mathscr{M}} \to \mathbb{R}^+$ such that

- $\mathscr{M}$ is the interior of $\hat{\mathscr{M}}$: $\hat{\mathscr{M}} = \mathscr{M} \sqcup \mathscr{I}$
- on $\mathscr{M}$, $\Omega > 0$ and $\hat{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$
- on $\mathscr{I}$, $\Omega = 0$ and $\mathbf{d}\Omega \neq 0$

Penrose reformulation of peeling (1965)

The peeling property of $\boldsymbol{\Psi}$ is equivalent to the conformally rescaled $\boldsymbol{\Psi}$ extending to a continuous field at $\mathscr{I}$.

This works well for Minkowski, which has a fully regular conformal compactification in the Einstein cylinder $\mathbb{R} \times \mathbb{S}^3$, including at spatial infinity i^0.
But, for a curved spacetime, i^0 is in general a singular point and it is not clear whether the peeling of massless fields holds for a sufficiently generic class of initial data...

The peeling à la Mason-Nicolas

Penrose peeling involves an expansion in powers of $1/r$ inherited from a Taylor expansion of the field that is assumed to be C^k at $\mathscr{I}^+$. Now, the initial value problem of hyperbolic equations is ill-posed in C^k spaces, so it is difficult to characterize the class of initial data that give rise to such a peeling.

In 2009, L. Mason & J.-P. Nicolas [J. Inst. Math. Jussieu **8**, 179] have reformulated the peeling in Schwarzschild spacetime by characterizing the regularity at $\mathscr{I}^+$ in terms of Sobolev-type spaces, via energy fluxes with respect to the Morawetz vector field.
Sobolev norms are adapted to the initial value problem for hyperbolic equations and Mason & Nicolas could provide a complete description of the class of initial data on a Cauchy hypersurface that give rise to a peeling at any order at $\mathscr{I}^+$.

We are going to consider Mason-Nicolas peeling in ERN spacetime

Outline

The conformal wave equation

Massless field equations are conformally invariant. Here we focus on the spin $s = 0$ case, i.e. a scalar field.

On spacetime $(\mathscr{M}, \boldsymbol{g})$, the conformal wave equation for a scalar field ϕ is

$$\Box_{\boldsymbol{g}}\phi - \frac{\mathcal{R}}{6}\,\phi = 0, \qquad (1)$$

where $\Box_{\boldsymbol{g}} := \nabla_\mu \nabla^\mu$ and $\mathcal{R}$ is the Ricci scalar of $\boldsymbol{g}$.

(1) is conformally invariant: if $\hat{\boldsymbol{g}} = \Omega^2 \boldsymbol{g}$,

$$(1) \iff \Box_{\hat{\boldsymbol{g}}}\hat{\phi} - \frac{\hat{\mathcal{R}}}{6}\,\hat{\phi} = 0 \text{ for } \hat{\phi} := \Omega^{-1}\phi$$

Conformal completion of ERN

ERN in outgoing EF ($u := t - r_*$): $\boldsymbol{g} = -\left(1 - \frac{M}{r}\right)^2 \mathbf{d}u^2 - 2\mathbf{d}u\mathbf{d}r + r^2\left(\mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\varphi^2\right)$

Choice of conformal factor: $\Omega = 1/r$

$\rightarrow$ same as for conformal completion of Schwarzschild introduced by Penrose (1965)

$\rightarrow$ different from the standard one for Minkowski: $\Omega = 2[(1 + (t - r)^2)(1 + (t + r)^2)]^{-1/2}$

In terms of the coordinates (u, R, θ, φ) where $R := 1/r$, $\Omega = R$ and the conformal metric is

$$\hat{\boldsymbol{g}} = R^2\boldsymbol{g} = -R^2(1 - MR)^2\mathbf{d}u^2 + 2\mathbf{d}u\mathbf{d}R + \mathbf{d}\theta^2 + \sin^2\theta\mathbf{d}\varphi^2$$

$\Longrightarrow \mathscr{I}^+$ is the hypersurface $R = 0$ and is spanned by the coordinates (u, θ, φ)

NB: i^0 remains at infinity, at $u \to -\infty$ on each slice $t = \mathrm{const}$.

The conformal wave equation on ERN

For ERN (as for any electrovacuum solution in 4-dim GR), $\mathcal{R} = 0$
$\Longrightarrow$ conformal wave equation (1) reduces to $\Box_{\boldsymbol{g}}\phi = 0$
But $\hat{R} = 12MR(MR-1) \neq 0$ and the conformal wave equation becomes

$$\Box_{\hat{\boldsymbol{g}}}\,\hat{\phi} + 2MR(1-MR)\hat{\phi} = 0, \qquad \hat{\phi} := R^{-1}\phi, \tag{2}$$

with

$$\Box_{\hat{\boldsymbol{g}}}\,\hat{\phi} = 2\frac{\partial^2\hat{\phi}}{\partial u\partial R} + \frac{\partial}{\partial R}\left(R^2(1-MR)^2\frac{\partial\hat{\phi}}{\partial R}\right) + \Delta_{\mathbb{S}^2}\hat{\phi}$$

Goal

Characterize the regularity at $\mathscr{I}^+$ of the solution ϕ in terms of the regularity and decay of the initial data on a Cauchy hypersurface Σ_0.

Focusing on a neighborhood of i^0

In order to control the regularity of $\hat{\phi}$ at $\mathscr{I}^+$, it suffices to control it in a neighborhood of i^0: provided the initial data have the correct regularity away from i^0, the regularity of $\hat{\phi}$ can be seen to extend to the whole of $\mathscr{I}^+$.

Choose the Cauchy hypersurface $\Sigma_0 = \{t = 0\}$

For $u_0 \ll -M$, define the neighborhood of i^0 in the future of Σ_0 by

$$\Omega_{u_0} = \{u \leq u_0, t \geq 0\}$$

The boundary of Ω_{u_0} is made of 3 parts:

$$\Sigma_{0,u_0} = \Sigma_0 \cap \Omega_{u_0}, \quad \mathscr{I}^+_{u_0} = \mathscr{I}^+ \cap \Omega_{u_0}, \quad \mathcal{S}_{u_0} = \{u = u_0\} \cap \Omega_{u_0}$$

The Morawetz vector field and the associated energy current

Consider the vector field

$$\boldsymbol{K} := u^2 \boldsymbol{\partial}_u - 2(1 + uR)\boldsymbol{\partial}_R$$

$\boldsymbol{K}$ has the same expression in terms of (u, R) coordinates as the conformal Killing vector $u^2\boldsymbol{\partial}_u + v^2\boldsymbol{\partial}_v$ of Minkowski spacetime introduced by C. Morawetz (1962) to establish decay properties of solutions to the wave equation in flat space.

One cas show that $\boldsymbol{K}$ is future-directed timelike in a neighborhood of i^0. Moreover, it is transverse to $\mathscr{I}^+$.

For Minkowski, $\boldsymbol{K}$ is a Killing vector of $\hat{\boldsymbol{g}} = R^2\hat{\boldsymbol{g}}$. Not here, except at $\mathscr{I}^+$ and i^0.

The Morawetz-based field energy through a hypersurface

Consider the energy-momentum tensor of the free wave equation $\Box_{\hat{\boldsymbol{g}}}\hat{\phi} = 0$, namely

$$T_{ab}(\hat{\phi}) = \hat{\nabla}_a\hat{\phi}\,\hat{\nabla}_b\hat{\phi} - \frac{1}{2}\hat{\nabla}_c\hat{\phi}\,\hat{\nabla}^c\hat{\phi}\,\hat{g}_{ab}$$

and define the current

$$J_a(\hat{\phi}) := T_{ab}(\hat{\phi})K^b$$

$\boldsymbol{J}(\hat{\phi})$ is not conserved since $\hat{\nabla}^a J_a(\hat{\phi}) = T_{ab}\hat{\nabla}^{(a}K^{b)} - \frac{\hat{\mathcal{R}}}{6}\hat{\phi}\,K^a\hat{\nabla}_a\phi$
but one can control the r.h.s. in a neighborhood of i^0.

Given an oriented hypersurface $\mathscr{S}$ of $\hat{\mathscr{M}}$, define the "energy" $\mathcal{E}_{\mathscr{S}}(\hat{\phi}) := \int_{\mathscr{S}} \star\boldsymbol{J}(\hat{\phi})$

The peeling property at $\mathscr{I}^+$

[Borthwick, EG, Nicolas, J. Hyper. Diff. Eq. **22**, 29 (2025)]

Let $k \in \mathbb{N}$ and $\hat{\phi}$ a solution to the conformal wave equation (2). Then $\mathcal{E}_{\mathscr{I}^+_{u_0}}(\partial_R^q \nabla_{\mathbb{S}^2}^p \hat{\phi}) + \mathcal{E}_{\mathcal{S}_{u_0}}(\partial_R^q \nabla_{\mathbb{S}^2}^p \hat{\phi}) < +\infty$ for all $p, q \in \mathbb{N}$, $p + q \leq k$ if and only if the initial data $(\hat{\phi}, \partial_t \hat{\phi}) = (\hat{\phi}_0, \hat{\phi}_1)$ on Σ_0 is chosen in the completion of $C_0^\infty([-u_0, +\infty[_{r_*} \times \mathbb{S}^2) \times C_0^\infty([-u_0, +\infty[_{r_*} \times \mathbb{S}^2)$ in the norm:

$$\left\| \begin{pmatrix} \hat{\phi}_0 \\ \hat{\phi}_1 \end{pmatrix} \right\|_k^2 = \sum_{p+q \leq k} \mathcal{E}_{\Sigma_{0,u_0}} \left(L^q \nabla_{\mathbb{S}^2}^p \begin{pmatrix} \hat{\phi}_0 \\ \hat{\phi}_1 \end{pmatrix} \right),$$

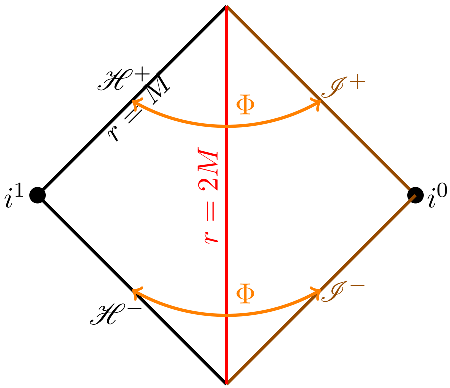
where L is the operator defined by $L = \begin{pmatrix} -\frac{r^2}{F(r)} \partial_{r_*} & -\frac{r^2}{F(r)} \\ -\frac{r^2}{F(r)} \partial_{r_*}^2 - \Delta_{\mathbb{S}^2} - \frac{2M}{r}\left(1 - \frac{M}{r}\right) & -\frac{r^2}{F(r)} \partial_{r_*} \end{pmatrix}$

In this case we say that $\hat{\phi}$ peels at order k at infinity.

Outline

1. Symmetries of the extreme Reissner-Nordström black hole
2. The peeling property and its variants
3. Peeling for the wave equation in ERN spacetime
4. Peeling at the ERN horizon

The Couch-Torrence inversion as an isometry of $\hat{\boldsymbol{g}}$



The Couch-Torrence inversion

$$\begin{array}{rccc}\Phi: & \widehat{\mathrm{Ext}} & \rightarrow & \widehat{\mathrm{Ext}} \\ & (t, R, \theta, \varphi) & \longmapsto & \left(t, \frac{1}{M} - R, \theta, \varphi\right)\end{array}$$

is an isometry of the ERN conformal metric $\hat{\boldsymbol{g}} = R^2 \boldsymbol{g}$:

$$\Phi^* \hat{\boldsymbol{g}} = \hat{\boldsymbol{g}},$$

which obeys

$$\Phi(\mathscr{H}^+) = \mathscr{I}^+$$

[Borthwick, EG, Nicolas, J. Hyper. Diff. Eq. **22**, 29 (2025)]

$\Longrightarrow$ allows one to establish a peeling property at the ERN horizon, from that already obtained at $\mathscr{I}^+$

Towards the peeling at $\mathscr{H}^+$: a neighborhood of i^1

Consider ingoing EF coordinates (v, R, θ, φ) on $\widehat{\mathrm{Ext}}$
The Couch-Torrence inversion maps i^0 to the internal infinity i^1
i^1 is the limit $v \to -\infty$ along $\mathscr{H}^+$
Choose the Cauchy hypersurface $\Sigma_0 = \{t = 0\}$
For $v_0 \ll -M$, define the neighborhood of i^1 in the future of Σ_0 by

$$\Omega_{v_0} = \{v \leq v_0, t \geq 0\}$$

The boundary of Ω_{v_0} is made of 3 parts:

$$\Sigma_{0,v_0} = \Sigma_0 \cap \Omega_{v_0}, \quad \mathscr{H}^+_{v_0} = \mathscr{H}^+ \cap \Omega_{v_0}, \quad \tilde{S}_{v_0} = \{v = v_0\} \cap \Omega_{v_0}$$

The peeling property at $\mathscr{H}^+$

[Borthwick, EG, Nicolas, J. Hyper. Diff. Eq. **22**, 29 (2025)]

Let $k \in \mathbb{N}$ and $\hat{\phi}$ a solution to the conformal wave equation (2). Then $\tilde{\mathcal{E}}_{\mathscr{H}^+_{v_0}}(\partial_R^q \nabla^p_{\mathbb{S}^2}\hat{\phi}) + \tilde{\mathcal{E}}_{\tilde{\mathcal{S}}_{v_0}}(\partial_R^q \nabla^p_{\mathbb{S}^2}\hat{\phi}) < +\infty$ for all $p, q \in \mathbb{N}$, $p+q \leq k$ if and only if the initial data $(\hat{\phi}, \partial_t\hat{\phi}) = (\hat{\phi}_0, \hat{\phi}_1)$ on Σ_0 is chosen in the completion of $C_0^\infty(]-\infty, v_0]_{r_*} \times \mathbb{S}^2) \times C_0^\infty(]-\infty, v_0]_{r_*} \times \mathbb{S}^2)$ in the norm:

$$\left\| \begin{pmatrix} \hat{\phi}_0 \\ \hat{\phi}_1 \end{pmatrix} \right\|_k^2 = \sum_{p+q\leq k} \mathcal{E}_{\Sigma_{0,v_0}} \left(\tilde{L}^q \nabla^p_{\mathbb{S}^2} \begin{pmatrix} \hat{\phi}_0 \\ \hat{\phi}_1 \end{pmatrix} \right),$$

where $\tilde{L}$ is the operator defined by $\tilde{L} = \begin{pmatrix} -\frac{r^2}{F(r)}\partial_{r_*} & \frac{r^2}{F(r)} \\ \frac{r^2}{F(r)}\partial^2_{r_*} + \Delta_{\mathbb{S}^2} + \frac{2M}{r}\left(1 - \frac{M}{r}\right) & -\frac{r^2}{F(r)}\partial_{r_*} \end{pmatrix}$

In this case we say that $\hat{\phi}$ peels at order k at the event horizon $\mathscr{H}^+$.

Perspectives

- Extend peeling at horizon to extreme Kerr black hole, but no Couch-Torrence (conformal) isometry in that case!
  NB: Peeling at $\mathscr{I}^+$ of Kerr has been obtained in [Nicolas & Xuan Pham, Ann. H. Poincaré **20**, 3419 (2019)]