A geometrical approach to relativistic magnetohydrodynamics

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Bon anniversaire !

to **Takashi Nakamura** and **Kei-ichi Maeda**

Plan

1. Introduction
2. Relativistic MHD with exterior calculus
3. Stationary and axisymmetric electromagnetic fields in general relativity
4. Stationary and axisymmetric MHD
5. Grad-Shafranov and transfield equations

Outline

1 Introduction

2 Relativistic MHD with exterior calculus

3 Stationary and axisymmetric electromagnetic fields in general relativity

4 Stationary and axisymmetric MHD

5 Grad-Shafranov and transfield equations

Short history of general relativistic MHD

focusing on stationary and axisymmetric spacetimes

- Lichnerowicz (1967): formulation of GRMHD
- Bekenstein & Oron (1978), Carter (1979) : development of GRMHD for stationary and axisymmetric spacetimes
- Mobarry & Lovelace (1986) : Grad-Shafranov equation for Schwarzschild spacetime
- Nitta, Takahashi & Tomimatsu (1991), Beskin & Par'ev (1993) : Grad-Shafranov equation for Kerr spacetime
- Ioka & Sasaki (2003) : Grad-Shafranov equation in the most general (i.e. *noncircular*) stationary and axisymmetric spacetimes

NB: not speaking about *numerical* GRMHD here
(see e.g. Shibata & Sekiguchi (2005))

Why a geometrical approach ?

- Previous studies made use of component expressions, the covariance of which is not obvious
  For instance, two of main quantities introduced by Bekenstein & Oron (1978) and employed by subsequent authors are

$$\omega := -\frac{F_{01}}{F_{31}} \qquad \text{and} \qquad C := \frac{F_{31}}{\sqrt{-g n u^2}}$$

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On the other side

- As well known, the electromagnetic field tensor $\boldsymbol{F}$ is fundamentally a 2-form and Maxwell equations are most naturally expressible in terms of the exterior derivative operator
- The equations of perfect hydrodynamics can also be recast in terms of exterior calculus, by introducing the fluid vorticity 2-form (Synge 1937, Lichnerowicz 1941)
- Cartan's exterior calculus makes calculations easier !

Exterior calculus in one slide

cf. Valeri Frolov's talk

- A p-form ($p = 0, 1, 2, \ldots$) is a multilinear form (i.e. a tensor 0-times contravariant and p-times covariant: $\omega_{\alpha_1 \ldots \alpha_p}$) that is fully *antisymmetric*
- *Index-free notation:* given a vector $\vec{v}$ and a p-form ω, $\boxed{\vec{v} \cdot \omega}$ and $\boxed{\omega \cdot \vec{v}}$ are the $(p-1)$-forms defined by

$$\vec{v} \cdot \omega := \omega(\vec{v}, ., \ldots, .) \qquad [\ (\vec{v} \cdot \omega)_{\alpha_1 \cdots \alpha_{p-1}} = v^\mu \omega_{\mu \alpha_1 \cdots \alpha_{p-1}}\ ]$$
$$\omega \cdot \vec{v} := \omega(., \ldots, ., \vec{v}) \qquad [(\omega \cdot \vec{v})_{\alpha_1 \cdots \alpha_{p-1}} = \omega_{\alpha_1 \cdots \alpha_{p-1} \mu} v^\mu\ ]$$

- Exterior derivative : p-form $\omega \longmapsto (p+1)$-form $\mathbf{d}\omega$ such that

$$\begin{aligned} &\text{0-form} \quad : \quad (\mathbf{d}\omega)_\alpha = \partial_\alpha \omega \\ &\text{1-form} \quad : \quad (\mathbf{d}\omega)_{\alpha\beta} = \partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha \\ &\text{2-form} \quad : \quad (\mathbf{d}\omega)_{\alpha\beta\gamma} = \partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta} \end{aligned}$$

The exterior derivative is nilpotent: $\boxed{\mathbf{dd}\omega = 0}$

- A very powerful tool : Cartan's identity expressing the Lie derivative of a p-form along a vector field: $\boxed{\mathcal{L}_{\vec{v}}\, \omega = \vec{v} \cdot \mathbf{d}\omega + \mathbf{d}(\vec{v} \cdot \omega)}$

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General framework and notations

Spacetime:

- $\mathscr{M}$: four-dimensional orientable real manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$, $\mathrm{sign}\, \boldsymbol{g} = (-,+,+,+)$
- $\boldsymbol{\epsilon}$: Levi-Civita tensor (volume element 4-form) associated with $\boldsymbol{g}$: for any orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$,

$$\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \pm 1$$

$\boldsymbol{\epsilon}$ gives rise to Hodge duality : p-form $\longmapsto$ $(4-p)$-form

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Notations:

- $\vec{\boldsymbol{v}}$ vector $\Longrightarrow$ $\boxed{\underline{\boldsymbol{v}}}$ 1-form associated to $\vec{\boldsymbol{v}}$ by the metric tensor:

$$\underline{\boldsymbol{v}} := \boldsymbol{g}(\vec{\boldsymbol{v}}, .) \qquad [\underline{\boldsymbol{v}} = \boldsymbol{v}^\flat] \qquad [u_\alpha = g_{\alpha\mu} u^\mu]$$

- $\boldsymbol{\omega}$ 1-form $\Longrightarrow$ $\boxed{\vec{\boldsymbol{\omega}}}$ vector associated to $\boldsymbol{\omega}$ by the metric tensor:

$$\boldsymbol{\omega} =: \boldsymbol{g}(\vec{\boldsymbol{\omega}}, .) \qquad [\vec{\boldsymbol{\omega}} = \boldsymbol{\omega}^\sharp] \qquad [\omega^\alpha = g^{\alpha\mu} \omega_\mu]$$

Maxwell equations

Electromagnetic field in $\mathscr{M}$: 2-form $\boldsymbol{F}$ which obeys to Maxwell equations:

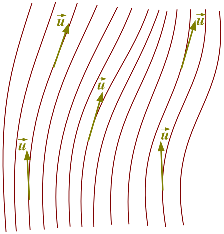
$$\boxed{\mathbf{d}\boldsymbol{F} = 0}$$

$$\boxed{\mathbf{d}\star\boldsymbol{F} = \mu_0 \star\underline{\boldsymbol{j}}}$$

- $\mathbf{d}\boldsymbol{F}$: exterior derivative of $\boldsymbol{F}$: $(\mathbf{d}\boldsymbol{F})_{\alpha\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$
- $\star\boldsymbol{F}$: Hodge dual of $\boldsymbol{F}$: $\star F_{\alpha\beta} := \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$
- $\star\underline{\boldsymbol{j}}$: 3-form Hodge-dual of the 1-form $\underline{\boldsymbol{j}}$ associated to the electric 4-current $\vec{\boldsymbol{j}}$: $\star\underline{\boldsymbol{j}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{j}}, ., ., .)$
- μ_0 : magnetic permeability of vacuum

Electric and magnetic fields in the fluid frame

Fluid : congruence of worldlines in $\mathscr{M}$ $\Longrightarrow$ 4-velocity $\vec{u}$



- **Electric field** in the fluid frame: 1-form $\boxed{\boldsymbol{e} = \boldsymbol{F} \cdot \vec{\boldsymbol{u}}}$
- **Magnetic field** in the fluid frame: vector $\vec{b}$ such that $\boxed{\underline{\boldsymbol{b}} = \vec{\boldsymbol{u}} \cdot \star\boldsymbol{F}}$

$\boldsymbol{e}$ and $\vec{b}$ are orthogonal to $\vec{u}$: $\boldsymbol{e} \cdot \vec{\boldsymbol{u}} = 0$ and $\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{u}} = 0$

$$\boldsymbol{F} = \underline{\boldsymbol{u}} \wedge \boldsymbol{e} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)$$

$$\star\boldsymbol{F} = -\underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{b}} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}, ., .)$$

Perfect conductor

Fluid is a perfect conductor $\iff$ $\vec{e} = 0$ $\iff$ $\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$

From now on, we assume that the fluid is a perfect conductor (ideal MHD)

The electromagnetic field is then entirely expressible in terms of vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{b}}$:

$$\boxed{\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)}$$

$$\boxed{\star\boldsymbol{F} = \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{u}}}$$

Alfvén's theorem

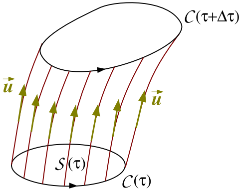
Cartan's identity applied to the 2-form $\boldsymbol{F}$:

$$\mathcal{L}_{\vec{\boldsymbol{u}}}\,\boldsymbol{F} = \vec{\boldsymbol{u}}\cdot\mathbf{d}\boldsymbol{F} + \mathbf{d}(\vec{\boldsymbol{u}}\cdot\boldsymbol{F})$$

Now $\mathbf{d}\boldsymbol{F} = 0$ (Maxwell eq.) and $\vec{\boldsymbol{u}}\cdot\boldsymbol{F} = 0$ (perfect conductor)
Hence the electromagnetic field is preserved by the flow:

$$\boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\,\boldsymbol{F} = 0}$$

Application: $\boxed{\dfrac{d}{d\tau}\oint_{\mathcal{C}(\tau)} \boldsymbol{A} = 0}$



- τ : fluid proper time
- $\mathcal{C}(\tau)$ = closed contour dragged along by the fluid
- $\boldsymbol{A}$: electromagnetic 4-potential : $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$

Proof: $\dfrac{d}{d\tau}\oint_{\mathcal{C}(\tau)}\boldsymbol{A} = \dfrac{d}{d\tau}\int_{\mathcal{S}(\tau)}\underbrace{\mathbf{d}\boldsymbol{A}}_{\boldsymbol{F}} = \dfrac{d}{d\tau}\int_{\mathcal{S}(\tau)}\boldsymbol{F} = \int_{\mathcal{S}(\tau)}\underbrace{\mathcal{L}_{\vec{\boldsymbol{u}}}\,\boldsymbol{F}}_{0} = 0$

Non-relativistic limit: $\displaystyle\int_{\mathcal{S}} \vec{\boldsymbol{b}}\cdot d\vec{\boldsymbol{S}} = \mathrm{const} \leftarrow$ Alfvén's theorem (mag. flux freezing)

Perfect fluid

From now on, we assume that the fluid is a perfect one: its energy-momentum tensor is

$$\boldsymbol{T}^{\rm fluid} = (\varepsilon + p)\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p\boldsymbol{g}$$

Simple fluid model: all thermodynamical quantities depend on

- s: entropy density in the fluid frame,
- n: baryon number density in the fluid frame

Equation of state : $\varepsilon = \varepsilon(s, n) \Longrightarrow \begin{cases} T := \dfrac{\partial \varepsilon}{\partial s} & \text{temperature} \\ \mu := \dfrac{\partial \varepsilon}{\partial n} & \text{baryon chemical potential} \end{cases}$

First law of thermodynamics $\Longrightarrow p = -\varepsilon + Ts + \mu n$

$\Longrightarrow$ enthalpy per baryon : $\boxed{h = \dfrac{\varepsilon + p}{n} = \mu + TS}$, with $S := \dfrac{s}{n}$ (entropy per baryon)

Conservation of energy-momentum

Conservation law for the total energy-momentum:

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{T}^{\text{fluid}}+\boldsymbol{T}^{\text{em}}\right)=0 \qquad (1)$$

- From Maxwell equations, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\text{em}}=-\boldsymbol{F}\cdot\vec{\boldsymbol{j}}$
- Using baryon number conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\text{fluid}}$ can be decomposed in two parts:
  - along $\vec{\boldsymbol{u}}$: $\vec{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\text{fluid}}=-nT\vec{\boldsymbol{u}}\cdot\mathbf{d}S$
  - orthogonal to $\vec{\boldsymbol{u}}$: $\perp_{\boldsymbol{u}}\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\text{fluid}}=n(\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S)$

    [Synge 1937] [Lichnerowicz 1941] [Taub 1959] [Carter 1979]

    $\boldsymbol{\Omega}:=\mathbf{d}(h\underline{\boldsymbol{u}})$ vorticity 2-form

Since $\vec{\boldsymbol{u}}\cdot\boldsymbol{F}\cdot\vec{\boldsymbol{j}}=0$, Eq. (1) is equivalent to the system

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}S=0} \qquad (2)$$

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S=\frac{1}{n}\boldsymbol{F}\cdot\vec{\boldsymbol{j}}} \qquad (3)$$

Eq. (3) is the MHD-Euler equation in *canonical form*

Example of application : Kelvin's theorem

$\mathcal{C}(\tau)$: closed contour dragged along by the fluid (proper time τ)

Fluid circulation around $\mathcal{C}(\tau)$: $C(\tau) := \oint_{\mathcal{C}(\tau)} h\underline{\boldsymbol{u}}$

Variation of the circulation as the contour is dragged by the fluid:

$$\frac{dC}{d\tau} = \frac{d}{d\tau}\oint_{\mathcal{C}(\tau)} h\underline{\boldsymbol{u}} = \oint_{\mathcal{C}(\tau)} \mathcal{L}_{\vec{\boldsymbol{u}}}\,(h\underline{\boldsymbol{u}}) = \oint_{\mathcal{C}(\tau)} \vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}}) + \oint_{\mathcal{C}(\tau)} \mathbf{d}(h\,\underbrace{\underline{\boldsymbol{u}}\cdot\vec{\boldsymbol{u}}}_{-1})$$

where the last equality follows from Cartan's identity

Now, since $\mathcal{C}(\tau)$ is closed, $\oint_{\mathcal{C}(\tau)} \mathbf{d}h = 0$

Using the MHD-Euler equation (3), we thus get

$$\boxed{\frac{dC}{d\tau} = \oint_{\mathcal{C}(\tau)} \left(T\mathbf{d}S + \frac{1}{n}\boldsymbol{F}\cdot\vec{\boldsymbol{j}}\right)}$$

If $\boldsymbol{F}\cdot\vec{\boldsymbol{j}} = 0$ (force-free MHD) and $T = \mathrm{const}$ or $S = \mathrm{const}$ on $\mathcal{C}(\tau)$, then C is conserved (Kelvin's theorem)

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Stationary and axisymmetric spacetimes

Assume that $(\mathscr{M}, \boldsymbol{g})$ is endowed with two symmetries:

1. **stationarity** : $\exists$ a group action of $(\mathbb{R}, +)$ on $\mathscr{M}$ such that
   - the orbits are timelike curves
   - $\boldsymbol{g}$ is invariant under the $(\mathbb{R}, +)$ action : if $\vec{\xi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\xi}}\, \boldsymbol{g} = 0} \quad (4)$$

2. **axisymmetry** : $\exists$ a group action of $\mathrm{SO}(2)$ on $\mathscr{M}$ such that
   - the set of fixed points is a 2-dimensional submanifold $\Delta \subset \mathscr{M}$ (called the *rotation axis*)
   - $\boldsymbol{g}$ is invariant under the $\mathrm{SO}(2)$ action : if $\vec{\chi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\chi}}\, \boldsymbol{g} = 0} \quad (5)$$

(4) and (5) are equivalent to *Killing equations*:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0} \quad \text{and} \quad \boxed{\nabla_\alpha \chi_\beta + \nabla_\beta \chi_\alpha = 0}$$

Stationary and axisymmetric spacetimes

No generality is lost by considering that the stationary and axisymmetric actions commute [Carter 1970] :
$(\mathscr{M}, \boldsymbol{g})$ is invariant under the action of the Abelian group $(\mathbb{R}, +) \times \mathrm{SO}(2)$, and not only under the actions of $(\mathbb{R}, +)$ and $\mathrm{SO}(2)$ separately. It is equivalent to say that the Killing vectors commute:

$$\boxed{[\vec{\xi}, \vec{\chi}] = 0}$$

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$$[\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}] = 0$$

$\Longrightarrow \exists$ coordinates $(x^\alpha) = (t, x^1, x^2, \varphi)$ on $\mathscr{M}$ such that $\vec{\boldsymbol{\xi}} = \dfrac{\partial}{\partial t}$ and $\vec{\boldsymbol{\chi}} = \dfrac{\partial}{\partial \varphi}$

Within them, $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)$

Adapted coordinates are not unique:
$$\begin{cases} t' &= t + F_0(x^1, x^2) \\ x'^1 &= F_1(x^1, x^2) \\ x'^2 &= F_2(x^1, x^2) \\ \varphi' &= \varphi + F_3(x^1, x^2) \end{cases}$$

Stationary and axisymmetric electromagnetic field

Assume that the electromagnetic field is both stationary and axisymmetric:

$$\boxed{\mathcal{L}_{\vec{\xi}}\,\boldsymbol{F} = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\chi}}\,\boldsymbol{F} = 0} \qquad (6)$$

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Cartan's identity and Maxwell eq. $\Longrightarrow \mathcal{L}_{\vec{\xi}} \boldsymbol{F} = \vec{\xi} \cdot \underbrace{\mathbf{d}\boldsymbol{F}}_{0} + \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F})$

Hence (6) is equivalent to

$$\mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = 0 \quad \text{and} \quad \mathbf{d}(\vec{\chi} \cdot \boldsymbol{F}) = 0$$

Poincaré lemma $\Longrightarrow \exists$ locally two scalar fields Φ and Ψ such that

$$\boxed{\vec{\xi} \cdot \boldsymbol{F} = -\mathbf{d}\Phi} \quad \text{and} \quad \boxed{\vec{\chi} \cdot \boldsymbol{F} = -\mathbf{d}\Psi}$$

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Link with the 4-potential $\boldsymbol{A}$: one may use the gauge freedom on $\boldsymbol{A}$ to set

$$\Phi = \boldsymbol{A}\cdot\vec{\xi} = A_t \quad \text{and} \quad \Psi = \boldsymbol{A}\cdot\vec{\chi} = A_\varphi$$

Symmetries of the scalar potentials

From the definitions of Φ and Ψ:

- $\mathcal{L}_{\vec{\xi}}\Phi = \vec{\xi}\cdot \mathrm{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\xi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\Psi = \vec{\chi}\cdot \mathrm{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\chi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\Phi = \vec{\chi}\cdot \mathrm{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\chi})$
- $\mathcal{L}_{\vec{\xi}}\Psi = \vec{\xi}\cdot \mathrm{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\xi}) = \boldsymbol{F}(\vec{\xi},\vec{\chi})$

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We have $\mathbf{d}[\boldsymbol{F}(\vec{\xi},\vec{\chi})] = \mathbf{d}[\vec{\xi}\cdot\mathbf{d}\Psi] = \mathcal{L}_{\vec{\xi}}\mathbf{d}\Psi - \vec{\xi}\cdot\underbrace{\mathbf{dd}\Psi}_{0} = \mathcal{L}_{\vec{\xi}}(\boldsymbol{F}\cdot\vec{\chi}) = 0$

Hence $\boldsymbol{F}(\vec{\xi},\vec{\chi}) = \mathrm{const}$

Assuming that $\boldsymbol{F}$ vanishes somewhere in $\mathscr{M}$ (for instance at spatial infinity), we conclude that

$$\boxed{\boldsymbol{F}(\vec{\xi},\vec{\chi}) = 0}$$

Then $\boxed{\mathcal{L}_{\vec{\xi}}\Phi = \mathcal{L}_{\vec{\chi}}\,\Phi = 0}$ and $\boxed{\mathcal{L}_{\vec{\xi}}\Psi = \mathcal{L}_{\vec{\chi}}\,\Psi = 0}$

i.e. the scalar potentials Φ and Ψ obey to the two spacetime symmetries

Most general stationary-axisymmetric electromagnetic field

$$\boldsymbol{F} = \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* + \mathbf{d}\Psi \wedge \boldsymbol{\chi}^* + \frac{I}{\sigma}\,\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .) \tag{7}$$

$$\star\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\nabla}\Phi, \vec{\xi^*}, ., .) + \boldsymbol{\epsilon}(\vec{\nabla}\Psi, \vec{\chi^*}, ., .) - \frac{I}{\sigma}\,\underline{\xi} \wedge \underline{\chi} \tag{8}$$

with

- $\boldsymbol{\xi}^* := \frac{1}{\sigma}\left(-X\,\underline{\xi} + W\underline{\chi}\right), \qquad \boldsymbol{\chi}^* := \frac{1}{\sigma}\left(W\,\underline{\xi} + V\underline{\chi}\right)$
- $V := -\underline{\xi}\cdot\vec{\xi}$, $\quad W := \underline{\xi}\cdot\vec{\chi}$, $\quad X := \underline{\chi}\cdot\vec{\chi}$, $\quad \sigma := VX + W^2$
  [Carter (1973) notations]
- $I := \star\boldsymbol{F}(\vec{\xi}, \vec{\chi})$ ← the only non-trivial scalar, apart from $\boldsymbol{F}(\vec{\xi}, \vec{\chi})$, one can form from $\boldsymbol{F}$, $\vec{\xi}$ and $\vec{\chi}$

$(\boldsymbol{\xi}^*, \boldsymbol{\chi}^*)$ is the dual basis of $(\vec{\xi}, \vec{\chi})$ in the 2-plane $\Pi := \mathrm{Vect}(\vec{\xi}, \vec{\chi})$:

$$\boldsymbol{\xi}^*\cdot\vec{\xi} = 1, \qquad \boldsymbol{\xi}^*\cdot\vec{\chi} = 0, \qquad \boldsymbol{\chi}^*\cdot\vec{\xi} = 0, \qquad \boldsymbol{\chi}^*\cdot\vec{\chi} = 1$$

$$\forall \vec{v} \in \Pi^\perp, \quad \boldsymbol{\xi}^*\cdot\vec{v} = 0 \quad \text{and} \quad \boldsymbol{\chi}^*\cdot\vec{v} = 0$$

Most general stationary-axisymmetric electromagnetic field

The proof

Consider the 2-form $\boldsymbol{H} := \boldsymbol{F} - \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* - \mathbf{d}\Psi \wedge \boldsymbol{\chi}^*$

It satisfies

$$\boldsymbol{H}(\vec{\xi},.) = \underbrace{\boldsymbol{F}(\vec{\xi},.)}_{-\mathbf{d}\Phi} - \underbrace{(\vec{\xi}\cdot\mathbf{d}\Phi)}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\xi}^*\cdot\vec{\xi})}_{1}\mathbf{d}\Phi - \underbrace{(\vec{\xi}\cdot\mathbf{d}\Psi)}_{0}\boldsymbol{\chi}^* + \underbrace{(\boldsymbol{\chi}^*\cdot\vec{\xi})}_{0}\mathbf{d}\Psi = 0$$

Similarly $\boldsymbol{H}(\vec{\chi},.) = 0$. Hence $\boldsymbol{H}|_{\Pi} = 0$

On $\Pi^\perp$, $\boldsymbol{H}|_{\Pi^\perp}$ is a 2-form. Another 2-form on $\Pi^\perp$ is $\boldsymbol{\epsilon}(\vec{\xi},\vec{\chi},.,.)\big|_{\Pi^\perp}$

Since $\dim \Pi^\perp = 2$ and $\boldsymbol{\epsilon}(\vec{\xi},\vec{\chi},.,.)\big|_{\Pi^\perp} \neq 0$, $\exists$ a scalar field I such that

$\boldsymbol{H}|_{\Pi^\perp} = \dfrac{I}{\sigma}\,\boldsymbol{\epsilon}(\vec{\xi},\vec{\chi},.,.)\big|_{\Pi^\perp}$. Because both $\boldsymbol{H}$ and $\boldsymbol{\epsilon}(\vec{\xi},\vec{\chi},.,.)$ vanish on Π, we can extend the equality to all space:

$$\boldsymbol{H} = \frac{I}{\sigma}\boldsymbol{\epsilon}(\vec{\xi},\vec{\chi},.,.)$$

Thus $\boldsymbol{F}$ has the form (7). Taking the Hodge dual gives the form (8) for $\star\boldsymbol{F}$, on which we readily check that $I = \star\boldsymbol{F}(\vec{\xi},\vec{\chi})$, thereby completing the proof.

Example: Kerr-Newman electromagnetic field

Using Boyer-Lindquist coordinates (t, r, θ, φ), the electromagnetic field of the Kerr-Newman solution (charged rotating black hole) is

$$\boldsymbol{F} = \frac{\mu_0 Q}{4\pi(r^2 + a^2\cos^2\theta)^2}\Big\{ \big[(r^2 - a^2\cos^2\theta)\,\mathbf{d}r - a^2 r\sin 2\theta\,\mathbf{d}\theta\big]\wedge\mathbf{d}t + \big[a(a^2\cos^2\theta - r^2)\sin^2\theta\,\mathbf{d}r + ar(r^2+a^2)\sin 2\theta\,\mathbf{d}\theta\big]\wedge\mathbf{d}\varphi\Big\}$$

Q: total electric charge, $a := J/M$: reduced angular momentum

For Kerr-Newman, $\boldsymbol{\xi}^* = \mathbf{d}t$ and $\boldsymbol{\chi}^* = \mathbf{d}\varphi$; comparison with (7) leads to

$$\Phi = -\frac{\mu_0 Q}{4\pi}\,\frac{r}{r^2 + a^2\cos^2\theta}, \quad \Psi = \frac{\mu_0 Q}{4\pi}\,\frac{ar\sin^2\theta}{r^2 + a^2\cos^2\theta}, \quad I = 0$$

Non-rotating limit $(a = 0)$: Reissner-Nordström solution: $\Phi = -\frac{\mu_0}{4\pi}\frac{Q}{r}$, $\Psi = 0$

Maxwell equations

First Maxwell equation: $\mathbf{d}\boldsymbol{F} = 0$

It is automatically satisfied by the form (7) of $\boldsymbol{F}$

Second Maxwell equation: $\mathbf{d} \star\boldsymbol{F} = \mu_0 \star\underline{\boldsymbol{j}}$

It gives the electric 4-current:

$$\mu_0\,\vec{\boldsymbol{j}} = a\,\vec{\boldsymbol{\xi}} + b\,\vec{\boldsymbol{\chi}} - \frac{1}{\sigma}\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, .) \qquad (9)$$

with

- $a := \nabla_\mu \left(\frac{X}{\sigma} \nabla^\mu \Phi - \frac{W}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[-X \mathscr{C}_{\boldsymbol{\xi}} + W \mathscr{C}_{\boldsymbol{\chi}} \right]$
- $b := -\nabla_\mu \left(\frac{W}{\sigma} \nabla^\mu \Phi + \frac{V}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[W \mathscr{C}_{\boldsymbol{\xi}} + V \mathscr{C}_{\boldsymbol{\chi}} \right]$
- $\mathscr{C}_{\boldsymbol{\xi}} := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\xi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \xi_\sigma$ (circularity factor)
- $\mathscr{C}_{\boldsymbol{\chi}} := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\chi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \chi_\sigma$ (circularity factor)

Remark: $\vec{\boldsymbol{j}}$ has no meridional component (i.e. $\vec{\boldsymbol{j}} \in \Pi$) $\iff$ $\mathbf{d}I = 0$

Simplification for circular spacetimes

Spacetime $(\mathscr{M}, \boldsymbol{g})$ is circular $\iff$ the planes $\Pi^{\perp}$ are integrable in 2-surfaces
$\iff$ $\mathscr{C}_{\boldsymbol{\xi}} = \mathscr{C}_{\boldsymbol{\chi}} = 0$

Generalized Papapetrou theorem [Papapetrou 1966] [Kundt & Trümper 1966] [Carter 1969] : a stationary and axisymmetric spacetime ruled by the Einstein equation is circular iff the total energy-momentum tensor $\boldsymbol{T}$ obeys to

$$\xi^{\mu} T_{\mu}^{\ [\alpha} \xi^{\beta} \chi^{\gamma]} = 0$$
$$\chi^{\mu} T_{\mu}^{\ [\alpha} \xi^{\beta} \chi^{\gamma]} = 0$$

Examples:

- circular spacetimes: Kerr-Newman, rotating star, magnetized rotating star with either purely poloidal magnetic field or purely toroidal magnetic field
- non-circular spacetimes: rotating star with meridional flow, magnetized rotating star with mixed magnetic field

In what follows, we do **not** assume that $(\mathscr{M}, \boldsymbol{g})$ is circular

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4 Stationary and axisymmetric MHD

5 Grad-Shafranov and transfield equations

Perfect conductor hypothesis (1/2)

$$\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0$$

with the fluid 4-velocity decomposed as

$$\vec{\boldsymbol{u}} = \lambda(\vec{\boldsymbol{\xi}} + \Omega\vec{\boldsymbol{\chi}}) + \vec{\boldsymbol{w}}, \qquad \vec{\boldsymbol{w}} \in \Pi^{\perp} \tag{10}$$

Ω is the rotational angular velocity and $\vec{\boldsymbol{w}}$ is the meridional velocity

$$\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1 \iff \lambda = \sqrt{\frac{1 + \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}}{V - 2\Omega W - \Omega^2 X}}$$

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We have

$$\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = 0 \quad \text{and} \quad \mathcal{L}_{\vec{\boldsymbol{u}}}\, \Psi = 0, \tag{11}$$

i.e. the scalar potentials Φ and Ψ are constant along the fluid lines.

Proof: $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = \vec{\boldsymbol{u}} \cdot \mathbf{d}\Phi = -\boldsymbol{F}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{u}}) = 0$ by the perfect conductor property.

Corollary: since we had already $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Phi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Phi = 0$ and $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Psi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Psi = 0$, it follows from (11) that

$$\vec{\boldsymbol{w}} \cdot \mathbf{d}\Phi = 0 \quad \text{and} \quad \vec{\boldsymbol{w}} \cdot \mathbf{d}\Psi = 0$$

Perfect conductor hypothesis (2/2)

Expressing the condition $\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0$ with the general form (7) of a stationary-axisymmetric electromagnetic field yields

$$\underbrace{(\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda}\mathbf{d}\Phi - \underbrace{(\mathbf{d}\Phi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda\Omega}\mathbf{d}\Psi - \underbrace{(\mathbf{d}\Psi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\chi}^* + \frac{I}{\sigma}\underbrace{\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, ., \vec{\boldsymbol{u}})}_{-\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} = 0$$

Hence

$$\boxed{\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma\lambda}\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} \qquad (12)$$

Conservation of baryon number and stream function

Baryon number conservation : $\boxed{\boldsymbol{\nabla} \cdot (n\vec{\boldsymbol{u}}) = 0} \iff \boxed{\mathbf{d}(n \star \underline{\boldsymbol{w}}) = 0}$

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$\rightarrow$Poincaré Lemma: $\exists$ a 2-form $\boldsymbol{H}$ such that $n \star \underline{\boldsymbol{w}} = \mathbf{d}\boldsymbol{H}$

Considering the scalar field $f := \boldsymbol{H}(\vec{\xi}, \vec{\chi})$, we get

$$\boxed{\mathbf{d}f = n\,\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{w}, .)} \iff \boxed{\vec{w} = -\frac{1}{\sigma n}\,\vec{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, .)} \qquad (13)$$

f is called the **(Stokes) stream function**

Conservation of baryon number and stream function

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It follows from (13) that

- $\vec{\xi}\cdot\mathbf{d}f=0$ and $\vec{\chi}\cdot\mathbf{d}f=0 \Longrightarrow f$ obeys to the spacetime symmetries
- $\vec{\boldsymbol{u}}\cdot\mathbf{d}f=0 \Longrightarrow f$ is constant along any fluid line

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- $\vec{\boldsymbol{u}}\cdot\mathbf{d}f=0 \implies$ f is constant along any fluid line

The perfect conductivity relation (12) is writable as

$$\boxed{\mathbf{d}\Phi=-\Omega\,\mathbf{d}\Psi+\frac{I}{\sigma n\lambda}\,\mathbf{d}f} \qquad (14)$$

Conserved quantities along the fluid lines

- If the fluid motion has no meridional component ($\vec{w} = 0$), then $\vec{u} = \lambda(\vec{\xi} + \Omega\vec{\chi})$ and any scalar quantity that obeys to the spacetime symmetries is conserved along the fluid lines
- To be non-trivial, we therefore assume in the following $\vec{w} \neq 0$ or equivalently $\boxed{\mathbf{d}f \neq 0}$

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Thanks to (13) the condition $\vec{w} \cdot \mathbf{d}\Psi = 0$ is equivalent to $\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, \vec{\nabla}\Psi) = 0$. Since $\mathbf{d}f \neq 0$, this implies the existence of a scalar field C such that

$$\boxed{\mathbf{d}\Psi = C\,\mathbf{d}f}$$

$$\Longrightarrow \mathbf{dd}\Psi = 0 = \mathbf{d}C \wedge \mathbf{d}f \Longrightarrow \boxed{C = C(f)}$$

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Similarly, $\exists$ scalar field $\boxed{D = D(f)}$ such that $\boxed{\mathbf{d}\Phi = D\,\mathbf{d}f}$

$C = C(f)$ and $D = D(f) \Longrightarrow$ C and D are constant along any fluid line

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Relation (14) yields to $C\Omega + D = \dfrac{I}{\sigma n \lambda}$

Conserved quantities of Bernoulli type

Applying the 1-form MHD-Euler equation (3) to the vector $\vec{\xi}$ leads to

$$\vec{u} \cdot \mathbf{d}(h\underline{u}) \cdot \vec{\xi} = \boldsymbol{F}(\vec{\xi}, \vec{j})/n$$

Now

- Cartan id. $\Longrightarrow \mathbf{d}(h\underline{u}) \cdot \vec{\xi} = -\vec{\xi} \cdot \mathbf{d}(h\underline{u}) = -\underbrace{\mathcal{L}_{\vec{\xi}}(h\underline{u})}_{0} + \mathbf{d}(h\underline{u} \cdot \vec{\xi}) = \mathbf{d}(h\underline{u} \cdot \vec{\xi})$
- Φ definition $\Longrightarrow \boldsymbol{F}(\vec{\xi}, \vec{j}) = -\vec{j} \cdot \mathbf{d}\Phi = (\mu_0 \sigma)^{-1} \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla}\Phi)$ by (9)
  $\vec{\nabla}\Phi = D\vec{\nabla} f \Longrightarrow \boldsymbol{F}(\vec{\xi}, \vec{j}) = \frac{D}{\mu_0 \sigma} \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla} f)$
  Eq. (13) $\Longrightarrow \boldsymbol{F}(\vec{\xi}, \vec{j}) = (Dn/\mu_0)\, \vec{w} \cdot \mathbf{d}I = (Dn/\mu_0)\, \vec{u} \cdot \mathbf{d}I = (Dn/\mu_0) \mathcal{L}_{\vec{u}}\, I$

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$$\vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) \cdot \vec{\xi} = \boldsymbol{F}(\vec{\xi}, \vec{j})/n$$

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- Cartan id. $\Longrightarrow$ $\mathbf{d}(h\underline{\boldsymbol{u}}) \cdot \vec{\xi} = -\vec{\xi} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) = -\underbrace{\mathcal{L}_{\vec{\xi}}\,(h\underline{\boldsymbol{u}})}_{0} + \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\xi}) = \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\xi})$
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  Eq. (13) $\Longrightarrow$ $\boldsymbol{F}(\vec{\xi}, \vec{j}) = (Dn/\mu_0)\, \vec{\boldsymbol{w}} \cdot \mathbf{d}I = (Dn/\mu_0)\, \vec{\boldsymbol{u}} \cdot \mathbf{d}I = (Dn/\mu_0) \mathcal{L}_{\vec{\boldsymbol{u}}}\, I$

Hence

$$\underbrace{\vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\xi})}_{\mathcal{L}_{\vec{\boldsymbol{u}}}\,(h\underline{\boldsymbol{u}} \cdot \vec{\xi})} = \frac{D}{\mu_0} \mathcal{L}_{\vec{\boldsymbol{u}}}\, I$$

Since $\mathcal{L}_{\vec{\boldsymbol{u}}}\, D = 0$, we get $\boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\, E = 0}$ $\longleftarrow$ E is constant along any fluid line

$$\boxed{E := -h\underline{\boldsymbol{u}} \cdot \vec{\xi} + \frac{DI}{\mu_0} = \lambda h(V - W\Omega) + \frac{DI}{\mu_0}} \qquad (15)$$

Conserved quantities of Bernoulli type

Similarly, applying the 1-form MHD-Euler equation (3) to the vector $\vec{\chi}$, instead of $\vec{\xi}$, leads to

$$\boxed{\mathcal{L}_{\vec{u}}\, L = 0}$$

with

$$\boxed{L := h\underline{\boldsymbol{u}} \cdot \vec{\chi} - \frac{CI}{\mu_0} = \lambda h(W + X\Omega) - \frac{CI}{\mu_0}} \qquad (16)$$

Non-relativistic limits

At the Newtonian limit and in standard isotropic spherical coordinates (t, r, θ, φ),

$$\begin{cases} V = 1 + 2\Phi_{\rm grav}, \qquad W = 0 \\ X = (1 - 2\Phi_{\rm grav}) r^2 \sin^2\theta \\ \sigma = r^2 \sin^2\theta, \end{cases}$$

where $\Phi_{\rm grav}$ is the Newtonian gravitational potential ($|\Phi_{\rm grav}| \ll 1$)

Moreover, introducing the *mass density* $\rho := m_{\rm b}\, n$ ($m_{\rm b}$ mean baryon mass) and *specific enthalpy* $H := \dfrac{\varepsilon_{\rm int} + p}{\rho}$, we get $h = m_{\rm b}(1 + H)$ with $H \ll 1$

Then

$$\boxed{\frac{E}{m_b} - 1 = H + \Phi_{\rm grav} + \frac{v^2}{2} + \frac{DI}{\mu_0 m_{\rm b}}}$$

(when $I = 0$, classical Bernoulli theorem)

$$\boxed{\frac{L}{m_b} = \Omega\, r^2 \sin^2\theta - \frac{CI}{\mu_0 m_{\rm b}}}$$

Conserved quantities: summary

- **For purely rotational fluid motion** $(\mathrm{d}f = 0)$: any scalar quantity which obeys to the spacetime symmetries is conserved along the fluid lines
- **For a fluid motion with meridional components** $(\mathrm{d}f \neq 0)$: there exist five scalar quantities which are constant along any given fluid line:

$$C, \quad D, \quad E, \quad L, \quad S$$

(S being the entropy per baryon, cf. Eq. (2))

If there is no electromagnetic field, $C = 0$, $D = 0$, $E = -h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}$ and the constancy of E along the fluid lines is the relativistic Bernoulli theorem [Synge 1937], [Lichnerowicz 1940]

Comparison with previous work

Bekenstein & Oron (1978)

The constancy of C, $\omega := -D/C$, E and L along the fluid lines has been shown first by Bekenstein & Oron (1978)

Bekenstein & Oron have provided coordinate-dependent definitions of ω and C, namely

$$\omega := -\frac{F_{01}}{F_{31}} \qquad \text{and} \qquad C := \frac{F_{31}}{\sqrt{-g}nu^2}$$

Besides, they have obtained expressions for E and L slightly more complicated than (15) and (16), namely

$$E = -\left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}} - \frac{C}{\mu_0}\left[\underline{\boldsymbol{u}} \cdot (\vec{\boldsymbol{\xi}} + \omega\vec{\boldsymbol{\chi}})\right] (\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{\xi}})$$

$$L = \left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}} + \frac{C}{\mu_0}\left[\underline{\boldsymbol{u}} \cdot (\vec{\boldsymbol{\xi}} + \omega\vec{\boldsymbol{\chi}})\right] (\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{\chi}})$$

It can be shown that these expressions are equivalent to (15) and (16)

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Integrating the MHD-Euler equation

With the writing (10) of $\vec{\boldsymbol{u}}$, (7) of $\boldsymbol{F}$ and (9) of $\vec{\boldsymbol{j}}$, the MHD-Euler equation

$$\vec{\boldsymbol{u}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}}) - T\mathbf{d}S = \frac{1}{n}\boldsymbol{F} \cdot \vec{\boldsymbol{j}}$$

can be recast as

$$\left[\vec{\boldsymbol{w}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}) - \frac{1}{\mu_0 \sigma n}\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, \vec{\boldsymbol{\nabla}}\Phi)\right] \boldsymbol{\xi}^*$$
$$+ \left[\vec{\boldsymbol{w}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}}) - \frac{1}{\mu_0 \sigma n}\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, \vec{\boldsymbol{\nabla}}\Psi)\right] \boldsymbol{\chi}^*$$
$$-\lambda\, \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}) - \lambda\Omega\, \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}}) + \frac{I}{\mu_0 \sigma n}\mathbf{d}I + \frac{1}{n}\left[q + \frac{\lambda h}{\sigma}(\mathscr{C}_{\boldsymbol{\xi}} + \Omega \mathscr{C}_{\boldsymbol{\chi}})\right] \mathbf{d}f$$
$$-\frac{\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{j}}}{n}\mathbf{d}\Phi - \frac{\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{j}}}{n}\mathbf{d}\Psi - T\,\mathbf{d}S = 0$$

with $q := -\nabla_\mu \left(\frac{h}{\sigma\, n}\nabla^\mu f\right)$

Integrating the MHD-Euler equation

The MHD-Euler equation is equivalent to the system

$$\vec{w}\cdot\mathbf{d}(h\underline{u}\cdot\vec{\xi}) - \frac{1}{\mu_0\sigma n}\epsilon(\vec{\xi},\vec{\chi},\vec{\nabla}I,\vec{\nabla}\Phi) = 0 \quad (17)$$

$$\vec{w}\cdot\mathbf{d}(h\underline{u}\cdot\vec{\chi}) - \frac{1}{\mu_0\sigma n}\epsilon(\vec{\xi},\vec{\chi},\vec{\nabla}I,\vec{\nabla}\Psi) = 0 \quad (18)$$

$$\lambda\,\mathbf{d}(h\underline{u}\cdot\vec{\xi}) + \lambda\Omega\,\mathbf{d}(h\underline{u}\cdot\vec{\chi}) - \frac{1}{n}\left[q + \frac{\lambda h}{\sigma}\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\mathscr{C}_{\boldsymbol{\chi}}\right)\right]\mathbf{d}f - \frac{I}{\mu_0\sigma n}\mathbf{d}I$$
$$+\frac{\boldsymbol{\xi}^*\cdot\vec{j}}{n}\mathbf{d}\Phi + \frac{\boldsymbol{\chi}^*\cdot\vec{j}}{n}\mathbf{d}\Psi + T\,\mathbf{d}S = 0. \quad (19)$$

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$$\vec{\boldsymbol{w}} \cdot \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}) - \frac{1}{\mu_0 \sigma n} \boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, \vec{\boldsymbol{\nabla}} \Phi) = 0 \tag{17}$$

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$$\lambda\, \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\xi}}) + \lambda\Omega\, \mathbf{d}(h\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{\chi}}) - \frac{1}{n}\left[q + \frac{\lambda h}{\sigma}\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega \mathscr{C}_{\boldsymbol{\chi}}\right)\right] \mathbf{d}f - \frac{I}{\mu_0 \sigma n}\mathbf{d}I$$
$$+ \frac{\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{j}}}{n}\mathbf{d}\Phi + \frac{\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{j}}}{n}\mathbf{d}\Psi + T\,\mathbf{d}S = 0. \tag{19}$$

To go further, one shall distinguish two case depending whether the fluid motion has some meridional component ($\mathbf{d}f \neq 0$), or not, ($\mathbf{d}f = 0$)

Pure rotational flow

Assumption: $\vec{w} = 0$, i.e. $\mathbf{d}f = 0$

Then the perfect conductor relation (14) reduces to $\boxed{\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi}$ and Eqs. (17)-(18) are equivalent to

$$\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla}\Psi) = 0 \qquad (20)$$

Two cases must be considered:

- generic case : $\mathbf{d}\Psi \neq 0$
- toroidal magnetic field: $\mathbf{d}\Psi = 0$

Pure rotational flow: generic case (1/2)

If $\mathbf{d}\Psi \neq 0$, the relation $\mathbf{d}\Phi = -\Omega\, \mathbf{d}\Psi$ implies

$$\boxed{\Omega = \Omega(\Psi)}$$ (relativistic generalization of Ferraro's law of isorotation)

Moreover Eq. (20) implies $\mathbf{d}I \propto \mathbf{d}\Psi$, resulting in

$$\boxed{I = I(\Psi)}$$

Assuming $S = S(\Psi)$, Eq. (19) admit the first integral

$$\boxed{\ln h + \frac{1}{2} \ln\left(V - 2W\Omega - X\Omega^2\right) + \int_0^{\Psi} G(\psi)\, d\psi = \mathrm{const}}$$

where $G = G(\Psi)$ obeys to...

Pure rotational flow: generic case (2/2)

$$\begin{aligned}&(V-2W\Omega-X\Omega^2)\Delta^*\Psi+\mathbf{d}(V-2W\Omega-X\Omega^2)\cdot\vec{\boldsymbol{\nabla}}\Psi\\&\quad+(W+X\Omega)\Omega'\,\mathbf{d}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi+I\left[I'-\frac{W+X\Omega}{\sigma}\mathscr{C}_{\boldsymbol{\xi}}-\frac{V-W\Omega}{\sigma}\mathscr{C}_{\boldsymbol{\chi}}\right]\\&\quad+\mu_0\sigma\left\{(\varepsilon+p)\left[\frac{(X\Omega+W)\Omega'}{V-2W\Omega-X\Omega^2}-G\right]-nTS'\right\}=0\end{aligned} \tag{21}$$

with $\boxed{\Delta^*\Psi:=\sigma\,\nabla_\mu\left(\frac{1}{\sigma}\nabla^\mu\Psi\right)}$

Eq. (21) is the relativistic generalization of the Grad-Shafranov equation, for the most general (i.e. noncircular) stationary and axisymmetric spacetimes

Pure rotational flow: generic case (2/2)

$$\begin{aligned}(V - 2W\Omega - X\Omega^2)\Delta^*\Psi + \mathbf{d}(V - 2W\Omega - X\Omega^2)\cdot\vec{\boldsymbol{\nabla}}\Psi \\ +(W + X\Omega)\Omega'\,\mathbf{d}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi + I\left[I' - \frac{W+X\Omega}{\sigma}\mathscr{C}_{\boldsymbol{\xi}} - \frac{V-W\Omega}{\sigma}\mathscr{C}_{\boldsymbol{\chi}}\right] \\ +\mu_0\sigma\left\{(\varepsilon + p)\left[\frac{(X\Omega+W)\Omega'}{V-2W\Omega-X\Omega^2} - G\right] - nTS'\right\} = 0\end{aligned} \quad (21)$$

with $\Delta^*\Psi := \sigma\,\nabla_\mu\left(\frac{1}{\sigma}\nabla^\mu\Psi\right)$

Eq. (21) is the relativistic generalization of the Grad-Shafranov equation, for the most general (i.e. noncircular) stationary and axisymmetric spacetimes

Newtonian limit:

$$\Delta^*\Psi + II' + \mu_0 r^2\sin^2\theta\left[\rho\left(\Omega\Omega' r^2\sin^2\theta - G\right) - nTS'\right] = 0$$

- $\Omega = 0$: Grad and Rubin (1958), Shafranov (1958)
- $\Omega \neq 0$: Chandrasekhar (1956)

Pure rotational flow: toroidal magnetic field (1/2)

Assumption: $\boxed{\mathbf{d}\Psi = 0}$

$\Longrightarrow$ $\boxed{\mathbf{d}\Phi = 0}$ [since $\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi$]

The electromagnetic field depends then only on I: $\boxed{\boldsymbol{F} = \frac{I}{\sigma}\,\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, ., .)}$

and $\vec{\boldsymbol{b}} \in \Pi$ (toroidal magnetic field)

Pure rotational flow: toroidal magnetic field (1/2)

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The first two equations of the MHD-Euler system (18)-(19) are automatically satisfied, while the third one reduces to

$$\mathbf{d}\ln\left(h\sqrt{V - 2W\Omega - X\Omega^2}\right) + \frac{X\Omega + W}{V - 2W\Omega - X\Omega^2}\,\mathbf{d}\Omega + \frac{I}{\mu_0\sigma(\varepsilon + p)}\,\mathbf{d}I - \frac{T}{h}\,\mathbf{d}S = 0$$

Sufficient conditions for the integrability are then $\boxed{\Omega = \Omega(I)}$ and $\boxed{S = S(I)}$

Pure rotational flow: toroidal magnetic field (2/2)

Assuming $\Omega = \Omega(I)$ and $S = S(I)$, the first integral is then

$$\ln h + \frac{1}{2} \ln \left(V - 2W\Omega - X\Omega^2 \right) + \int_0^I Q(i)\, di = \mathrm{const}$$

where $Q = Q(I)$ such that

$$\frac{I}{\mu_0 \sigma (\varepsilon + p)} - Q(I) + \frac{(X\Omega + W)\Omega'}{V - 2W\Omega - X\Omega^2} - \frac{TS'}{h} = 0$$

In the special case $\Omega = \mathrm{const}$ and $S = \mathrm{const}$, we recover results by Kiuchi & Yoshida (2008)

Newtonian limit: Miketinac (1973)

Flow with meridional component

Assumption: $\vec{\boldsymbol{w}} \neq 0$, i.e. $\mathbf{d}f \neq 0$

As shown above, we may then write $\boxed{\mathbf{d}\Phi = D(f)\,\mathbf{d}f}$ and $\boxed{\mathbf{d}\Psi = C(f)\,\mathbf{d}f}$

Moreover Eq. (2) implies $\boxed{S = S(f)}$

The MHD-Euler system (17)-(19) is then equivalent to

$$\vec{\boldsymbol{w}} \cdot \mathbf{d}E = 0 \tag{22}$$

$$\vec{\boldsymbol{w}} \cdot \mathbf{d}L = 0 \tag{23}$$

$$\begin{aligned}&\left\{ n\lambda \left[\Omega L' - E' + \frac{I}{\mu_0}(D' + \Omega C') \right] + D\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{j}} + C\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{j}} - q \right. \\ &\qquad \left. - \frac{\lambda h}{\sigma} \left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\, \mathscr{C}_{\boldsymbol{\chi}} \right) + nTS' \right\} \mathbf{d}f = 0\end{aligned} \tag{24}$$

We recognize in (22)-(23) the Bernoulli-type conservation laws established earlier [cf. Eqs. (15)-(16)]

Eq. (24) is equivalent to the vanishing of the factor in front of $\mathbf{d}f$

Transfield equation

Substituting expression (9) for $\vec{j}$ in Eq. (24) we obtain the transfield equation:

$$\begin{aligned} &\left(1-\frac{A}{M^2}\right)\Delta^* f+\frac{n}{h}\left[\mathbf{d}\left(\frac{h}{n}\right)-\frac{1}{\mu_0}\left(C^2\,\mathbf{d}V+2CD\,\mathbf{d}W-D^2\,\mathbf{d}X\right)\right]\cdot\vec{\boldsymbol{\nabla}}f \\ &-\frac{n}{\mu_0 h}\left[(VC+WD)C'+(WC-XD)D'\right]\mathbf{d}f\cdot\vec{\boldsymbol{\nabla}}f \\ &+\frac{\sigma n^2}{h}\left\{\lambda\left[\Omega L'-E'+\frac{I}{\mu_0}(D'+\Omega C')\right]+TS'\right\}-\lambda n\left(\mathscr{C}_{\boldsymbol{\xi}}+\Omega\,\mathscr{C}_{\boldsymbol{\chi}}\right) \\ &+\frac{In}{\mu_0\sigma h}\left[(WC-XD)\mathscr{C}_{\boldsymbol{\xi}}+(VC+WD)\mathscr{C}_{\boldsymbol{\chi}}\right]=0 \end{aligned} \tag{25}$$

with

- $A := V + 2W\dfrac{D}{C} - X\dfrac{D^2}{C^2}$, $\qquad \Delta^* f := \sigma\,\nabla_\mu\left(\dfrac{1}{\sigma}\nabla^\mu f\right)$
- $M^2 := \dfrac{\mu_0 h}{C^2 n}$ *(poloidal Alfvén Mach-number)*

Eq. (25) is called *transfield* for it expresses the component along $\mathbf{d}f$ of the MHD-Euler equation and $\mathbf{d}f$ is *transverse* to the magnetic field $\vec{b}$ in the fluid frame, in the sense that $\vec{b}\cdot\mathbf{d}f = 0$

Poloidal wind equation

The transfield equation has to be supplemented by the poloidal wind equation, arising from the 4-velocity normalization $\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$, with λ and Ω expressed in terms of C, D, E, L and h :

$$\begin{aligned}(A - M^2)^2 \left(\frac{h^2}{n^2} \mathbf{d}f \cdot \vec{\boldsymbol{\nabla}} f + \sigma h^2 \right) - M^4 (XE^2 + 2WEL - VL^2) \\ - \frac{\sigma}{C^2} (A - 2M^2)(CE + DL)^2 = 0\end{aligned} \quad (26)$$

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Notice that I, λ and Ω in Eq. (25) can be expressed in terms of C, D, E, L, n and h

Then

Given

- the metric (represented by V, X, W, σ and $\boldsymbol{\nabla}$),
- the EOS $h = h(n, S)$,
- the five functions $C(f)$, $D(f)$, $E(f)$, $L(f)$ and $S(f)$,

Eqs. (25)-(26) constitute a system of 2 PDEs for the 2 unknowns (f, n)

Solving it provides a complete solution of the MHD-Euler equation

Comparison with previous works

Newtonian limit:

- The transfield equation (25) reduces to the equation obtained by Solov'ev (1967)
- Pure hydrodynamical limit (vanishing electromagnetic field) :

  $$\Delta^* f - \frac{1}{n}\mathbf{d}n \cdot \vec{\nabla} f + r^2 \sin^2\theta \frac{n^2}{m_{\rm b}} \left(\Omega L' - E' + TS'\right) = 0$$

  Special case $n = \mathrm{const}$, $\Omega = 0$, $T = 0$: Stokes (1880).

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Relativistic studies:

All previous GRMHD studies derived the transfield equation for the flux function Ψ, instead of the stream function f ($\mathbf{d}f = C^{-1}\mathbf{d}\Psi$)

$\Longrightarrow$ the transfield equation can be then seen as a generalization to meridional flows of the Grad-Shafranov equation (21)

Drawback: no straightforward hydrodynamical limit

- Schwarzschild spacetime : Mobarry & Lovelace (1986)
- Kerr spacetime (circular spacetimes): Nitta, Takahashi & Tomimatsu (1991), Beskin & Par'ev (1993)
- noncircular spacetimes: Ioka & Sasaki (2003)

Comparison with Ioka & Sasaki (2003)

To deal with noncircular stationary axisymmetric spacetimes, Ioka & Sasaki used a (2+1)+1 formalism developed by Gourgoulhon & Bonazzola (1993), similar to the (2+1)+1 formalism introduced by **K. Maeda**[1], M. Sasaki, **T. Nakamura**[1] & S. Miyama (1980)

This (2+1)+1 formalism is based on a foliation by 2-surfaces ("meridional surfaces") transverse to the 2-surfaces of transitivity of the group action $\mathrm{R} \times \mathrm{SO}(2)$

In noncircular spacetimes, there is no unique choice for the meridional surfaces

To be general, the work of Ioka & Sasaki is covariant with respect to that choice (described by means of "spatial lapse" and "meridional shift" functions)

[1] Happy birthday !

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Another difference: case of pure rotational flow and toroidal magnetic field not included in Ioka & Sasaki's treatment

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Conclusions

- Ideal GRMHD is well amenable to a treatment based on exterior calculus.
- This simplifies calculations with respect to the traditional tensor calculus, notably via the massive use of Cartan's identity.
- For stationary and axisymmetric GRMHD, we have developed a systematic treatment based on such an approach. This provides some insight on previously introduced quantities and leads to the formulation of very general laws, recovering previous ones as subcases and obtaining new ones in some specific limits.

A related work

GRMHD for neutron star-neutron star and neutron star-black hole binary systems on close circular orbits (helical symmetry)

See poster no. 77 by Koji Uryu !

Bibliography

- J. D. Bekenstein & E. Oron : *New conservation laws in general-relativistic magnetohydrodynamics*, Phys. Rev. D **18**, 1809 (1978)
- B. Carter : *Perfect fluid and magnetic field conservations laws in the theory of black hole accretion rings*, in *Active Galactic Nuclei*, Eds. C. Hazard & S. Mitton, Cambridge University Press (Cambridge), p. 273 (1979)
- E. Gourgoulhon : *An introduction to the theory of rotating relativistic stars*, arXiv:1003.5015 (2010)
- E. Gourgoulhon, C. Markakis & K. Uryu : *Magnetohydrodynamics in stationary and axisymmetric spacetimes: a fully covariant approach*, in preparation
- K. Ioka & M. Sasaki : *Grad-Shafranov equation in noncircular stationary axisymmetric spacetimes*, Phys. Rev. D **67**, 12026 (2003)
- A. Lichnerowicz : *Relativistic hydrodynamics and magnetohydrodynamics*, Benjamin (New York) (1967)
- S. Nitta, M. Takahashi & A. Tomimatsu : *Effects of magnetohydrodynamics accretion flows on global structure of a Kerr black-hole magnetosphere*, Phys. Rev. D **44**, 2295 (1991)