The black hole no-hair theorem

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Outline

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The no-hair theorem: a major achievement in general relativity

Black hole uniqueness theorem ("no-hair")
(Dorochkevitch, Novikov & Zeldovitch 1965, Israel 1967, Carter 1971, Hawking 1972, Robinson 1975)

Within 4-dimensional general relativity and modulo some "reasonable" hypotheses (to be discussed below), any isolated *stationary* black hole is a Kerr-Newman black hole, which is entirely described by only three numbers:

- the mass M
- the angular momentum J
- the electric charge Q

Special cases:

- $Q = 0$: Kerr BH (1963)
- $J = 0$: Reissner-Nordström BH (1916)
- $Q = 0, \ J = 0$: Schwarzschild BH (1915)

$\Longrightarrow$ *"A black hole has no hair"* (John A. Wheeler, 1971)

The Kerr black hole

Kerr solution to the vacuum Einstein equation (1963)

Expression in Boyer-Lindquist coordinates (t, r, θ, φ):

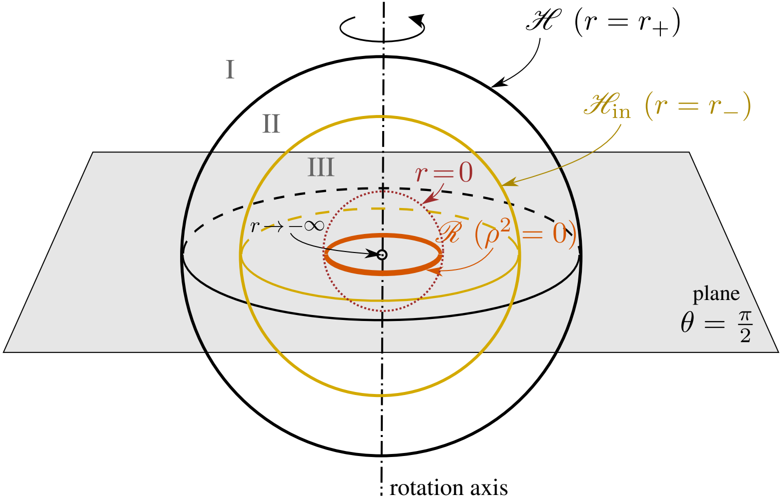
$$\begin{aligned} \mathrm{d}s^2 = & -\left(1 - \frac{2Mr}{\rho^2}\right) \mathrm{d}t^2 - \frac{4Mar\sin^2\theta}{\rho^2} \mathrm{d}t\, \mathrm{d}\varphi + \frac{\rho^2}{\Delta} \mathrm{d}r^2 + \rho^2 \mathrm{d}\theta^2 \\ & + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right) \sin^2\theta\, \mathrm{d}\varphi^2 \end{aligned}$$

where $a := J/M$, $\rho^2 := r^2 + a^2\cos^2\theta$, $\Delta := r^2 - 2Mr + a^2$

→ spacetime manifold: $\mathscr{M} = \mathbb{R}^2 \times \mathbb{S}^2 \setminus \{r = 0 \ \& \ \theta = \pi/2\}$;
NB: $r \in (-\infty, \infty)$

→ describes a rotating black hole with the event horizon $\mathscr{H}$ located at $r = r_+ := M + \sqrt{M^2 - a^2}$ (in $c = 1$ and $G = 1$ units)

The Kerr black hole



View of a section $t = \mathrm{const}$ of the $\mathbb{R}^2 \times \mathbb{S}^2$ manifold in O'Neill coord. (R, θ, φ) with $R := \mathrm{e}^r$

Physical meaning of the two parameters M and J

- Mass M: *not* a measure of the "amount of matter" inside the black hole, but rather a *characteristic of the external gravitational field*
  $\rightarrow$ measurable from the orbital period of a test particle in remote circular orbit around the black hole *(Kepler's third law)*

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- Angular momentum $J = aM$: characterizes the *gravito-magnetic* part of the gravitational field
  $\rightarrow$ measurable from the precession of a gyroscope orbiting the black hole *(Lense-Thirring effect)*

The no-hair theorem: a precise mathematical statement (for $Q = 0$)

Let $(\mathscr{M}, \boldsymbol{g})$ be a spacetime (Lorentzian manifold) that

- is 4-dimensional,
- is asymptotically flat,
- is stationary,
- obeys the vacuum Einstein equation: $\mathbf{Ric}(\boldsymbol{g}) = 0$,
- contains a black hole with a connected regular horizon that is either (i) rotating or (ii) non-rotating and non-degenerate,
- is analytic,
- has no closed timelike curve in the domain of outer communications (DOC) (= black hole exterior).

Then $(\mathscr{M}, \boldsymbol{g})$ has a DOC that is isometric to the DOC of Kerr spacetime.

Discussion: weak points in the theorem

From an (astro)physical point of view, it would be more satisfactory if the last two hypotheses of the theorem would appear as *consequences*, not as *premises*:

- the Lorentzian manifold $(\mathscr{M}, \boldsymbol{g})$ is analytic
- there is no closed timelike curve in the black hole exterior.

Analyticity hypothesis

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is **(real) analytic** iff for each $x_0 \in I$, there exists a neighborhood of x_0 in which $f(x)$ is expressible as a convergent series:

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$$

NB: f analytic $\Longrightarrow$ f smooth (C^∞), but the converse is false (for real functions).

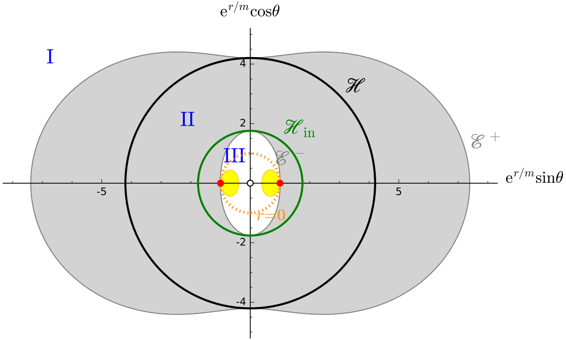
Analyticity is a very rigid concept and in physics, one usually considers smooth fields, not analytic ones.

To date, the analyticity hypothesis has been removed from the no-hair theorem only under the assumption of slow rotation [Alexakis, Ionescu & Klainerman, Duke Math. J. **163**, 2603 (2014)]

Hypothesis of no closed timelike curves in the BH exterior

...sounds a "reasonable" requirement: who wants to break causality in the black hole exterior? However, general relativity by itself offers no guarantee against causality violation. So it is not superfluous to state the no-CTC hypothesis if it is required in the proof.

An example of causality violation in general relativity comes from the Kerr spacetime itself! In its *deep interior* (probably unstable), it contains the Carter time machine.



← meridional cut of Kerr spacetime with $a = 0.9\,M$

- grey area: ergoregion
- red dots: ring singularity
- yellow area: Carter time machine

Outline

Black holes with scalar hair in general relativity

Herdeiro & Radu discovery (2014)

A black hole can have a complex scalar hair

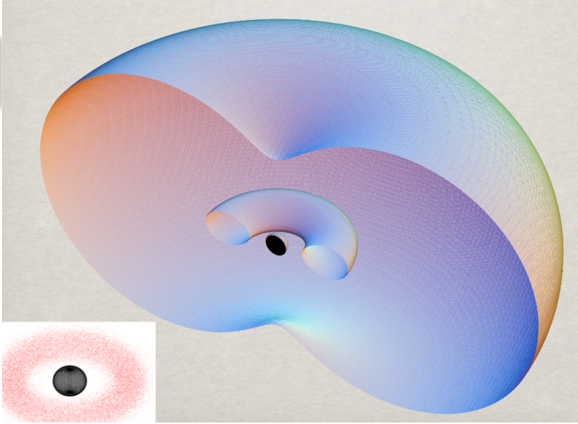
Stationary axisymmetric configurations with a massive complex scalar field Φ minimally coupled to general relativity

$\Phi(t, r, \theta, \varphi) = \Phi_0(r, \theta) e^{i(\omega t + k\varphi)}$

$\omega = k\Omega_{\mathrm{H}}, \quad k \in \mathbb{N}$

$\Longrightarrow$ the energy-momentum tensor of Φ is independent from t

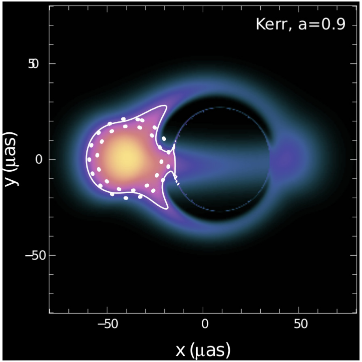
$\Longrightarrow$ *stationary* solution $\boldsymbol{g}$ to the Einstein equation, distinct from the Kerr metric

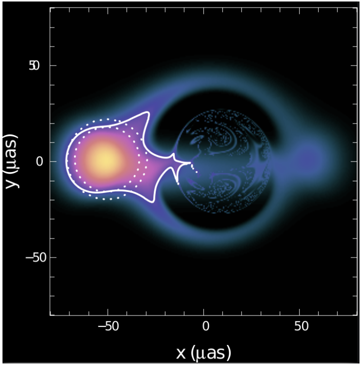


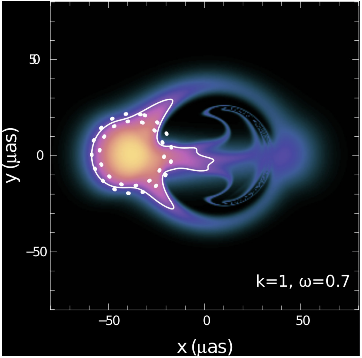
[Herdeiro & Radu, PRL **112**, 221101 (2014)]

Difference in the images of an accretion torus

Kerr BH	Herdeiro-Radu BH [1]	boson star (no BH) [2]
$a/M = 0.9$	$a/M = 0.9$	$k=1,\ \omega=0.7\, m/\hbar$







[[1] Vincent, Gourgoulhon, Herdeiro & Radu, Phys. Rev. D **94**, 084045 (2016)]

[[2] Vincent, Meliani, Grandclément, Gourgoulhon & Straub, Class. Quantum Grav. **33**, 105015 (2016)]

No-hair theorem in scalar-tensor theories

Scalar-tensor theories

Modified gravity theories involving a (real) scalar field ϕ in addition to the metric tensor $\boldsymbol{g}$, with a non-minimal coupling.

Basically the no-hair theorem remains true for a large class of scalar tensor theories (including Brans-Dicke, as shown by Hawking in 1972), provided that ϕ is stationary $\left(\frac{\partial \phi}{\partial t} = 0\right)$, as $\boldsymbol{g}$: Kerr metric is the only black hole solution. In particular, BHs have no scalar hair.

See [Capuano, Santoni & Barausse, PRD **108**, 064058 (2023)] for recent results.

No-hair theorem not obeyed by

- Einstein-scalar-Gauss-Bonnet gravity
- scalar-tensor theories with $\phi(t, r, \theta) = qt + \psi(r, \theta)$ and action depending only on $\nabla_\alpha \phi$

Cf. Nicolas Lecœur's recent thesis (2024) for a review [arXiv:2406.11095]

General relativity in dimension higher than 4

For spacetime dimension $n > 4$, Kerr solution generalizes to Myers-Perry solution (1986)

However, the no-hair theorem does not hold for $n > 4$: Myers-Perry BHs are not the only stationary BH solution to the vacuum Einstein equation.

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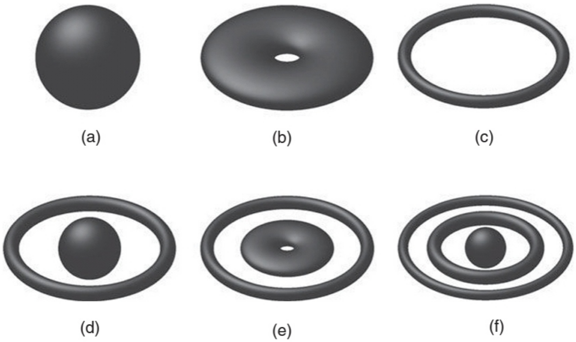


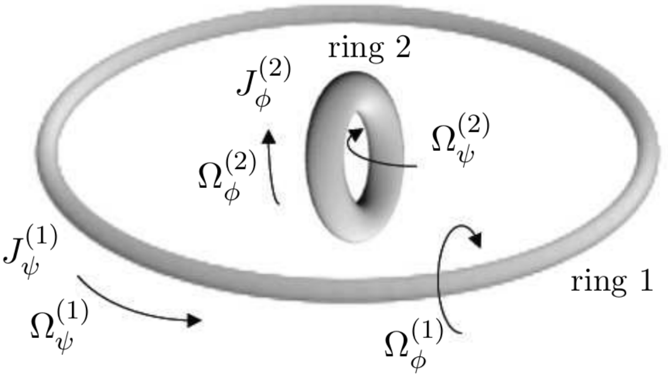
Fig. 10.11 Schematical illustration of 5D black objects: (a) MP black hole; (b) Fat black ring; (c) Thin black ring; (d) Black Saturn; (e) Di-ring; (f) Black Saturn with several rings.

[Frolov & Zelnikov: *Introduction to Black Hole Physics*, Oxford Univ. Press (2011)]

BH solutions for $n = 5$

Myers-Perry BHs are only those with an event horizon having cross-sections of $\mathbb{S}^3$ topology (fig. a), but there exist $\mathbb{S}^2 \times \mathbb{S}^1$ topologies (figs. b and c), as well as disconnected topologies (figs. d, e and f).

Bicycling black rings in dimension 5



[Elvang & Rodriguez, JHEP04(2008)045]

Outline

The Kerr metric is specific to black holes

Spherically symmetric (non-rotating) bodies:

Birkhoff theorem

Within 4-dimensional general relativity, the spacetime outside any spherically symmetric body is described by Schwarzschild metric

$\Longrightarrow$ No possibility to distinguish a non-rotating black hole from a non-rotating dark star by monitoring orbital motion or fitting accretion disk spectra

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Rotating axisymmetric bodies: *no Birkhoff-like theorem*

Moreover, no "reasonable" matter source has ever been found for the Kerr metric

The only known source consists of two counter-rotating thin disks of collisionless particles

[Bicak & Ledvinka, PRL **71**, 1669 (1993)]

$\Longrightarrow$ The Kerr metric is specific to rotating black holes
(in 4-dimensional general relativity)

Lowest order no-hair theorem: quadrupole moment

Asymptotic expansion (large r) of the metric in terms of multipole moments $(\mathcal{M}_k, \mathcal{J}_k)_{k\in\mathbb{N}}$ [Geroch (1970), Hansen (1974)]:

- $\mathcal{M}_k$: mass 2^k-pole moment
- $\mathcal{J}_k$: angular momentum 2^k-pole moment

$\Longrightarrow$ For the Kerr metric, all the multipole moments are determined by (M, a):

- $\mathcal{M}_0 = M$
- $\mathcal{J}_1 = aM = J$
- $\mathcal{M}_2 = -a^2 M \Longrightarrow \boxed{\mathcal{M}_2 = -\frac{J^2}{M}}$ (1) $\leftarrow$ mass quadrupole moment
- $\mathcal{J}_3 = -a^3 M$
- $\mathcal{M}_4 = a^4 M$
- $\cdots$

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Measuring the three quantities M, J, $\mathcal{M}_2$ provides a compatibility test w.r.t. the Kerr metric, by checking (1)

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Conclusions

- The no-hair theorem is a striking result of general relativity; it has no equivalent in other areas of physics.
- As a mathematical theorem, it relies on some technical hypotheses that one would like to get rid of (analyticity and causality in the black hole exterior).
- The no-hair theorem may not hold if some physical hypotheses are violated:
    - the black hole is not isolated (e.g. surrounded by some massive complex scalar field or dark matter);
    - the theory of gravity is not general relativity, nor a "simple" scalar-tensor theory;
    - the spacetime dimension is not 4.
- GRAVITY+ (or GRAVITY++ ?) could perform the lowest order test of the no-hair theorem by measuring the mass quadrupole moment of Sgr A*, in addition to its mass and spin.
- The ultimate test: monitoring extreme mass-ratio inspirals (EMRIs) with LISA?