Construction of initial data for 3+1 numerical relativity

Part 4

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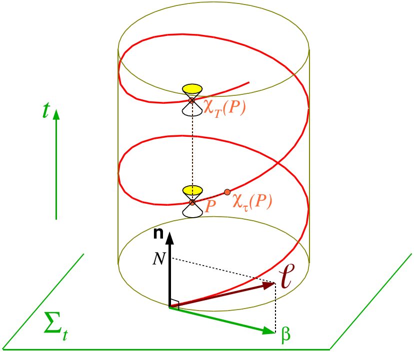
Plan

Outline

Helical symmetry

Physical assumption: when the two objects are sufficiently far apart, the radiation reaction can be neglected $\Rightarrow$ closed orbits
Gravitational radiation reaction circularizes the orbits $\Rightarrow$ circular orbits
Geometrical translation: spacetime possesses some **helical symmetry**



Helical Killing vector ξ:

(i) timelike near the system,
(ii) spacelike far from it, but such that $\exists$ a smaller $T > 0$ such that the separation between any point P and and its image $\chi_T(P)$ under the symmetry group is timelike
[Bonazzola, Gourgoulhon & Marck, PRD **56**, 7740 (1997)]
[Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)]

Helical symmetry: discussion

Helical symmetry is exact

- in **Newtonian gravity** and in **2nd order Post-Newtonian gravity**
- in the **Isenberg-Wilson-Mathews** approximation to General Relativity [Baumgarte et al., PRL **79**, 1182 (1997)]
- in general relativity for a non-axisymmetric system (binary) only with **standing gravitational waves** [Detweiler, PRD **50**, 4929 (1994)]

A spacetime with a helical Killing vector and standing gravitational waves cannot be asymptotically flat in full GR [Gibbons & Stewart 1983].

Helical symmetry and extended conformal thin sandwich (XCTS)

Choose coordinates (t, x^i) adapted to the helical Killing vector: $\frac{\partial}{\partial t} = \boldsymbol{\xi}$.
$\Longrightarrow$ the "velocity" part of the freely specifiable data of the XCTS approach is fully determined:

$$\boxed{\dot{\tilde{\gamma}}^{ij} = \frac{\partial \tilde{\gamma}^{ij}}{\partial t} = 0} \quad \text{and} \quad \boxed{\dot{K} = \frac{\partial K}{\partial t} = 0}$$

No such direct translation of helical symmetry in the CTT scheme

In addition, choose maximal slicing $K = 0$
The XCTS system becomes then

$$\tilde{D}_i \tilde{D}^i \Psi - \frac{\tilde{R}}{8}\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} + 2\pi \tilde{E}\Psi^{-3} = 0$$

$$\tilde{D}_j \left(\frac{1}{\tilde{N}} (\tilde{L}\beta)^{ij} \right) = 16\pi \tilde{p}^i$$

$$\tilde{D}_i \tilde{D}^i (\tilde{N}\Psi^7) - (\tilde{N}\Psi^7)\left[\frac{1}{8}\tilde{R} + \frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-8} + 2\pi(\tilde{E} + 2\tilde{S})\Psi^{-4} \right] = 0$$

Helical symmetry and extended conformal thin sandwich (XCTS)

Case of flat conformal metric: if one choose, as part of free data, $\tilde{\gamma}_{ij} = f_{ij}$, the helical-symmetry XCTS equations reduce to

$$\Delta\Psi + \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\,\Psi^{-7} + 2\pi\tilde{E}\Psi^{-3} = 0$$

$$\Delta\beta^i + \frac{1}{3}\mathcal{D}^i\mathcal{D}_j\beta^j - (L\beta)^{ij}\tilde{D}_j \ln\tilde{N} = 16\pi\tilde{N}\tilde{p}^i$$

$$\Delta(\tilde{N}\Psi^7) - (\tilde{N}\Psi^7)\left[\frac{7}{8}\hat{A}_{ij}\hat{A}^{ij}\Psi^{-8} + 2\pi(\tilde{E} + 2\tilde{S})\Psi^{-4}\right] = 0$$

$\Delta := \mathcal{D}_i\mathcal{D}^i$ flat Laplacian, $\mathcal{D}_i$ flat connection ($\mathcal{D}_i = \partial_i$ in Cartesian coord.),
$\hat{A}^{ij} = \frac{1}{2\tilde{N}}(L\beta)^{ij}$, $\quad (L\beta)^{ij} := \mathcal{D}^i\beta^j + \mathcal{D}^j\beta^i - \frac{2}{3}\mathcal{D}_k\beta^k\, f^{ij}$

Helical symmetry and IWM approximation

Isenberg-Wilson-Mathews approximation: waveless approximation to General Relativity based on a conformally flat spatial metric: $\boldsymbol{\gamma} = \Psi^4 \boldsymbol{f}$

[Isenberg (1978)], [Wilson & Mathews (1989)]

$\Rightarrow$ spacetime metric : $ds^2 = -N^2 dt^2 + \Psi^4 f_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

Amounts to solve only 5 of the 10 Einstein equations:

- Hamiltonian constraint
- momentum constraint (3 equations)
- trace of the evolution equation for the extrinsic curvature

Within the helical symmetry, the IWM equations reduce to the XCTS equations with choice $\tilde{\boldsymbol{\gamma}} = \boldsymbol{f}$

but note

XCTS is not some approximation to general relativity (contrary to IWM): it provides exact initial data

Outline

Basic framework

- $\boldsymbol{\partial}_t = \boldsymbol{\xi}$ helical Killing vector $\Longrightarrow$ XCTS scheme with $\dot{\tilde{\gamma}}^{ij} = 0$ and $\dot{K} = 0$
- each black hole is a non-expanding horizon

From Lecture 3 : Non-expanding horizon boundary conditions:

- $4\tilde{s}^i \tilde{D}_i \Psi + \tilde{D}_i \tilde{s}^i \, \Psi + \frac{1}{2\tilde{N}} (\tilde{L}\beta)_{ij} \tilde{s}^i \tilde{s}^j \, \Psi^{-3} - \frac{2}{3} K \Psi^3 \stackrel{\mathcal{S}}{=} 0$
- $\beta^\perp \stackrel{\mathcal{S}}{=} N$
- $\boldsymbol{V}$ conformal Killing vector of $\tilde{\boldsymbol{q}}$, e.g. $\boldsymbol{V} \stackrel{\mathcal{S}}{=} \omega \, \boldsymbol{\partial}_{\varphi_\star}$ $\quad (\omega = \text{const})$

$\mathcal{S} = \mathcal{S}_1$ or $\mathcal{S}_2$ (one the two excised surfaces)

Hence $\boldsymbol{\beta} \stackrel{\mathcal{S}}{=} \beta^\perp \boldsymbol{s} - \boldsymbol{V}$ yields $\boxed{\boldsymbol{\beta} \stackrel{\mathcal{S}}{=} N \boldsymbol{s} - \omega \, \boldsymbol{\partial}_{\varphi_\star}}$

The remains to choose, for each hole,

- the lapse N (choice of foliation)
- the conformal Killing vector $\boldsymbol{\partial}_{\varphi_\star}$ on $\mathcal{S}$ (choice of the direction of spin)
- the constant ω (choice of spin amplitude, cf. below)

Boundary conditions at spatial infinity

At spatial infinity:

$$\boldsymbol{\xi} = \boldsymbol{\partial}_{t_0} + \Omega \boldsymbol{\partial}_{\varphi_0}$$

where $(t_0, r_0, \theta_0, \varphi_0)$ = coordinate system associated with an asymptotically inertial observer

Ω : constant = **orbital angular velocity**

Hence the boundary conditions:

- asymptotic flatness : $\Psi|_{r\to\infty} = 1$ and $\tilde{N}|_{r\to\infty} = 1$
- $\boldsymbol{\partial}_t = N\boldsymbol{n} + \boldsymbol{\beta} = \boldsymbol{\xi}$ helical vector: $\boldsymbol{\beta}|_{r\to\infty} = \Omega \boldsymbol{\partial}_{\varphi_0}$

Rotation state of the holes

1. Corotating state (synchronized configuration)

the BHs are corotating $\iff$ the null generators of the non-expanding horizons are colinear to the helical Killing vector

$\iff$ the non-expanding horizons are Killing horizons

$\iff$ the helical Killing vector $\boldsymbol{\partial}_t$ is null at $\mathcal{S}_1$ and $\mathcal{S}_2$

$$\iff (N\boldsymbol{n} + \boldsymbol{\beta}) \cdot (N\boldsymbol{n} + \boldsymbol{\beta}) \stackrel{\mathcal{S}}{=} 0$$

$$\iff -N^2 + \boldsymbol{\beta} \cdot \boldsymbol{\beta} \stackrel{\mathcal{S}}{=} 0$$

$$\iff -N^2 + N^2 + \omega^2 \, \boldsymbol{\partial}_{\varphi_0} \cdot \boldsymbol{\partial}_{\varphi_0} \stackrel{\mathcal{S}}{=} 0$$

$$\iff \boxed{\omega \stackrel{\mathcal{S}}{=} 0}$$

Remark: From a full spacetime point of view, the corotating state is the only rotation state fully compatible with the helical symmetry (rigidity property)

[Friedman, Uryu & Shibata, PRD **65**, 064035 (2002)]

Rotation state of the holes

2. Irrotational state

Spin of a non-expanding horizon [Ashtekar, Beetle & Lewandowski, CQG **19**, 1195 (2002)]

$$S_{(\phi)} := \frac{1}{8\pi} \oint_{\mathcal{S}} \langle \boldsymbol{L}, \boldsymbol{\phi} \rangle \sqrt{q} d^2 x$$

where

- $\boldsymbol{L}$ is the 1-form defined by $L_a = K_{ij} s^i q^j{}_a$ (cf. Lecture 3)
- $\boldsymbol{\phi}$ is a Killing vector of $(\mathcal{S}, \boldsymbol{q})$

In terms of conformal quantities: $S_{(\phi)} := \frac{1}{16\pi} \oint_{\mathcal{S}} \frac{1}{\tilde{N}} (\tilde{L}\beta)_{ij} \tilde{s}^i \phi^j \sqrt{\tilde{q}}\, d^2 x$

Problem: find a (approximate) Killing vector on $\mathcal{S}$

numerical method: [Dreyer, Krishnan, Shoemaker & Schnetter, PRD **67**, 024018 (2003)]

Definition of **irrotationality** [Caudill, Cook, Grigsby & Pfeiffer, PRD **74**, 064011 (2006)] :

$$S_{(\phi)} = 0$$

$\Longrightarrow$ choose ω to ensure $S_{(\phi)} = 0$

Global quantities

- **Orbital angular velocity:** Ω / $\boldsymbol{\xi} = \boldsymbol{\partial}_{t_0} + \Omega \boldsymbol{\partial}_{\varphi_0}$
- **ADM mass:** $M_{\rm ADM} = -\frac{1}{2\pi} \oint_\infty s^i \left(\mathcal{D}_i \Psi - \frac{1}{8} \mathcal{D}^j \tilde{\gamma}_{ij} \right) \sqrt{q} \, d^2 x$
- **Total angular momentum:** $J = \frac{1}{8\pi} \oint_\infty (K_{ij} - K \gamma_{ij}) \, (\boldsymbol{\partial}_{\varphi_0})^i \, s^j \sqrt{q} \, d^2 x$
- **Irreducible masses:** $M_{{\rm irr}\, i} := \sqrt{\frac{A_i}{16\pi}}$ $(i = 1, 2)$

  A_i = area of surface $\mathcal{S}_i$ (measured with induced metric $\boldsymbol{q}$)

Determination of Ω

1. Effective potential method

Origin: [Cook, PRD **50**, 5025 (1994)], improved by [Caudill, Cook, Grigsby & Pfeiffer, PRD **74**, 064011 (2006)]

- Define the binding energy by $E := M_{\rm ADM} - M_{\rm irr1} - M_{\rm irr2}$
- Define a circular orbit as an extremum of E with respect to proper separation l at fixed angular momentum, irreducible masses and spins:

$$\left.\frac{\partial E}{\partial l}\right|_{J, M_{\rm irr1}, M_{\rm irr2}, S_1, S_2} = 0$$

Determination of Ω

2. Virial theorem method

Origin: [Gourgoulhon, Grandclément & Bonazzola, PRD **65**, 044020 (2002)],

Virial assumption: $O(r^{-1})$ part of the metric ($r \to \infty$) same as Schwarzschild
[The only quantity "felt" at the $O(r^{-1})$ level by a distant observer is the total mass of the system.]

A priori

$$\Psi \sim 1 + \frac{M_{\rm ADM}}{2r} \qquad \text{and} \qquad N \sim 1 - \frac{M_{\rm K}}{r}$$

Hence

$$\text{(virial assumption)} \iff M_{\rm ADM} = M_{\rm K}$$

Note

$$\text{(virial assumption)} \iff \Psi^2 N \sim 1 + \frac{\alpha}{r^2}$$

Determination of Ω

2. Virial theorem method (con't)

Link with the classical virial theorem

Einstein equations ⇒

$$\Delta \ln(\Psi^2 N) = \Psi^4 \left[4\pi S_i^{\ i} + \frac{3}{4} \tilde{A}_{ij} \tilde{A}^{ij} \right] - \frac{1}{2} \left[\mathcal{D}_i \ln N \mathcal{D}^i \ln N + \mathcal{D}_i \ln(\Psi^2 N) \mathcal{D}^i \ln(\Psi^2 N) \right]$$

No monopolar $1/r$ term in $\Psi^2 N$ ⟺

$$\int_{\Sigma_t} \left\{ 4\pi S_i^{\ i} + \frac{3}{4} \tilde{A}_{ij} \tilde{A}^{ij} - \frac{\Psi^{-4}}{2} \left[\mathcal{D}_i \ln N \mathcal{D}^i \ln N + \mathcal{D}_i \ln(\Psi^2 N) \mathcal{D}^i \ln(\Psi^2 N) \right] \right\} \Psi^4 \sqrt{f}\, d^3x = 0$$

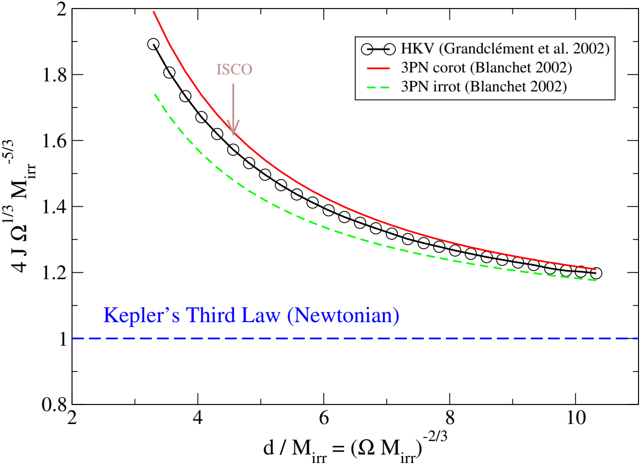
Newtonian limit is the classical virial theorem:

$$2E_{\rm kin} + 3P + E_{\rm grav} = 0$$

Determination of Ω

2. Virial theorem method : validation

recovering Kepler's third law



Determination of Ω: comparison of the two methods

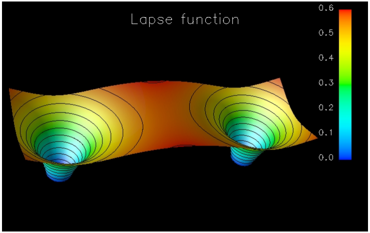
Agreement betweent the effective potential method and the virial theorem method: *very good*

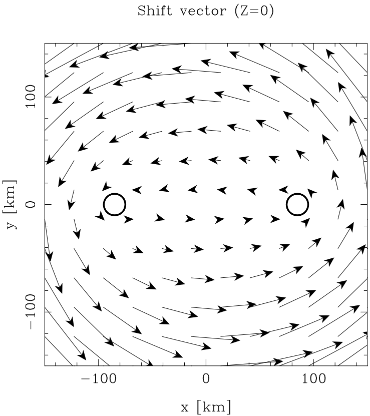
[Skoge & Baumgarte, PRD **66**, 107501 (2002)]), [Caudill, Cook, Grigsby & Pfeiffer, PRD **74**, 064011 (2006)]

Numerical implementation

- [Grandclément, Gourgoulhon & Bonazzola, PRD **65**, 044021 (2002)] : corotating BH
- [Cook & Pfeiffer, PRD **70**, 104016 (2004)] : corotating and "quasi-irrotational" BH
- [Ansorg, PRD **72**, 024018 (2005)] : corotating BH
- [Caudill, Cook, Grigsby & Pfeiffer, PRD **74**, 064011 (2006)] : corotating and irrotational BH

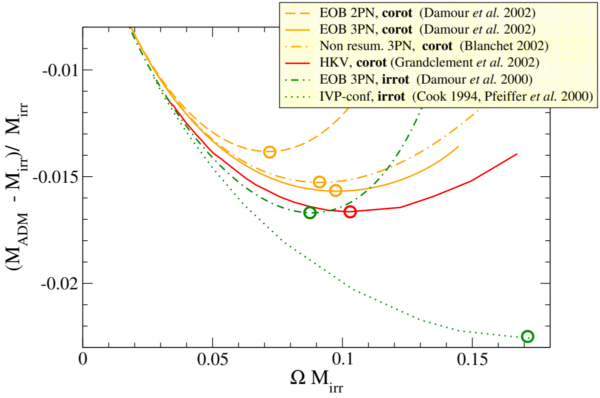
All are using $\tilde{\gamma} = f$ and $K = 0$





Results

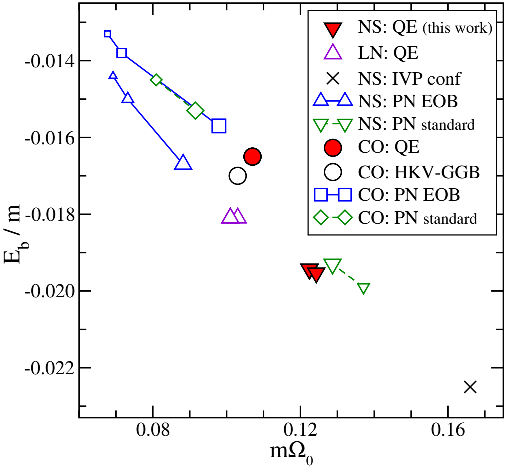
Binding energy along an evolutionary sequence of equal-mass binary black holes



[Damour, Gourgoulhon, Grandclément, PRD **66**, 024007 (2002)]

Results

ISCO configurations



[Caudill, Cook, Grigsby & Pfeiffer, PRD **74**, 064011 (2006)]

Punctured initial data

Choice of the initial data 3-dimensional manifold: twice-punctured $\mathbb{R}^3$:
$\Sigma_0 = \mathbb{R}^3 \backslash \{O_1, O_2\}$
Problem: incompatible with XCTS [Hannam, Evans, Cook & Baumgarte, PRD **68**, 064003 (2003)]

$\Longrightarrow$ computations within the CTT framework
then no way to implement helical symmetry
instead select

- $\hat{A}^{ij}_{\mathrm{TT}} = 0$
- $\hat{A}^{ij} = (LX)^{ij}$ with X = Bowen-York solution $\leftarrow$ ad hoc solution (no link with helical symmetry)

Punctured initial data: numerical implementations

- [Baumgarte, PRD **62**, 024018 (2000)]
- [Baker, Campanelli, Lousto & Takashi, PRD **65**, 124012 (2002)]
- [Ansorg, Brügman & Tichy, PRD **70**, 064011 (2004)]

Used in the UTB and NASA/Goddard binary BH merger computations

Good agreement with XCTS at large separation
but deviation from XCTS and post-Newtonian at close separation

Bowen-York extrinsic curvature is bad for binary systems in quasi-equilibrium

Post-Newtonian based initial data

- [Tichy, Brügman, Campanelli & Diener, PRD **67**, 064008 (2003)] : punctures + CTT method with free data $(\tilde{\gamma}_{ij}, \hat{A}^{ij}_{\mathrm{TT}})$ given by the 2PN metric
- [Nissanke, PRD **73**, 124002 (2006)]: provides 2PN free data for both CTT and XCTS schemes

Outline

Fluid equation of motion

Neutron star fluid = perfect fluid : $\boldsymbol{T} = (e+p)\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p\boldsymbol{g}$.
Carter-Lichnerowicz equation of motion for zero-temperature fluids:

$$\nabla \cdot \boldsymbol{T} = 0 \iff \begin{cases} \boldsymbol{u} \cdot \mathbf{d}\boldsymbol{w} = 0 & (1) \\ \nabla \cdot (n\boldsymbol{u}) = 0 & (2) \end{cases}$$

$\boldsymbol{w} := h\underline{\boldsymbol{u}}$: co-momentum 1-form
$\mathbf{d}\boldsymbol{w}$: vorticity 2-form

with n = baryon number density and $h = (e+p)/(m_{\rm B} n)$ specific enthalpy.
Cartan identity : Killing vector $\boldsymbol{\xi} \Longrightarrow \mathcal{L}_{\boldsymbol{\xi}}\, \boldsymbol{w} = 0 = \boldsymbol{\xi} \cdot \mathbf{d}\boldsymbol{w} + \mathbf{d}(\boldsymbol{\xi} \cdot \boldsymbol{w})$ (3)
Two cases with a first integral : $\boxed{\boldsymbol{\xi} \cdot \boldsymbol{w} = \text{const}}$ (4)

- **Rigid motion:** $\boldsymbol{u} = \lambda \boldsymbol{\xi}$: (3) + (1) ⇔ (4) ; (2) automatically satisfied

- **Irrotational motion:** $\mathbf{d}\boldsymbol{w} = 0 \Leftrightarrow \boldsymbol{w} = \boldsymbol{\nabla}\Phi$: (3) ⇔ (4) ; (1) automatically satisfied

$$(2) \Leftrightarrow \frac{n}{h} \nabla \cdot \nabla\Phi + \nabla \left(\frac{n}{h} \right) \cdot \nabla\Phi = 0$$

[Bonazzola, Gourgoulhon & Marck, PRD **56**, 7740 (1997)], [Asada, PRD **57**, 7292 (1998)], [Shibata, PRD **58**, 024012 (1998)], [Teukolsky, ApJ **504**, 442 (1998)]
review: [Gourgoulhon, gr-qc/0603009]

Astrophysical relevance of the two rotation states

- **Rigid motion (synchronized binaries)** (also called **corotating binaries**) : the viscosity of neutron star matter is far too low to ensure synchronization of the stellar spins with the orbital motion
  [Kochanek, ApJ **398**, 234 (1992)], [Bildsten & Cutler, ApJ **400**, 175 (1992)]
  $\Longrightarrow$ unrealistic state of rotation
- **Irrotational motion:** good approximation for neutron stars which are not initially millisecond rotators, because then $\Omega_{\rm spin} \ll \Omega_{\rm orb}$ at the late stages.

Fluid equations to be solved

Baryon number conservation for irrotational flows:

$$n\Delta\Phi + \mathcal{D}_i n\, \mathcal{D}^i\Phi = \cdots$$

$\longrightarrow$ singular ($n = 0$ at the stellar surface) elliptic equation to be solved for Φ.

First integral of fluid motion $\boldsymbol{\xi}\cdot\boldsymbol{w} = \mathrm{const}$ writes $\quad hN\frac{\Gamma}{\Gamma_0} = \mathrm{const} \quad (5)$

with Γ : Lorentz factor between fluid co-moving observer and co-orbiting obser
($= 1$ for synchronized binaries)

Γ_0 : Lorentz factor between co-orbiting observer and asymptotically inertial observer

$\longrightarrow$ solve (5) for the specific enthalpy h.

From h compute the fluid proper energy density e, pressure p and baryon number n via an equation of state:

$$e = e(h), \qquad p = p(h), \qquad n = n(h)$$

Determination of Ω

First integral of fluid motion:

$$hN\frac{\Gamma}{\Gamma_0} = \mathrm{const}$$

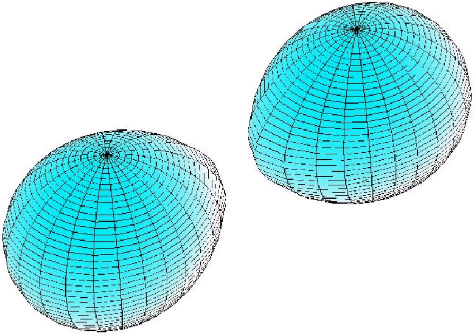
The Lorentz factor Γ_0 contains Ω: at the Newtonian limit, $\ln\Gamma_0$ is nothing but the centrifugal potential: $\ln\Gamma_0 \sim \frac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{r})^2$.

At each step of the iterative procedure, Ω and the location of the rotation axis are then determined so that the stellar centers (density maxima) remain at fixed coordinate distance from each other.

Numerical results

- Polytropic EOS corotating : [Baumgarte et al., PRL **79**, 1182 (1997)], [Baumgarte et al., PRD **57**, 7299 (1998)], [Marronetti, Mathews & Wilson, PRD **58**, 107503 (1998)]
- Polytropic EOS irrotational : [Bonazzola, Gourgoulhon & Marck, PRL **82**, 892 (1999)], [Gourgoulhon et al., PRD **63**, 064029 (2001)], [Marronetti, Mathews & Wilson, PRD **60**, 087301 (2000)], [Uryu & Eriguchi, PRD **61**, 124023 (2000)], [Uryu & Eriguchi, PRD **62**, 104015 (2000)], [Taniguchi & Gourgoulhon, PRD **66**, 104019 (2002)], [Taniguchi & Gourgoulhon, PRD **68**, 124025 (2003)]
- Nuclear matter EOS : [Bejger, Gondek-Rosińska, Gourgoulhon, Haensel, Taniguchi & Zdunik, A&A **431**, 297 (2005)], [Oechslin, Janka & Marek, astro-ph/0611047]
- Strange quark stars: [Oechslin, Uryu, Poghosyan & Thielemann, MNRAS **349**, 1469 (2004)], [Limousin, Gondek-Rosińska & Gourgoulhon, PRD **71**, 064012 (2005)]

Results



[Taniguchi & Gourgoulhon, PRD **66**, 104019 (2002)]

First non conformally flat initial data for binary NS:

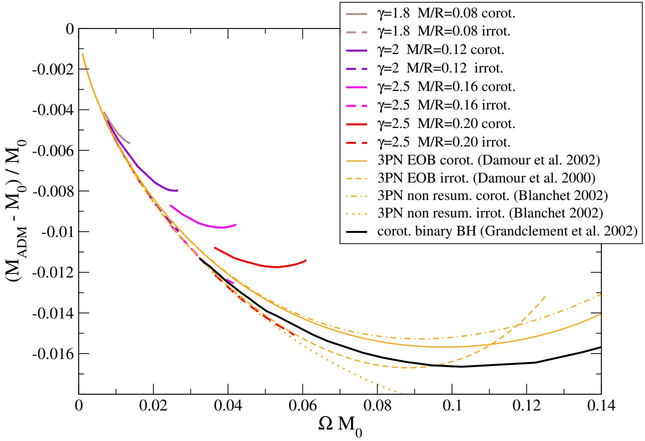
[Uryu, Limousin, Friedman, Gourgoulhon, & Shibata, PRL **97**, 171101 (2006)]

Comparing BH and NS evolutionary sequences

Evolutionary sequence:

BH : configurations of decreasing separation with fixed irreducible mass

NS : configurations of decreasing separation with fixed baryon mass



[Taniguchi & Gourgoulhon, PRD **68**, 124025 (2003)]

Outline

1. Helical symmetry for binary systems
2. Initial data for orbiting binary black holes
3. Initial data for orbiting binary neutron stars
4. References

Review articles

In chronological order:

- J.W. York : *Kinematics and dynamics of general relativity*, in *Sources of Gravitational Radiation*, edited by L.L. Smarr, Cambridge University Press, Cambridge (1979), p. 83.
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- R. Bartnik and J. Isenberg : *The Constraint Equations*, in *The Einstein Equations and the Large Scale Behavior of Gravitational Fields — 50 years of the Cauchy Problem in General Relativity*, edited by P.T. Chruściel and H. Friedrich, Birkhäuser Verlag, Basel (2004), p. 1.

Historical articles

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