Magnetohydrodynamics in stationary and axisymmetric spacetimes: a geometrical approach

Eric Gourgoulhon[1], Charalampos Markakis[2], Kōji Uryū[3] & Yoshiharu Eriguchi[4]

[1] LUTH, Observatoire de Paris / CNRS / Univ. Paris Diderot, Meudon, France

[2] Dept. of Physics, University of Wisconsin, Milwaukee, USA

[3] Dept. of Physics, University of the Ryukyus, Okinawa, Japan

[4] Dept. of Earth Science and Astronomy, University of Tokyo, Japan

Black Holes, General Relativity, Waves
Station biologique de Roscoff,
8-10 November 2010

Plan

Outline

Short history of general relativistic MHD
focusing on stationary and axisymmetric spacetimes

- Lichnerowicz (1967): formulation of GRMHD
- Bekenstein & Oron (1978), Carter (1979) : development of GRMHD for stationary and axisymmetric spacetimes
- Mobarry & Lovelace (1986) : Grad-Shafranov equation for Schwarzschild spacetime
- Nitta, Takahashi & Tomimatsu (1991), Beskin & Pariev (1993) : Grad-Shafranov equation for Kerr spacetime
- Ioka & Sasaki (2003) : Grad-Shafranov equation in the most general (i.e. *noncircular*) stationary and axisymmetric spacetimes

NB: not speaking about *numerical* GRMHD here
(see e.g. Shibata & Sekiguchi (2005))

Why a geometrical approach ?

- Previous studies made use of component expressions, the covariance of which is not obvious
  For instance, two of main quantities introduced by Bekenstein & Oron (1978) and employed by subsequent authors are

$$\omega := -\frac{F_{01}}{F_{31}} \qquad \text{and} \qquad C := \frac{F_{31}}{\sqrt{-g}nu^2}$$

- GRMHD calculations can be cumbersome by means of standard tensor calculus

Why a geometrical approach ?

- Previous studies made use of component expressions, the covariance of which is not obvious
  For instance, two of main quantities introduced by Bekenstein & Oron (1978) and employed by subsequent authors are

$$\omega := -\frac{F_{01}}{F_{31}} \qquad \text{and} \qquad C := \frac{F_{31}}{\sqrt{-gnu^2}}$$

- GRMHD calculations can be cumbersome by means of standard tensor calculus

On the other side

- As well known, the electromagnetic field tensor $\boldsymbol{F}$ is fundamentally a 2-form and Maxwell equations are most naturally expressible in terms of the exterior derivative operator
- The equations of perfect hydrodynamics can also be recast in terms of exterior calculus, by introducing the fluid vorticity 2-form (Synge 1937, Lichnerowicz 1941)
- Cartan's exterior calculus makes calculations easier !

Exterior calculus in one slide

- A p-form ($p = 0, 1, 2, \ldots$) is a multilinear form (i.e. a tensor 0-times contravariant and p-times covariant: $\omega_{\alpha_1 \ldots \alpha_p}$) that is fully *antisymmetric*
- *Index-free notation:* given a vector $\vec{\boldsymbol{v}}$ and a p-form $\boldsymbol{\omega}$, $\boxed{\vec{\boldsymbol{v}} \cdot \boldsymbol{\omega}}$ and $\boxed{\boldsymbol{\omega} \cdot \vec{\boldsymbol{v}}}$ are the $(p-1)$-forms defined by

$$\vec{\boldsymbol{v}} \cdot \boldsymbol{\omega} := \boldsymbol{\omega}(\vec{\boldsymbol{v}}, ., \ldots, .) \qquad \left[\ (\vec{\boldsymbol{v}} \cdot \boldsymbol{\omega})_{\alpha_1 \cdots \alpha_{p-1}} = v^\mu \omega_{\mu \alpha_1 \cdots \alpha_{p-1}}\ \right]$$
$$\boldsymbol{\omega} \cdot \vec{\boldsymbol{v}} := \boldsymbol{\omega}(., \ldots, ., \vec{\boldsymbol{v}}) \qquad \left[(\boldsymbol{\omega} \cdot \vec{\boldsymbol{v}})_{\alpha_1 \cdots \alpha_{p-1}} = \omega_{\alpha_1 \cdots \alpha_{p-1} \mu} v^\mu\ \right]$$

- Exterior derivative : p-form $\boldsymbol{\omega} \longmapsto (p+1)$-form $\mathbf{d}\boldsymbol{\omega}$ such that

$$
\begin{aligned}
&\text{0-form} \quad : \quad (\mathbf{d}\boldsymbol{\omega})_\alpha = \partial_\alpha \omega \\
&\text{1-form} \quad : \quad (\mathbf{d}\boldsymbol{\omega})_{\alpha\beta} = \partial_\alpha \omega_\beta - \partial_\beta \omega_\alpha \\
&\text{2-form} \quad : \quad (\mathbf{d}\boldsymbol{\omega})_{\alpha\beta\gamma} = \partial_\alpha \omega_{\beta\gamma} + \partial_\beta \omega_{\gamma\alpha} + \partial_\gamma \omega_{\alpha\beta}
\end{aligned}
$$

  The exterior derivative is nilpotent: $\boxed{\mathbf{dd}\boldsymbol{\omega} = 0}$

- A very powerful tool : Cartan's identity expressing the Lie derivative of a p-form along a vector field: $\boxed{\mathcal{L}_{\vec{\boldsymbol{v}}}\, \boldsymbol{\omega} = \vec{\boldsymbol{v}} \cdot \mathbf{d}\boldsymbol{\omega} + \mathbf{d}(\vec{\boldsymbol{v}} \cdot \boldsymbol{\omega})}$

Outline

General framework and notations

Spacetime:

- $\mathscr{M}$: four-dimensional orientable real manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$, $\mathrm{sign}\, \boldsymbol{g} = (-,+,+,+)$
- $\boldsymbol{\epsilon}$: Levi-Civita tensor (volume element 4-form) associated with $\boldsymbol{g}$: for any orthonormal basis $(\vec{e}_\alpha)$,

$$\boldsymbol{\epsilon}(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3) = \pm 1$$

$\boldsymbol{\epsilon}$ gives rise to Hodge duality : p-form $\longmapsto$ $(4-p)$-form

General framework and notations

Spacetime:

- $\mathscr{M}$: four-dimensional orientable real manifold
- $\boldsymbol{g}$: Lorentzian metric on $\mathscr{M}$, $\mathrm{sign}\, \boldsymbol{g} = (-,+,+,+)$
- $\boldsymbol{\epsilon}$: Levi-Civita tensor (volume element 4-form) associated with $\boldsymbol{g}$: for any orthonormal basis $(\vec{\boldsymbol{e}}_\alpha)$,

$$\boldsymbol{\epsilon}(\vec{\boldsymbol{e}}_0, \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2, \vec{\boldsymbol{e}}_3) = \pm 1$$

$\boldsymbol{\epsilon}$ gives rise to Hodge duality : p-form $\longmapsto$ $(4-p)$-form

Notations:

- $\vec{\boldsymbol{v}}$ vector $\Longrightarrow$ $\boxed{\underline{\boldsymbol{v}}}$ 1-form associated to $\vec{\boldsymbol{v}}$ by the metric tensor:

$$\underline{\boldsymbol{v}} := \boldsymbol{g}(\vec{\boldsymbol{v}}, .) \qquad [\underline{\boldsymbol{v}} = \boldsymbol{v}^\flat] \qquad [u_\alpha = g_{\alpha\mu} u^\mu]$$

- $\boldsymbol{\omega}$ 1-form $\Longrightarrow$ $\boxed{\vec{\boldsymbol{\omega}}}$ vector associated to $\boldsymbol{\omega}$ by the metric tensor:

$$\boldsymbol{\omega} =: \boldsymbol{g}(\vec{\boldsymbol{\omega}}, .) \qquad [\vec{\boldsymbol{\omega}} = \boldsymbol{\omega}^\sharp] \qquad [\omega^\alpha = g^{\alpha\mu}\omega_\mu]$$

Maxwell equations

Electromagnetic field in $\mathscr{M}$: 2-form $\boldsymbol{F}$ which obeys to Maxwell equations:

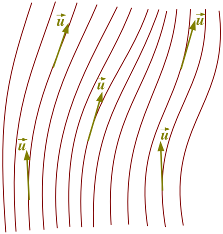
$$\boxed{\mathbf{d}\boldsymbol{F} = 0}$$

$$\boxed{\mathbf{d} \star\boldsymbol{F} = \mu_0 \star\underline{\boldsymbol{j}}}$$

- $\mathbf{d}\boldsymbol{F}$: exterior derivative of $\boldsymbol{F}$: $(\mathbf{d}\boldsymbol{F})_{\alpha\beta\gamma} = \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta}$
- $\star\boldsymbol{F}$: Hodge dual of $\boldsymbol{F}$: $\star F_{\alpha\beta} := \dfrac{1}{2}\epsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$
- $\star\underline{\boldsymbol{j}}$: 3-form Hodge-dual of the 1-form $\underline{\boldsymbol{j}}$ associated to the electric 4-current $\vec{\boldsymbol{j}}$: $\star\underline{\boldsymbol{j}} := \boldsymbol{\epsilon}(\vec{\boldsymbol{j}}, ., ., .)$
- μ_0 : magnetic permeability of vacuum

Electric and magnetic fields in the fluid frame

Fluid : congruence of worldlines in $\mathscr{M}$ $\Longrightarrow$ 4-velocity $\vec{u}$



- **Electric field** in the fluid frame: 1-form $\boxed{\boldsymbol{e} = \boldsymbol{F} \cdot \vec{\boldsymbol{u}}}$
- **Magnetic field** in the fluid frame: vector $\vec{b}$ such that $\boxed{\underline{\boldsymbol{b}} = \vec{\boldsymbol{u}} \cdot \star\boldsymbol{F}}$

$\boldsymbol{e}$ and $\vec{b}$ are orthogonal to $\vec{u}$: $\boldsymbol{e} \cdot \vec{\boldsymbol{u}} = 0$ and $\underline{\boldsymbol{b}} \cdot \vec{\boldsymbol{u}} = 0$

$$\boldsymbol{F} = \underline{\boldsymbol{u}} \wedge \boldsymbol{e} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)$$

$$\star\boldsymbol{F} = -\underline{\boldsymbol{u}} \wedge \underline{\boldsymbol{b}} + \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{e}}, ., .)$$

Perfect conductor

Fluid is a perfect conductor $\iff$ $\vec{e} = 0$ $\iff$ $\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$

From now on, we assume that the fluid is a perfect conductor (ideal MHD)

The electromagnetic field is then entirely expressible in terms of vectors $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{b}}$:

$$\boxed{\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{u}}, \vec{\boldsymbol{b}}, ., .)}$$

$$\boxed{\star\boldsymbol{F} = \underline{\boldsymbol{b}} \wedge \underline{\boldsymbol{u}}}$$

Alfvén's theorem

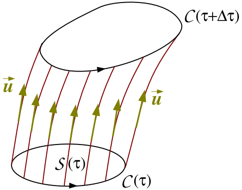
Cartan's identity applied to the 2-form $\boldsymbol{F}$:

$$\mathcal{L}_{\vec{u}}\,\boldsymbol{F} = \vec{u}\cdot\mathbf{d}\boldsymbol{F} + \mathbf{d}(\vec{u}\cdot\boldsymbol{F})$$

Now $\mathbf{d}\boldsymbol{F} = 0$ (Maxwell eq.) and $\vec{u}\cdot\boldsymbol{F} = 0$ (perfect conductor)
Hence the electromagnetic field is preserved by the flow:

$$\boxed{\mathcal{L}_{\vec{u}}\,\boldsymbol{F} = 0}$$

Application: $\boxed{\frac{d}{d\tau}\oint_{\mathcal{C}(\tau)} \boldsymbol{A} = 0}$



- τ : fluid proper time
- $\mathcal{C}(\tau)$ = closed contour dragged along by the fluid
- $\boldsymbol{A}$: electromagnetic 4-potential : $\boldsymbol{F} = \mathbf{d}\boldsymbol{A}$

Proof: $\frac{d}{d\tau}\oint_{\mathcal{C}(\tau)} \boldsymbol{A} = \frac{d}{d\tau}\int_{\mathcal{S}(\tau)} \underbrace{\mathbf{d}\boldsymbol{A}}_{\boldsymbol{F}} = \frac{d}{d\tau}\int_{\mathcal{S}(\tau)} \boldsymbol{F} = \int_{\mathcal{S}(\tau)} \underbrace{\mathcal{L}_{\vec{u}}\,\boldsymbol{F}}_{0} = 0$

Non-relativistic limit: $\int_{\mathcal{S}} \vec{b}\cdot d\vec{S} = \mathrm{const} \leftarrow$ Alfvén's theorem (mag. flux freezing)

Perfect fluid

From now on, we assume that the fluid is a perfect one: its energy-momentum tensor is

$$\boxed{\boldsymbol{T}^{\rm fluid} = (\varepsilon + p)\underline{\boldsymbol{u}} \otimes \underline{\boldsymbol{u}} + p\boldsymbol{g}}$$

Simple fluid model: all thermodynamical quantities depend on

- s: entropy density in the fluid frame,
- n: baryon number density in the fluid frame

Equation of state : $\varepsilon = \varepsilon(s, n) \Longrightarrow \begin{cases} T := \dfrac{\partial \varepsilon}{\partial s} & \text{temperature} \\ \mu := \dfrac{\partial \varepsilon}{\partial n} & \text{baryon chemical potential} \end{cases}$

First law of thermodynamics $\Longrightarrow p = -\varepsilon + Ts + \mu n$

$\Longrightarrow$ enthalpy per baryon : $\boxed{h = \dfrac{\varepsilon + p}{n} = \mu + TS}$, with $S := \dfrac{s}{n}$ (entropy per baryon)

Conservation of energy-momentum

Conservation law for the total energy-momentum:

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{T}^{\rm fluid}+\boldsymbol{T}^{\rm em}\right)=0 \qquad (1)$$

- From Maxwell equations, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm em}=-\boldsymbol{F}\cdot\vec{\boldsymbol{j}}$
- Using baryon number conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}$ can be decomposed in two parts:
  - along $\vec{\boldsymbol{u}}$: $\vec{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}=-nT\vec{\boldsymbol{u}}\cdot\mathbf{d}S$
  - orthogonal to $\vec{\boldsymbol{u}}$: $\perp_{\boldsymbol{u}}\boldsymbol{\nabla}\cdot\boldsymbol{T}^{\rm fluid}=n[\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S]$

[Synge 1937] [Lichnerowicz 1941] [Taub 1959] [Carter 1979]

$\boldsymbol{\Omega}:=\mathbf{d}(h\underline{\boldsymbol{u}})$ vorticity 2-form

Since $\vec{\boldsymbol{u}}\cdot\boldsymbol{F}\cdot\vec{\boldsymbol{j}}=0$, Eq. (1) is equivalent to the system

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}S=0} \qquad (2)$$

$$\boxed{\vec{\boldsymbol{u}}\cdot\mathbf{d}(h\underline{\boldsymbol{u}})-T\mathbf{d}S=\frac{1}{n}\boldsymbol{F}\cdot\vec{\boldsymbol{j}}} \qquad (3)$$

Eq. (3) is the MHD-Euler equation in *canonical form*

Outline

Stationary and axisymmetric spacetimes

Assume that $(\mathscr{M}, \boldsymbol{g})$ is endowed with two symmetries:

1. **stationarity** : $\exists$ a group action of $(\mathbb{R}, +)$ on $\mathscr{M}$ such that
   - the orbits are timelike curves
   - $\boldsymbol{g}$ is invariant under the $(\mathbb{R}, +)$ action :
     if $\vec{\xi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\xi}}\, \boldsymbol{g} = 0} \qquad (4)$$

2. **axisymmetry** : $\exists$ a group action of $\mathrm{SO}(2)$ on $\mathscr{M}$ such that
   - the set of fixed points is a 2-dimensional submanifold $\Delta \subset \mathscr{M}$ (called the *rotation axis*)
   - $\boldsymbol{g}$ is invariant under the $\mathrm{SO}(2)$ action :
     if $\vec{\chi}$ is a generator of the group action,

$$\boxed{\mathcal{L}_{\vec{\chi}}\, \boldsymbol{g} = 0} \qquad (5)$$

(4) and (5) are equivalent to *Killing equations*:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0} \quad \text{and} \quad \boxed{\nabla_\alpha \chi_\beta + \nabla_\beta \chi_\alpha = 0}$$

Stationary and axisymmetric spacetimes

No generality is lost by considering that the stationary and axisymmetric actions commute [Carter 1970] :
$(\mathscr{M}, \boldsymbol{g})$ is invariant under the action of the Abelian group $(\mathbb{R}, +) \times \mathrm{SO}(2)$, and not only under the actions of $(\mathbb{R}, +)$ and $\mathrm{SO}(2)$ separately. It is equivalent to say that the Killing vectors commute:

$$\boxed{[\vec{\xi}, \vec{\chi}] = 0}$$

Stationary and axisymmetric spacetimes

No generality is lost by considering that the stationary and axisymmetric actions commute [Carter 1970] :
$(\mathscr{M}, \boldsymbol{g})$ is invariant under the action of the Abelian group $(\mathbb{R}, +) \times \mathrm{SO}(2)$, and not only under the actions of $(\mathbb{R}, +)$ and $\mathrm{SO}(2)$ separately. It is equivalent to say that the Killing vectors commute:

$$\boxed{[\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}] = 0}$$

$\Longrightarrow$ $\exists$ coordinates $(x^\alpha) = (t, x^1, x^2, \varphi)$ on $\mathscr{M}$ such that $\vec{\boldsymbol{\xi}} = \dfrac{\partial}{\partial t}$ and $\vec{\boldsymbol{\chi}} = \dfrac{\partial}{\partial \varphi}$

Within them, $g_{\alpha\beta} = g_{\alpha\beta}(x^1, x^2)$

Adapted coordinates are not unique:
$$\begin{cases} t' & = & t + F_0(x^1, x^2) \\ x'^1 & = & F_1(x^1, x^2) \\ x'^2 & = & F_2(x^1, x^2) \\ \varphi' & = & \varphi + F_3(x^1, x^2) \end{cases}$$

Stationary and axisymmetric electromagnetic field

Assume that the electromagnetic field is both stationary and axisymmetric:

$$\boxed{\mathcal{L}_{\vec{\xi}}\, \boldsymbol{F} = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\chi}}\, \boldsymbol{F} = 0} \qquad (6)$$

Stationary and axisymmetric electromagnetic field

Assume that the electromagnetic field is both stationary and axisymmetric:

$$\boxed{\mathcal{L}_{\vec{\xi}} \boldsymbol{F} = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\chi}} \boldsymbol{F} = 0} \tag{6}$$

Cartan's identity and Maxwell eq. $\Longrightarrow \mathcal{L}_{\vec{\xi}} \boldsymbol{F} = \vec{\xi} \cdot \underbrace{\mathbf{d}\boldsymbol{F}}_{0} + \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = \mathbf{d}(\vec{\xi} \cdot \boldsymbol{F})$

Hence (6) is equivalent to

$$\mathbf{d}(\vec{\xi} \cdot \boldsymbol{F}) = 0 \quad \text{and} \quad \mathbf{d}(\vec{\chi} \cdot \boldsymbol{F}) = 0$$

Poincaré lemma $\Longrightarrow \exists$ locally two scalar fields Φ and Ψ such that

$$\boxed{\vec{\xi} \cdot \boldsymbol{F} = -\mathbf{d}\Phi} \quad \text{and} \quad \boxed{\vec{\chi} \cdot \boldsymbol{F} = -\mathbf{d}\Psi}$$

Stationary and axisymmetric electromagnetic field

Assume that the electromagnetic field is both stationary and axisymmetric:

$$\boxed{\mathcal{L}_{\vec{\xi}}\,\boldsymbol{F} = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\chi}}\,\boldsymbol{F} = 0} \qquad (6)$$

Cartan's identity and Maxwell eq. $\Longrightarrow$ $\mathcal{L}_{\vec{\xi}}\,\boldsymbol{F} = \vec{\xi}\cdot\underbrace{\mathbf{d}\boldsymbol{F}}_{0} + \mathbf{d}(\vec{\xi}\cdot\boldsymbol{F}) = \mathbf{d}(\vec{\xi}\cdot\boldsymbol{F})$

Hence (6) is equivalent to

$$\mathbf{d}(\vec{\xi}\cdot\boldsymbol{F}) = 0 \quad \text{and} \quad \mathbf{d}(\vec{\chi}\cdot\boldsymbol{F}) = 0$$

Poincaré lemma $\Longrightarrow$ $\exists$ locally two scalar fields Φ and Ψ such that

$$\boxed{\vec{\xi}\cdot\boldsymbol{F} = -\mathbf{d}\Phi} \quad \text{and} \quad \boxed{\vec{\chi}\cdot\boldsymbol{F} = -\mathbf{d}\Psi}$$

Link with the 4-potential $\boldsymbol{A}$: one may use the gauge freedom on $\boldsymbol{A}$ to set

$$\Phi = \boldsymbol{A}\cdot\vec{\xi} = A_t \quad \text{and} \quad \Psi = \boldsymbol{A}\cdot\vec{\chi} = A_\varphi$$

Symmetries of the scalar potentials

From the definitions of Φ and Ψ:

- $\mathcal{L}_{\vec{\xi}}\,\Phi = \vec{\xi}\cdot\mathbf{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\xi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\,\Psi = \vec{\chi}\cdot\mathbf{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\chi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\,\Phi = \vec{\chi}\cdot\mathbf{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\chi})$
- $\mathcal{L}_{\vec{\xi}}\,\Psi = \vec{\xi}\cdot\mathbf{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\xi}) = \boldsymbol{F}(\vec{\xi},\vec{\chi})$

Symmetries of the scalar potentials

From the definitions of Φ and Ψ:

- $\mathcal{L}_{\vec{\xi}}\,\Phi = \vec{\xi}\cdot \mathbf{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\xi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\,\Psi = \vec{\chi}\cdot \mathbf{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\chi}) = 0$
- $\mathcal{L}_{\vec{\chi}}\,\Phi = \vec{\chi}\cdot \mathbf{d}\Phi = -\boldsymbol{F}(\vec{\xi},\vec{\chi})$
- $\mathcal{L}_{\vec{\xi}}\,\Psi = \vec{\xi}\cdot \mathbf{d}\Psi = -\boldsymbol{F}(\vec{\chi},\vec{\xi}) = \boldsymbol{F}(\vec{\xi},\vec{\chi})$

We have $\mathbf{d}[\boldsymbol{F}(\vec{\xi},\vec{\chi})] = \mathbf{d}[\vec{\xi}\cdot\mathbf{d}\Psi] = \mathcal{L}_{\vec{\xi}}\mathbf{d}\Psi - \vec{\xi}\cdot\underbrace{\mathbf{dd}\Psi}_{0} = \mathcal{L}_{\vec{\xi}}(\boldsymbol{F}\cdot\vec{\chi}) = 0$

Hence $\boldsymbol{F}(\vec{\xi},\vec{\chi}) = \mathrm{const}$

Assuming that $\boldsymbol{F}$ vanishes somewhere in $\mathscr{M}$ (for instance at spatial infinity), we conclude that

$$\boxed{\boldsymbol{F}(\vec{\xi},\vec{\chi}) = 0}$$

Then $\boxed{\mathcal{L}_{\vec{\xi}}\,\Phi = \mathcal{L}_{\vec{\chi}}\,\Phi = 0}$ and $\boxed{\mathcal{L}_{\vec{\xi}}\,\Psi = \mathcal{L}_{\vec{\chi}}\,\Psi = 0}$

i.e. the scalar potentials Φ and Ψ obey to the two spacetime symmetries

Most general stationary-axisymmetric electromagnetic field

$$\boxed{\boldsymbol{F} = \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* + \mathbf{d}\Psi \wedge \boldsymbol{\chi}^* + \frac{I}{\sigma}\,\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)} \quad (7)$$

$$\boxed{\star\boldsymbol{F} = \boldsymbol{\epsilon}(\vec{\boldsymbol{\nabla}}\Phi, \vec{\xi}^*, ., .) + \boldsymbol{\epsilon}(\vec{\boldsymbol{\nabla}}\Psi, \vec{\chi}^*, ., .) - \frac{I}{\sigma}\underline{\xi} \wedge \underline{\chi}} \quad (8)$$

with

- $\boldsymbol{\xi}^* := \dfrac{1}{\sigma}\left(-X\,\underline{\xi} + W\underline{\chi}\right)$, $\quad \boldsymbol{\chi}^* := \dfrac{1}{\sigma}\left(W\,\underline{\xi} + V\underline{\chi}\right)$
- $\boxed{V := -\underline{\xi}\cdot\vec{\xi}}$, $\boxed{W := \underline{\xi}\cdot\vec{\chi}}$, $\boxed{X := \underline{\chi}\cdot\vec{\chi}}$, $\boxed{\sigma := VX + W^2}$
  [Carter (1973) notations]
- $\boxed{I := \star\boldsymbol{F}(\vec{\xi}, \vec{\chi})}$ $\leftarrow$ the only non-trivial scalar, apart from $\boldsymbol{F}(\vec{\xi}, \vec{\chi})$, one can form from $\boldsymbol{F}$, $\vec{\xi}$ and $\vec{\chi}$

$(\boldsymbol{\xi}^*, \boldsymbol{\chi}^*)$ is the dual basis of $(\vec{\xi}, \vec{\chi})$ in the 2-plane $\boxed{\Pi := \mathrm{Vect}(\vec{\xi}, \vec{\chi})}$:

$$\boldsymbol{\xi}^*\cdot\vec{\xi} = 1, \qquad \boldsymbol{\xi}^*\cdot\vec{\chi} = 0, \qquad \boldsymbol{\chi}^*\cdot\vec{\xi} = 0, \qquad \boldsymbol{\chi}^*\cdot\vec{\chi} = 1$$

$$\forall\vec{v}\in\Pi^\perp, \quad \boldsymbol{\xi}^*\cdot\vec{v} = 0 \quad \text{and} \quad \boldsymbol{\chi}^*\cdot\vec{v} = 0$$

Most general stationary-axisymmetric electromagnetic field
The proof

Consider the 2-form $\boldsymbol{H} := \boldsymbol{F} - \mathbf{d}\Phi \wedge \boldsymbol{\xi}^* - \mathbf{d}\Psi \wedge \boldsymbol{\chi}^*$
It satisfies

$$\boldsymbol{H}(\vec{\xi}, .) = \underbrace{\boldsymbol{F}(\vec{\xi}, .)}_{-\mathbf{d}\Phi} - \underbrace{(\vec{\xi} \cdot \mathbf{d}\Phi)}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\xi}^* \cdot \vec{\xi})}_{1}\mathbf{d}\Phi - \underbrace{(\vec{\xi} \cdot \mathbf{d}\Psi)}_{0}\boldsymbol{\chi}^* + \underbrace{(\boldsymbol{\chi}^* \cdot \vec{\xi})}_{0}\mathbf{d}\Psi = 0$$

Similarly $\boldsymbol{H}(\vec{\chi}, .) = 0$. Hence $\boldsymbol{H}|_{\Pi} = 0$

On $\Pi^\perp$, $\boldsymbol{H}|_{\Pi^\perp}$ is a 2-form. Another 2-form on $\Pi^\perp$ is $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp}$

Since $\dim \Pi^\perp = 2$ and $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp} \neq 0$, $\exists$ a scalar field I such that $\boldsymbol{H}|_{\Pi^\perp} = \frac{I}{\sigma}\,\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)\Big|_{\Pi^\perp}$. Because both $\boldsymbol{H}$ and $\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)$ vanish on Π, we can extend the equality to all space:

$$\boldsymbol{H} = \frac{I}{\sigma}\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, ., .)$$

Thus $\boldsymbol{F}$ has the form (7). Taking the Hodge dual gives the form (8) for $\star\boldsymbol{F}$, on which we readily check that $I = \star\boldsymbol{F}(\vec{\xi}, \vec{\chi})$, thereby completing the proof.

Example: Kerr-Newman electromagnetic field

Using Boyer-Lindquist coordinates (t, r, θ, φ), the electromagnetic field of the Kerr-Newman solution (charged rotating black hole) is

$$\begin{aligned} \boldsymbol{F} &= \frac{\mu_0 Q}{4\pi(r^2 + a^2\cos^2\theta)^2} \Big\{ \left[(r^2 - a^2\cos^2\theta)\,\mathbf{d}r - a^2 r \sin 2\theta\,\mathbf{d}\theta\right] \wedge \mathbf{d}t \\ &\quad + \left[a(a^2\cos^2\theta - r^2)\sin^2\theta\,\mathbf{d}r + ar(r^2 + a^2)\sin 2\theta\,\mathbf{d}\theta\right] \wedge \mathbf{d}\varphi \Big\} \end{aligned}$$

Q: total electric charge, $a := J/M$: reduced angular momentum

For Kerr-Newman, $\boldsymbol{\xi}^* = \mathbf{d}t$ and $\boldsymbol{\chi}^* = \mathbf{d}\varphi$; comparison with (7) leads to

$$\boxed{\Phi = -\frac{\mu_0 Q}{4\pi}\,\frac{r}{r^2 + a^2\cos^2\theta}}, \quad \boxed{\Psi = \frac{\mu_0 Q}{4\pi}\,\frac{a r \sin^2\theta}{r^2 + a^2\cos^2\theta}}, \quad \boxed{I = 0}$$

Non-rotating limit $(a = 0)$: Reissner-Nordström solution: $\Phi = -\dfrac{\mu_0}{4\pi}\dfrac{Q}{r}$, $\Psi = 0$

Maxwell equations

First Maxwell equation: $\mathbf{d}\boldsymbol{F} = 0$

It is automatically satisfied by the form (7) of $\boldsymbol{F}$

Second Maxwell equation: $\mathbf{d}\star\boldsymbol{F} = \mu_0 \star\underline{\boldsymbol{j}}$

It gives the electric 4-current:

$$\boxed{\mu_0 \vec{\boldsymbol{j}} = a\,\vec{\boldsymbol{\xi}} + b\,\vec{\boldsymbol{\chi}} - \frac{1}{\sigma}\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} I, .)} \qquad (9)$$

with

- $a := \nabla_\mu \left(\frac{X}{\sigma} \nabla^\mu \Phi - \frac{W}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[-X \mathscr{C}_{\boldsymbol{\xi}} + W \mathscr{C}_{\boldsymbol{\chi}} \right]$
- $b := -\nabla_\mu \left(\frac{W}{\sigma} \nabla^\mu \Phi + \frac{V}{\sigma} \nabla^\mu \Psi \right) + \frac{I}{\sigma^2} \left[W \mathscr{C}_{\boldsymbol{\xi}} + V \mathscr{C}_{\boldsymbol{\chi}} \right]$
- $\mathscr{C}_{\boldsymbol{\xi}} := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\xi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \xi_\sigma$ (1st twist scalar)
- $\mathscr{C}_{\boldsymbol{\chi}} := \star(\underline{\boldsymbol{\xi}} \wedge \underline{\boldsymbol{\chi}} \wedge \mathbf{d}\underline{\boldsymbol{\chi}}) = \epsilon^{\mu\nu\rho\sigma} \xi_\mu \chi_\nu \nabla_\rho \chi_\sigma$ (2nd twist scalar)

Remark: $\vec{\boldsymbol{j}}$ has no meridional component (i.e. $\vec{\boldsymbol{j}} \in \Pi$) $\iff$ $\mathbf{d}I = 0$

Simplification for circular spacetimes

Spacetime $(\mathscr{M}, \boldsymbol{g})$ is circular $\iff$ the planes $\Pi^\perp$ are integrable in 2-surfaces
$\iff$ $\mathscr{C}_{\boldsymbol{\xi}} = \mathscr{C}_{\boldsymbol{\chi}} = 0$

Generalized Papapetrou theorem [Papapetrou 1966] [Kundt & Trümper 1966] [Carter 1969] :
a stationary and axisymmetric spacetime ruled by the Einstein equation is circular iff the total energy-momentum tensor $\boldsymbol{T}$ obeys to

$$\begin{aligned} \xi^\mu T_\mu^{\ [\alpha} \xi^\beta \chi^{\gamma]} &= 0 \\ \chi^\mu T_\mu^{\ [\alpha} \xi^\beta \chi^{\gamma]} &= 0 \end{aligned}$$

Examples:

- circular spacetimes: Kerr-Newman, rotating star, magnetized rotating star with either purely poloidal magnetic field or purely toroidal magnetic field
- non-circular spacetimes: rotating star with meridional flow, magnetized rotating star with mixed magnetic field

In what follows, we do **not** assume that $(\mathscr{M}, \boldsymbol{g})$ is circular

Outline

Perfect conductor hypothesis (1/2)

$$\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$$

with the fluid 4-velocity decomposed as

$$\boxed{\vec{\boldsymbol{u}} = \lambda(\vec{\boldsymbol{\xi}} + \Omega \vec{\boldsymbol{\chi}}) + \vec{\boldsymbol{w}}, \qquad \vec{\boldsymbol{w}} \in \Pi^{\perp}} \tag{10}$$

Ω is the rotational angular velocity and $\vec{\boldsymbol{w}}$ is the meridional velocity

$$\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1 \iff \lambda = \sqrt{\frac{1 + \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}}{V - 2\Omega W - \Omega^2 X}}$$

Perfect conductor hypothesis (1/2)

$$\boxed{\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0}$$

with the fluid 4-velocity decomposed as

$$\boxed{\vec{\boldsymbol{u}} = \lambda(\vec{\boldsymbol{\xi}} + \Omega\vec{\boldsymbol{\chi}}) + \vec{\boldsymbol{w}}, \qquad \vec{\boldsymbol{w}} \in \Pi^\perp} \qquad (10)$$

Ω is the rotational angular velocity and $\vec{\boldsymbol{w}}$ is the meridional velocity

$$\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1 \iff \lambda = \sqrt{\frac{1 + \underline{\boldsymbol{w}} \cdot \vec{\boldsymbol{w}}}{V - 2\Omega W - \Omega^2 X}}$$

We have

$$\boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = 0} \quad \text{and} \quad \boxed{\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Psi = 0}, \qquad (11)$$

i.e. the scalar potentials Φ and Ψ are constant along the fluid lines.

Proof: $\mathcal{L}_{\vec{\boldsymbol{u}}}\, \Phi = \vec{\boldsymbol{u}} \cdot \mathbf{d}\Phi = -\boldsymbol{F}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{u}}) = 0$ by the perfect conductor property.

Corollary: since we had already $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Phi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Phi = 0$ and $\mathcal{L}_{\vec{\boldsymbol{\xi}}}\, \Psi = \mathcal{L}_{\vec{\boldsymbol{\chi}}}\, \Psi = 0$, it follows from (11) that

$$\boxed{\vec{\boldsymbol{w}} \cdot \mathbf{d}\Phi = 0} \quad \text{and} \quad \boxed{\vec{\boldsymbol{w}} \cdot \mathbf{d}\Psi = 0} \qquad (12)$$

Perfect conductor hypothesis (2/2)

Expressing the condition $\boldsymbol{F} \cdot \vec{\boldsymbol{u}} = 0$ with the general form (7) of a stationary-axisymmetric electromagnetic field yields

$$\underbrace{(\boldsymbol{\xi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda}\mathbf{d}\Phi - \underbrace{(\mathbf{d}\Phi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\xi}^* + \underbrace{(\boldsymbol{\chi}^* \cdot \vec{\boldsymbol{u}})}_{\lambda\Omega}\mathbf{d}\Psi - \underbrace{(\mathbf{d}\Psi \cdot \vec{\boldsymbol{u}})}_{0}\boldsymbol{\chi}^* + \frac{I}{\sigma}\underbrace{\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, ., \vec{\boldsymbol{u}})}_{-\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}},\vec{\boldsymbol{\chi}},\vec{\boldsymbol{w}},.)} = 0$$

Hence

$$\boxed{\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma\lambda}\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} \tag{13}$$

Conservation of baryon number and stream function

Baryon number conservation : $\boxed{\boldsymbol{\nabla}\cdot(n\vec{\boldsymbol{u}}) = 0} \iff \boxed{\mathbf{d}(n\star\underline{\boldsymbol{w}}) = 0}$

Conservation of baryon number and stream function

Baryon number conservation : $\boxed{\nabla \cdot (n\vec{u}) = 0}$ $\iff$ $\boxed{\mathbf{d}(n \star \underline{w}) = 0}$

$\rightarrow$Poincaré Lemma: $\exists$ a 2-form $\boldsymbol{H}$ such that $n \star \underline{w} = \mathbf{d}\boldsymbol{H}$

Considering the scalar field $f := \boldsymbol{H}(\vec{\xi}, \vec{\chi})$, we get

$$\boxed{\mathbf{d}f = n\,\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{w}, .)} \iff \boxed{\vec{w} = -\frac{1}{\sigma n}\,\vec{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, .)} \qquad (14)$$

f is called the **(Stokes) stream function**

Conservation of baryon number and stream function

Baryon number conservation : $\boxed{\boldsymbol{\nabla}\cdot(n\vec{\boldsymbol{u}}) = 0}$ $\iff$ $\boxed{\mathbf{d}(n \star \underline{\boldsymbol{w}}) = 0}$

$\rightarrow$Poincaré Lemma: $\exists$ a 2-form $\boldsymbol{H}$ such that $n \star \underline{\boldsymbol{w}} = \mathbf{d}\boldsymbol{H}$

Considering the scalar field $f := \boldsymbol{H}(\vec{\xi}, \vec{\chi})$, we get

$$\boxed{\mathbf{d}f = n\,\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} \iff \boxed{\vec{\boldsymbol{w}} = -\frac{1}{\sigma n}\,\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} f, .)} \qquad (14)$$

f is called the **(Stokes) stream function**

It follows from (14) that

- $\vec{\xi}\cdot\mathbf{d}f = 0$ and $\vec{\chi}\cdot\mathbf{d}f = 0 \Longrightarrow$ f obeys to the spacetime symmetries
- $\vec{u}\cdot\mathbf{d}f = 0 \Longrightarrow$ f is constant along any fluid line

Conservation of baryon number and stream function

Baryon number conservation : $\boxed{\boldsymbol{\nabla} \cdot (n\vec{\boldsymbol{u}}) = 0}$ $\iff$ $\boxed{\mathbf{d}(n \star \underline{\boldsymbol{w}}) = 0}$

$\rightarrow$Poincaré Lemma: $\exists$ a 2-form $\boldsymbol{H}$ such that $n \star \underline{\boldsymbol{w}} = \mathbf{d}\boldsymbol{H}$

Considering the scalar field $f := \boldsymbol{H}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}})$, we get

$$\boxed{\mathbf{d}f = n\,\boldsymbol{\epsilon}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{w}}, .)} \iff \boxed{\vec{\boldsymbol{w}} = -\frac{1}{\sigma n}\,\vec{\boldsymbol{\epsilon}}(\vec{\boldsymbol{\xi}}, \vec{\boldsymbol{\chi}}, \vec{\boldsymbol{\nabla}} f, .)} \qquad (14)$$

f is called the **(Stokes) stream function**

It follows from (14) that

- $\vec{\boldsymbol{\xi}} \cdot \mathbf{d}f = 0$ and $\vec{\boldsymbol{\chi}} \cdot \mathbf{d}f = 0 \Longrightarrow$ f obeys to the spacetime symmetries
- $\vec{\boldsymbol{u}} \cdot \mathbf{d}f = 0 \Longrightarrow$ f is constant along any fluid line

The perfect conductivity relation (13) is writable as

$$\boxed{\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma n \lambda}\,\mathbf{d}f} \qquad (15)$$

Integrating the MHD-Euler equation

With the writing (10) of $\vec{u}$, (7) of $\boldsymbol{F}$ and (9) of $\vec{j}$, the MHD-Euler equation

$$\vec{u} \cdot \mathbf{d}(h\underline{u}) - T\mathbf{d}S = \frac{1}{n}\boldsymbol{F} \cdot \vec{j}$$

can be shown to be equivalent to the system

$$\vec{w} \cdot \mathbf{d}(h\underline{u} \cdot \vec{\xi}) - \frac{1}{\mu_0 \sigma n}\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla}\Phi) = 0 \qquad (16)$$

$$\vec{w} \cdot \mathbf{d}(h\underline{u} \cdot \vec{\chi}) - \frac{1}{\mu_0 \sigma n}\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} I, \vec{\nabla}\Psi) = 0 \qquad (17)$$

$$\lambda\, \mathbf{d}(h\underline{u} \cdot \vec{\xi}) + \lambda\Omega\, \mathbf{d}(h\underline{u} \cdot \vec{\chi}) - \frac{1}{n}\left[q + \frac{\lambda h}{\sigma}\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\mathscr{C}_{\boldsymbol{\chi}}\right)\right]\mathbf{d}f - \frac{I}{\mu_0 \sigma n}\mathbf{d}I$$
$$+ \frac{\boldsymbol{\xi}^* \cdot \vec{j}}{n}\mathbf{d}\Phi + \frac{\boldsymbol{\chi}^* \cdot \vec{j}}{n}\mathbf{d}\Psi + T\,\mathbf{d}S = 0. \qquad (18)$$

with $q := -\nabla_\mu \left(\dfrac{h}{\sigma\, n}\nabla^\mu f\right)$

Introducing the master potential (1/2)

As a consequence of the perfect conductivity properties (12) and the baryon number conservation relation (14), one has

$$\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\boldsymbol{\nabla}} f, \vec{\boldsymbol{\nabla}}\Phi) = 0 \quad \text{and} \quad \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\boldsymbol{\nabla}} f, \vec{\boldsymbol{\nabla}}\Psi) = 0$$

Along with Eq. (15) above, i.e.

$$\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma n \lambda}\,\mathbf{d}f$$

this implies that

The gradient 1-forms $\mathbf{d}\Phi$, $\mathbf{d}\Psi$ and $\mathbf{d}f$ are colinear to each other

Introducing the master potential (1/2)

As a consequence of the perfect conductivity properties (12) and the baryon number conservation relation (14), one has

$$\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, \vec{\nabla}\Phi) = 0 \quad \text{and} \quad \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, \vec{\nabla}\Psi) = 0$$

Along with Eq. (15) above, i.e.

$$\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma n \lambda}\,\mathbf{d}f$$

this implies that

The gradient 1-forms $\mathbf{d}\Phi$, $\mathbf{d}\Psi$ and $\mathbf{d}f$ are colinear to each other

Standard approach in GRMHD: privilege Ψ and write $\mathbf{d}\Phi = -\omega\,\mathbf{d}\Psi$, $\mathbf{d}f = a\,\mathbf{d}\Psi$

Introducing the master potential (1/2)

As a consequence of the perfect conductivity properties (12) and the baryon number conservation relation (14), one has

$$\boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\boldsymbol{\nabla}} f, \vec{\boldsymbol{\nabla}} \Phi) = 0 \quad \text{and} \quad \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\boldsymbol{\nabla}} f, \vec{\boldsymbol{\nabla}} \Psi) = 0$$

Along with Eq. (15) above, i.e.

$$\mathbf{d}\Phi = -\Omega\, \mathbf{d}\Psi + \frac{I}{\sigma n \lambda}\, \mathbf{d}f$$

this implies that

The gradient 1-forms $\mathbf{d}\Phi$, $\mathbf{d}\Psi$ and $\mathbf{d}f$ are colinear to each other

Standard approach in GRMHD: privilege Ψ and write $\mathbf{d}\Phi = -\omega\, \mathbf{d}\Psi$, $\mathbf{d}f = a\, \mathbf{d}\Psi$
Drawback: This is degenerate if $\mathbf{d}\Psi = 0$

Introducing the master potential (1/2)

As a consequence of the perfect conductivity properties (12) and the baryon number conservation relation (14), one has

$$\epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, \vec{\nabla}\Phi) = 0 \quad \text{and} \quad \epsilon(\vec{\xi}, \vec{\chi}, \vec{\nabla} f, \vec{\nabla}\Psi) = 0$$

Along with Eq. (15) above, i.e.

$$\mathbf{d}\Phi = -\Omega\,\mathbf{d}\Psi + \frac{I}{\sigma n \lambda}\,\mathbf{d}f$$

this implies that

The gradient 1-forms $\mathbf{d}\Phi$, $\mathbf{d}\Psi$ and $\mathbf{d}f$ are colinear to each other

Standard approach in GRMHD: privilege Ψ and write $\mathbf{d}\Phi = -\omega\,\mathbf{d}\Psi$, $\mathbf{d}f = a\,\mathbf{d}\Psi$

Drawback: This is degenerate if $\mathbf{d}\Psi = 0$

Here: we follow the approach of Tkalich (1959) and Soloviev (1967) for Newtonian MHD, i.e. we introduce a fourth potential Υ such that

1. Υ obeys to both spacetime symmetries
2. $\mathbf{d}\Upsilon \neq 0$
3. $\exists$ three scalar fields α, β and γ such that

$$\boxed{\mathbf{d}\Phi = \alpha\,\mathbf{d}\Upsilon, \quad \mathbf{d}\Psi = \beta\,\mathbf{d}\Upsilon, \quad \mathbf{d}f = \gamma\,\mathbf{d}\Upsilon}$$

Introducing the master potential (2/2)

$$\boxed{\mathbf{d}\Phi = \alpha\,\mathbf{d}\Upsilon, \quad \mathbf{d}\Psi = \beta\,\mathbf{d}\Upsilon, \quad \mathbf{d}f = \gamma\,\mathbf{d}\Upsilon}$$

- All potentials can be considered as functions of Υ:

$$\begin{aligned} \Phi &= \Phi(\Upsilon), \quad \Psi = \Psi(\Upsilon), \quad f = f(\Upsilon), \\ \alpha &= \Phi'(\Upsilon), \quad \beta = \Psi'(\Upsilon), \quad \gamma = f'(\Upsilon) \end{aligned}$$

Proof: $\mathbf{dd}\Phi = 0 = \mathbf{d}\alpha \wedge \mathbf{d}\Upsilon \Longrightarrow \alpha = \alpha(\Upsilon) \Longrightarrow \Phi = \Phi(\Upsilon)$ with $\alpha = \Phi'$

- Υ is conserved along the fluid lines (since f is)
- the perfect conductor property (13) leads to the relation

$$\alpha + \Omega\beta = \frac{\gamma I}{\sigma n \lambda}$$

Integrating the first two equations of the MHD-Euler system

Expressing $\vec{w}$ in terms of $\mathbf{d}f$ via (14) and using $\mathbf{d}f = \gamma\,\mathbf{d}\Upsilon$ as well as $\mathbf{d}\Phi = \alpha\,\mathbf{d}\Upsilon$ enables us to write the first equation of the MHD-Euler system [Eq. (16)] in the equivalent form

$$\boldsymbol{\epsilon}\left(\vec{\xi},\ \vec{\chi},\ \vec{\nabla}\Upsilon,\ -\gamma\vec{\nabla}(h\underline{\boldsymbol{u}}\cdot\vec{\xi}) + \frac{\alpha}{\mu_0}\vec{\nabla}I\right) = 0$$

$\Longrightarrow$ $-\gamma h\underline{\boldsymbol{u}}\cdot\vec{\xi} + \alpha I/\mu_0$ must be a function of Υ, $\Sigma(\Upsilon)$ say:

$$\boxed{\Sigma(\Upsilon) = -\gamma h\underline{\boldsymbol{u}}\cdot\vec{\xi} + \frac{\alpha I}{\mu_0} = \gamma\lambda h(V - W\Omega) + \frac{\alpha I}{\mu_0}}$$

Integrating the first two equations of the MHD-Euler system

Expressing $\vec{w}$ in terms of $\mathbf{d}f$ via (14) and using $\mathbf{d}f = \gamma\,\mathbf{d}\Upsilon$ as well as $\mathbf{d}\Phi = \alpha\,\mathbf{d}\Upsilon$ enables us to write the first equation of the MHD-Euler system [Eq. (16)] in the equivalent form

$$\boldsymbol{\epsilon}\left(\vec{\xi},\ \vec{\chi},\ \vec{\nabla}\Upsilon,\ -\gamma\vec{\nabla}(h\underline{\boldsymbol{u}}\cdot\vec{\xi}) + \frac{\alpha}{\mu_0}\vec{\nabla}I\right) = 0$$

$\Longrightarrow -\gamma h\underline{\boldsymbol{u}}\cdot\vec{\xi} + \alpha I/\mu_0$ must be a function of Υ, $\Sigma(\Upsilon)$ say:

$$\boxed{\Sigma(\Upsilon) = -\gamma h\underline{\boldsymbol{u}}\cdot\vec{\xi} + \frac{\alpha I}{\mu_0} = \gamma\lambda h(V - W\Omega) + \frac{\alpha I}{\mu_0}}$$

Similarly, the second equation of the MHD-Euler system [Eq. (17)] is equivalent to the existence of a function $\Lambda(\Upsilon)$ such that

$$\boxed{\Lambda(\Upsilon) = \gamma h\underline{\boldsymbol{u}}\cdot\vec{\chi} - \frac{\beta I}{\mu_0} = \gamma\lambda h(W + X\Omega) - \frac{\beta I}{\mu_0}}$$

Interpretation as Bernoulli-like theorems

- If the fluid motion is purely rotational, $\vec{u} = \lambda(\vec{\xi} + \Omega\vec{\chi})$ and any scalar quantity obeying to the two spacetime symmetries is conserved along the fluid lines
- If the fluid flow has some meridional component, $\vec{w} \neq 0 \iff \mathbf{d}f \neq 0$: we may choose $\boxed{\Upsilon = f}$; then $\gamma = 1$ and

$$\boxed{\Sigma = \lambda h(V - W\Omega) + \frac{\alpha I}{\mu_0}} \qquad \boxed{\Lambda = \lambda h(W + X\Omega) - \frac{\beta I}{\mu_0}}$$

We recover the two streamline-constants of motion found by Bekenstein & Oron (1978) in a slightly more complicated form:

$$\Sigma = -\left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\xi} - \frac{\beta}{\mu_0}\left[\underline{\boldsymbol{u}} \cdot \left(\vec{\xi} - \frac{\alpha}{\beta}\vec{\chi}\right)\right] (\underline{\boldsymbol{b}} \cdot \vec{\xi})$$

$$\Lambda = \left(h + \frac{|b|^2}{\mu_0 n}\right) \underline{\boldsymbol{u}} \cdot \vec{\chi} + \frac{\beta}{\mu_0}\left[\underline{\boldsymbol{u}} \cdot \left(\vec{\xi} - \frac{\alpha}{\beta}\vec{\chi}\right)\right] (\underline{\boldsymbol{b}} \cdot \vec{\chi})$$

Non-relativistic limit

At the Newtonian limit and in standard isotropic spherical coordinates (t, r, θ, φ),

$$\begin{cases} V = 1 + 2\Phi_{\rm grav}, \qquad W = 0 \\ X = (1 - 2\Phi_{\rm grav}) r^2 \sin^2\theta \\ \sigma = r^2 \sin^2\theta, \end{cases}$$

where $\Phi_{\rm grav}$ is the Newtonian gravitational potential ($|\Phi_{\rm grav}| \ll 1$)

Moreover, introducing the *mass density* $\rho := m_{\rm b}\, n$ ($m_{\rm b}$ mean baryon mass) and *specific enthalpy* $H := \dfrac{\varepsilon_{\rm int} + p}{\rho}$, we get $h = m_{\rm b}(1 + H)$ with $H \ll 1$

Then

$$\boxed{\frac{\Sigma}{m_b} - 1 = H + \Phi_{\rm grav} + \frac{v^2}{2} + \frac{\alpha I}{\mu_0 m_{\rm b}}}$$ (when $I = 0$, classical Bernoulli theorem)

$$\boxed{\frac{\Lambda}{m_b} = \Omega\, r^2 \sin^2\theta - \frac{\beta I}{\mu_0 m_{\rm b}}}$$

Entropy as a function of the master potential

Equation (2) (resulting from $\boldsymbol{\nabla} \cdot \boldsymbol{T} = 0$) implies successively

$$\vec{\boldsymbol{u}} \cdot \mathbf{d}S = 0 \Longrightarrow \vec{\boldsymbol{w}} \cdot \mathbf{d}S = 0 \Longrightarrow \boldsymbol{\epsilon}(\vec{\xi}, \vec{\chi}, \vec{\boldsymbol{\nabla}} f, \vec{\boldsymbol{\nabla}} S) = 0$$

If $\mathbf{d}f \neq 0$, this implies $S = S(f)$, i.e.

$$\boxed{S = S(\Upsilon)} \tag{19}$$

If $\mathbf{d}f = 0$ (purely rotational flow), we assume that (19) still holds

The master transfield equation

Thanks to the existence of $\Sigma(\Upsilon)$, $\Lambda(\Upsilon)$ and $S(\Upsilon)$, the remaining part of the MHD-Euler equation [Eq. (18)] can be rewritten as $\boxed{\mathscr{A}\,\mathbf{d}\Upsilon = 0}$. Since $\mathbf{d}\Upsilon \neq 0$, it is equivalent to $\mathscr{A} = 0$. Expliciting $\mathscr{A}$, we get the master transfield equation:

$$\boxed{\begin{aligned} & A\,\Delta^*\Upsilon + \frac{n}{h}\left[\gamma^2\mathbf{d}\left(\frac{h}{n}\right) - \frac{1}{\mu_0}\left(\beta^2\mathbf{d}V + 2\alpha\beta\mathbf{d}W - \alpha^2\mathbf{d}X\right)\right]\cdot\vec{\boldsymbol{\nabla}}\Upsilon \\ & + \left\{\gamma\gamma' - \frac{n}{\mu_0 h}\left[V\beta\beta' + W(\alpha'\beta + \alpha\beta') - X\alpha\alpha'\right]\right\}\mathbf{d}\Upsilon\cdot\vec{\boldsymbol{\nabla}}\Upsilon \\ & + \frac{\sigma n^2}{h}\left\{\frac{\lambda}{\gamma}\left[\Omega\Lambda' - \Sigma' + \frac{I}{\mu_0}(\alpha' + \Omega\beta') + \gamma'\lambda h(V - 2W\Omega - X\Omega^2)\right] + TS'\right\} \\ & - \gamma\lambda n\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\mathscr{C}_{\boldsymbol{\chi}}\right) + \frac{In}{\mu_0\sigma h}\left[(W\beta - X\alpha)\mathscr{C}_{\boldsymbol{\xi}} + (W\alpha + V\beta)\mathscr{C}_{\boldsymbol{\chi}}\right] = 0 \end{aligned}} \tag{20}$$

with $A := \gamma^2 - \dfrac{n}{\mu_0 h}(V\beta^2 + 2W\alpha\beta - X\alpha^2)$ and $\boxed{\Delta^*\Upsilon := \sigma\,\nabla_\mu\left(\dfrac{1}{\sigma}\nabla^\mu\Upsilon\right)}$

Eq. (20) is called *transfield* for it expresses the component along $\mathbf{d}\Upsilon$ of the MHD-Euler equation and $\mathbf{d}\Upsilon$ is *transverse* to the magnetic *field* in the fluid frame $\vec{b}$, in the sense that $\vec{b}\cdot\mathbf{d}\Upsilon = 0$

Poloidal wind equation

The master transfield eq. has to be supplemented by the poloidal wind equation, arising from the 4-velocity normalization $\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$, with λ and Ω expressed in terms of α, β, γ, Σ, Λ and h :

$$\boxed{\begin{aligned} & h^2 \left(\sigma + \frac{\gamma^2}{n^2} \mathbf{d}\Upsilon \cdot \vec{\boldsymbol{\nabla}}\Upsilon \right) - \frac{1}{\gamma^2} \left(X\Sigma^2 + 2W\Sigma\Lambda - V\Lambda^2 \right) \\ & \qquad + \frac{n}{\mu_0 h} \frac{A+\gamma^2}{A^2\gamma^2} \left[(X\alpha - W\beta)\Sigma + (V\beta + W\alpha)\Lambda \right]^2 = 0 \end{aligned}} \tag{21}$$

Poloidal wind equation

The master transfield eq. has to be supplemented by the poloidal wind equation, arising from the 4-velocity normalization $\underline{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = -1$, with λ and Ω expressed in terms of α, β, γ, Σ, Λ and h :

$$\boxed{\begin{aligned} & h^2 \left(\sigma + \frac{\gamma^2}{n^2} \mathbf{d}\Upsilon \cdot \vec{\boldsymbol{\nabla}} \Upsilon \right) - \frac{1}{\gamma^2} \left(X\Sigma^2 + 2W\Sigma\Lambda - V\Lambda^2 \right) \\ & \quad + \frac{n}{\mu_0 h} \frac{A+\gamma^2}{A^2\gamma^2} \left[(X\alpha - W\beta)\Sigma + (V\beta + W\alpha)\Lambda \right]^2 = 0 \end{aligned}} \qquad (21)$$

Notice that I, λ and Ω in Eq. (20) can be expressed in terms of α, β, γ, Σ, Λ, n and h. Then

Given

- the metric (represented by V, X, W, σ and $\boldsymbol{\nabla}$),
- the EOS $h = h(n, S)$,
- the six functions $\alpha(\Upsilon)$, $\beta(\Upsilon)$, $\gamma(\Upsilon)$, $\Sigma(\Upsilon)$, $\Lambda(\Upsilon)$ and $S(\Upsilon)$,

Eqs. (20)-(21) constitute a system of 2 PDEs for the 2 unknowns (Υ, n)

Solving it provides a complete solution of the MHD-Euler equation

Outline

Subcase 1 : Newtonian limit

Expression of Σ and Λ:

$$\begin{aligned}\Sigma &= \gamma m_{\rm b}\left(1 + H + \Phi_{\rm grav} + \frac{v^2}{2}\right) + \frac{\alpha I}{\mu_0} \\ \Lambda &= \gamma m_{\rm b} r^2 \sin^2\theta\,\Omega - \frac{\beta I}{\mu_0}\end{aligned}$$

Master transfield equation:

$$\boxed{\begin{aligned}&A\Delta^*\Upsilon - \frac{\gamma^2}{n}\,\mathbf{d}n\cdot\vec{\nabla}\Upsilon + \left(\gamma\gamma' - \frac{n}{\mu_0 m_{\rm b}}\beta\beta'\right)\mathbf{d}\Upsilon\cdot\vec{\nabla}\Upsilon \\ &\quad + r^2\sin^2\theta\,\frac{n^2}{m_{\rm b}}\left\{\frac{1}{\gamma}\left[\Omega\Lambda' - \Sigma' + \frac{I}{\mu_0}(\alpha' + \Omega\beta') + \gamma' m_{\rm b}\right] + TS'\right\} = 0\end{aligned}}$$

with $\Delta^*\Upsilon = \partial_r^2\Upsilon + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\Upsilon\right)$

We recover the equation obtained by Soloviev (1967)

Subcase 2 : relativistic Grad-Shafranov equation

Assume $\mathbf{d}\Psi \neq 0$ and choose $\Upsilon = \Psi$ (i.e. $\beta = 1$).
The master transfield equation reduces then to

$$\boxed{\begin{aligned}&\left(1 - \frac{V - 2W\omega - X\omega^2}{M^2}\right)\Delta^*\Psi + \left[\frac{n}{h}\mathbf{d}\left(\frac{h}{n}\right) - \frac{1}{M^2}\left(\mathbf{d}V - 2\omega\mathbf{d}W - \omega^2\mathbf{d}X\right)\right]\cdot\vec{\boldsymbol{\nabla}}\Psi \\ &+ \left[\frac{\omega'}{M^2}(W + X\omega) - \frac{C'}{C}\right]\mathbf{d}\Psi\cdot\vec{\boldsymbol{\nabla}}\Psi \\ &+ \frac{\mu_0\sigma n}{M^2}\left\{\lambda\left[\Omega L' - E' + \frac{I}{\mu_0}\left(C'(\Omega - \omega) - C\omega'\right)\right] + TS'\right\} \\ &- \lambda n C\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\,\mathscr{C}_{\boldsymbol{\chi}}\right) + \frac{I}{\sigma M^2}\left[(W + X\omega)\mathscr{C}_{\boldsymbol{\xi}} + (V - W\omega)\mathscr{C}_{\boldsymbol{\chi}}\right] = 0\end{aligned}}$$

where $C := \gamma^{-1}$, $\omega := -\alpha$, $E := \Sigma/\gamma$ and $L := \Lambda/\gamma$

This is the relativistic Grad-Shafranov equation, in the most general form (i.e. including meridional flow and for non-circular spacetimes)

Subcase 2 : relativistic Grad-Shafranov equation

History of the relativistic Grad-Shafranov equation:

- Camenzind (1987) : Minkowski spacetime
- Lovelace, Mehanian, Mobarry & Sulkanen (1986) : weak gravitational fields
- Lovelace & Mobarry (1986) : Schwarzschild spacetime
- Nitta, Takahashi & Tomimatsu (1991) : Kerr spacetime (pressureless matter)
- Beskin & Pariev (1993) : Kerr spacetime
- Ioka & Sasaki (2003) : non-circular spacetimes

Remark: Grad-Shafranov equation not fully explicited by Ioka & Sasaki (2003) + use of additional structure ((2+1)+1 formalism)

Subcase 3 : pure hydrodynamical flow

No electromagnetic field

$$\mathbf{d}\Phi = 0 \ (\iff \alpha = 0),\ \mathbf{d}\Psi = 0 \ (\iff \beta = 0) \text{ and } I = 0$$

$\Longrightarrow \Sigma = \gamma\, E$ with $E := \lambda h(V - W\Omega)$ and $\Lambda = \gamma\, L$ with $L := \lambda h(W + X\Omega)$

Master transfield + poloidal wind equations:

$$\gamma^2 \Delta^* \Upsilon + \gamma^2 \frac{n}{h} \mathbf{d}\left(\frac{h}{n}\right) \cdot \vec{\nabla}\Upsilon + \gamma\gamma' \mathbf{d}\Upsilon \cdot \vec{\nabla}\Upsilon + \frac{\sigma n^2}{h}\left[\lambda(\Omega L' - E') + TS'\right] - \gamma\lambda n \left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega \mathscr{C}_{\boldsymbol{\chi}}\right) = 0$$

$$\frac{\gamma^2 h^2}{n^2} \mathbf{d}\Upsilon \cdot \vec{\nabla}\Upsilon + \sigma h^2 - XE^2 - 2WEL + VL^2 = 0$$

Subcase 3 : pure hydrodynamical flow

Case of purely rotational motion : $\gamma = 0$

The master transfield equation reduces to

$$\boxed{\Omega L' - E' + \frac{T}{\lambda} S' = 0}$$

A wide class of solutions is found by assuming

$$\Omega = \Omega(\Upsilon) \text{ and } \frac{T}{\lambda} = \bar{T}(\Upsilon) \text{ with } \bar{T}' = -\bar{T} \frac{\lambda L}{h} \Omega'$$

For $\Omega = \mathrm{const}$, this leads to the well-known first integral of motion [$T = 0$: Boyer (1965)]

$$\boxed{\frac{\mu}{\lambda} = \frac{h - TS}{\lambda} = \mathrm{const}}$$

For $\Omega \neq \mathrm{const}$, we obtain instead

$$\boxed{\ln\left(\frac{\mu}{\lambda}\right) + \int_0^{\Omega} \mathcal{F}(\tilde{\Omega})\, d\tilde{\Omega} = \mathrm{const}}$$

with $\mathcal{F}(\Omega) = \dfrac{W + X\Omega}{V - 2W\Omega - X\Omega^2}$ (relativistic Poincaré-Wavre theorem)

Subcase 3 : pure hydrodynamical flow

Case of flow with meriodional component : $\gamma \neq 0$

Then $\mathbf{d}f \neq 0$ and a natural choice for Υ is $\boxed{\Upsilon = f}$

The master transfield + poloidal wind equations reduces to

$$\boxed{\Delta^* f + \frac{n}{h}\mathbf{d}\left(\frac{h}{n}\right)\cdot\vec{\boldsymbol{\nabla}} f + \frac{\sigma n^2}{h}\left[\lambda(\Omega L' - E') + TS'\right] - \lambda n\left(\mathscr{C}_{\boldsymbol{\xi}} + \Omega\mathscr{C}_{\boldsymbol{\chi}}\right) = 0} \tag{22}$$

$$\boxed{\frac{h^2}{n^2}\mathbf{d}f\cdot\vec{\boldsymbol{\nabla}} f + \sigma h^2 - XE^2 - 2WEL + VL^2 = 0} \tag{23}$$

with $\Omega = \dfrac{VL - WE}{XE + WL}$

Given the three functions $E(f)$, $L(f)$ and $S(f)$ and the EOS $h = h(S, n)$, $T = T(S, n)$, (22)-(23) forms a system of coupled PDE for (f, n)

At the Newtonian limit, (22) is the Stokes equation

Outline

Conclusions

- Ideal GRMHD is well amenable to a treatment based on exterior calculus.
- This simplifies calculations with respect to the traditional tensor calculus, notably via the massive use of Cartan's identity.
- For stationary and axisymmetric GRMHD, we have developed a systematic treatment based on such an approach. This provides some insight on previously introduced quantities and leads to the formulation of very general laws, recovering previous ones as subcases and obtaining new ones in some specific limits.

Bibliography

- J. D. Bekenstein & E. Oron : *New conservation laws in general-relativistic magnetohydrodynamics*, Phys. Rev. D **18**, 1809 (1978)
- B. Carter : *Perfect fluid and magnetic field conservations laws in the theory of black hole accretion rings*, in *Active Galactic Nuclei*, Eds. C. Hazard & S. Mitton, Cambridge University Press (Cambridge), p. 273 (1979)
- E. Gourgoulhon : *An introduction to the theory of rotating relativistic stars*, arXiv:1003.5015 (2010)
- E. Gourgoulhon, C. Markakis, K. Uryu & Y. Eriguchi : *Magnetohydrodynamics in stationary and axisymmetric spacetimes: a fully covariant approach*, in preparation
- K. Ioka & M. Sasaki : *Grad-Shafranov equation in noncircular stationary axisymmetric spacetimes*, Phys. Rev. D **67**, 12026 (2003)
- A. Lichnerowicz : *Relativistic hydrodynamics and magnetohydrodynamics*, Benjamin (New York) (1967)
- S. Nitta, M. Takahashi & A. Tomimatsu : *Effects of magnetohydrodynamics accretion flows on global structure of a Kerr black-hole magnetosphere*, Phys. Rev. D **44**, 2295 (1991)